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Calculus of Variations

Compactness and existence results of the prescribing fractional Q-curvature problem on $\mathbb{S}^n$

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Abstract

This paper is devoted to establishing the compactness and existence results of the solutions to the prescribing fractional Q-curvature problem of order 2σ on n-dimensional standard sphere when $n - 2\sigma = 2$, $\sigma = 1 + m/2$, $m \in \mathbb{N}_+$. The compactness results are novel and optimal. In addition, we prove a degree-counting formula of all solutions to achieve the existence. From our results, we can know where blow up occur. Furthermore, the sequence of solutions that blow up precisely at any finite distinct location can be constructed. It is worth noting that our results include the case of multiple harmonic.

Mathematics Subject Classification 35R09 · 35B44 · 35J35

1 Introduction

The study of the prescribing scalar curvature problem on Riemannian manifolds, which dates back to [34–36], has received a lot of attention. In the case of n-dimensional standard sphere $(\mathbb{S}^n, g_0)$, this is known as Nirenberg problem. The classical Nirenberg problem is as follows: which function K on $(\mathbb{S}^n, g_0)$ is the scalar curvature (Gauss curvature in dimension $n = 2$) of a metric g that is conformal to g_0? If we denote $g = e^{2v} g_0$ in the two dimensional case and $g = v^{\frac{4}{n-2}} g_0$ in the n ($n \geq 3$) dimensional case, this problem is equivalent to solving the

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following nonlinear elliptic equations:

$$-\Delta_{g_0} v + 1 = K e^{2v} \quad \text{on } \mathbb{S}^2, \tag{1.1}$$

and

$$-\Delta_{g_0} v + c(n) R_0 v = c(n) K v^{\frac{n+2}{n-2}} \quad \text{on } \mathbb{S}^n, \quad n \geq 3, \tag{1.2}$$

where Δ_{g_0} is the Laplace–Beltrami operator, $c(n) = \frac{n-2}{4(n-1)}$, $R_0 = n(n-1)$ is the scalar curvature associated to g_0.

A first answer to the Nirenberg problem was given by Koutroufiotis [38], which established the existence of the solutions to (1.1) by assuming that K is an antipodally symmetric function which close to 1. Morse [45] proved the existence of antipodally symmetric solutions to (1.1) for all antipodally symmetric functions K which are positive somewhere. Chang and Yang [11] further extended this existence result to the case of K without making any symmetry assumptions. Moreover, Bahri and Coron [6] presented a sufficient condition for the existence of solutions to (1.2) in dimension $n = 3$. As for the compactness of all solutions in dimensions $n = 2, 3$, Chang et al. [12], Han [25], and Schoen and Zhang [50] proved that a sequence of solutions cannot blow up at more than one point. Li [40, 41] established the compactness and existence results for (1.2). In these two papers, the compactness result is very different from the previous low-dimensional case. In fact, when $n = 2$ or $n = 3$, a sequence of solutions cannot blow up at more than one point. However, if $n > 3$, there could be blow up at many points, which considerably complicates the study of the problem. There have been many papers on the problem and related ones, see e.g., [9, 13, 20, 26, 28, 48, 49, 51].

The linear operators defined on left-hand side of (1.1) and (1.2) are called the conformal Laplacian associated to the metric g_0 and are denoted as $P_1^{g_0}$. For any Riemannian manifold (M, g), let R_g be the scalar curvature of (M, g), and the conformal Laplacian be defined as $P_1^g = -\Delta_g + \frac{n-2}{4(n-1)} R_g$. The Paneitz operator P_2^g is another conformal invariant operator, which was discovered by Paneitz [46]. Graham et al. [23] constructed a sequence of conformally covariant elliptic operators $\{P_k^g\}$ on Riemannian manifolds for all positive integers k if n is odd, and for $k \in \{1, \cdots, n/2\}$ if n is even, which are called GJMS operators. Juhl [32, 33] found an explicit formula and a recursive formula for GJMS operators and Q-curvature (see also Fefferman and Graham [22]). Graham and Zworski [24] presented a family of fractional order conformally invariant operators P_σ^g of non-integer order $\sigma \in (0, n/2)$ on the conformal infinity of asymptotically hyperbolic manifolds. In addition, Chang and González [10] showed that the operator P_σ^g, $\sigma \in (0, n/2)$ can be defined as a Dirichlet-to-Neumann operator of a conformally compact Einstein manifold by using localization method in [8], they also provided some new interpretations and properties of those fractional operators and their associated fractional Q-curvature.

Regarded as a generalization of Nirenberg problem, the prescribing fractional Q-curvature problem of order 2σ on $\mathbb{S}^n$ can be described as: which function K on $\mathbb{S}^n$ is the fractional Q-curvature of a metric g on $\mathbb{S}^n$ conformally equivalent to g_0? If we denote $g = v^{4/(n-2\sigma)} g_0$, this problem can be represented as finding the solution of the following nonlinear equation with critical exponent:

$$P_\sigma^{g_0}(v) = c(n, \sigma) K v^{\frac{n+2\sigma}{n-2\sigma}} \quad \text{on } \mathbb{S}^n, \tag{1.3}$$

where $n \geq 2$, $0 < \sigma < n/2$, $c(n, \sigma) = \Gamma(\frac{n}{2} + \sigma) / \Gamma(\frac{n}{2} - \sigma)$, Γ is the Gamma function, K is a function defined on $\mathbb{S}^n$, $P_\sigma^{g_0}$ is an intertwining operator of 2σ-order:

$$P_\sigma^{g_0} = \frac{\Gamma(B + \frac{1}{2} + \sigma)}{\Gamma(B + \frac{1}{2} - \sigma)}, \quad B = \sqrt{-\Delta_{g_0} + \Big(\frac{n-1}{2}\Big)^2}.$$

In what follows, $P_\sigma^{g_0}$ is simply written as P_σ. It can be viewed as the pull back operator of the σ power of the Laplacian $(-\Delta)^\sigma$ on $\mathbb{R}^n$ via the stereographic projection:

$$(P_\sigma(v)) \circ F = |J_F|^{-\frac{n+2\sigma}{2n}} (-\Delta)^\sigma (|J_F|^{\frac{n-2\sigma}{2n}} (v \circ F)) \quad \text{for } v \in C^{2\sigma}(\mathbb{S}^n),$$

where F is the inverse of the stereographic projection and $|J_F|$ is the determinant of the Jacobian of F. In addition, the Green function of P_σ is the spherical Riesz potential, i.e.,

$$P_\sigma^{-1} f(\xi) = c_{n,\sigma} \int_{\mathbb{S}^n} \frac{f(\zeta)}{|\xi - \zeta|^{n-2\sigma}} \mathrm{d}vol_{g_0}(\zeta) \quad \text{for } f \in L^p(\mathbb{S}^n), \tag{1.4}$$

where $c_{n,\sigma} = \frac{\Gamma(\frac{n-2\sigma}{2})}{2^{2\sigma}\pi^{n/2}\Gamma(\sigma)}$, $p > 1$, and $|\cdot|$ is the Euclidean distance in $\mathbb{R}^{n+1}$.

Many research have been conducted on the fractional operators P_σ^g and their associated fractional Q-curvature, for instance, see [2, 3, 15–19, 21, 29–31, 43, 44]. The flatness of the prescribing fractional Q-curvature function K plays a crucial role in the study of this problem. We begin with the definition of the β-flatness condition that characterizes flatness.

β-**flatness condition:** Let $K \in C^1(\mathbb{S}^n)$ ($K \in C^{1,1}(\mathbb{S}^n)$ if $0 < \sigma \le 1/2$) be a positive function and β is a positive constant, we say that K satisfies the β-flatness condition if for every critical point ξ_0 of K, in some geodesic normal coordinates $\{y_1, \cdots, y_n\}$ centered at ξ_0, there exists a small neighborhood $\mathscr{O}$ of 0 and $a_j(\xi_0) \neq 0$, $\sum_{j=1}^n a_j(\xi_0) \neq 0$, such that

$$K(y) = K(0) + \sum_{j=1}^{n} a_j(\xi_0)|y_j|^\beta + R(y) \quad \text{in } \mathscr{O},$$

where $\sum_{s=0}^{[\beta]} |\nabla^s R(y)||y|^{-\beta+s} \to 0$ as $y \to 0$, here ∇^s denotes all possible derivatives of order s and $[\beta]$ is the integer part of β. We call β the flatness order.

For $\sigma \in (0, 1)$ and $\beta \in (n - 2\sigma, n)$, Jin et al. [29, 30] proved the existence of the solutions to (1.3) and derived some compactness properties when K satisfies the β-flatness condition by using the approach based on approximation of the solutions to (1.3) by a blow up subcritical method. For $\sigma \in (0, n/2)$ and $\beta \in (n - 2\sigma, n)$, Jin et al. [31] developed a unified approach to establish blow up profiles, compactness and existence of positive solutions to (1.3) when K satisfies β-flatness condition by making use of integral representations. Since their conclusions are valid only when the flatness order $n - 2\sigma < \beta < n$, some very interesting functions K are excluded. In fact, note that an important class of functions, which is worth including in the results of existence and compactness for (1.3), are the Morse functions with only non-degenerate critical points. Such functions satisfy the 2-flatness condition.

Existence results of the solutions to (1.3) were given when $\beta \in (1, n - 2\sigma]$ by Abdelhedi et al. [3], and when $\beta \in [n - 2\sigma, n)$ by Chtioui and Abdelhedi [16]. Under a so-called "non-degenerate condition", Khadijah and Chtioui [37] studied the lack of compactness and provided the existence results for (1.3) when $\beta = n - 2\sigma = 2$, $\sigma \in (0, n/2)$.

However, under the assumption of the flatness order $\beta = n - 2\sigma$ of prescribing curvature function K, the precise compactness results of the solutions to (1.3) are unknown. When $\sigma = 1$ and $n = 2\sigma + 2 = 4$, Li [41] obtained the optimal compactness and a degree-counting formula of the solutions to (1.2) when K is some special class of functions satisfying condition $2 = n - 2\sigma$-flatness condition. Therefore, a quite natural question arises: can we establish the optimal compactness results and provide a degree-counting formula of the solutions to (1.3) when the curvature function K is specified as a special function satisfying the β-flatness condition with $\beta = n - 2\sigma = 2$? The main target of this article is to give an affirmative answer to this question.

In the present paper, we are interested to the prescribing fractional Q-curvature problem (1.3), $n = 2\sigma + 2$, $\sigma = 1 + m/2$, $m \in \mathbb{N}_+$. Our aim is to establish the optimal compactness and existence results of the solutions, when the prescribing curvature function K is some special function satisfying $2 = n - 2\sigma$-flatness condition. In order to obtain an existence result, we will prove a degree-counting formula of the solutions to (1.3). This counting formula, together with the compactness results completely describes where blow up occur. Especially, from our results, we can construct a sequence of solutions to (1.3) that blow up precisely at these points for any finite distinct points on $\mathbb{S}^n$.

First of all, Eq. (1.3) is not always solvable. Indeed, we have the Kazdan-Warner type obstruction: for any conformal Killing vector field X on $\mathbb{S}^n$, there holds

$$\int_{\mathbb{S}^n} (\nabla_X K) v^{\frac{2n}{n-2\sigma}} \mathrm{d}vol_{g_{\mathbb{S}^n}} = 0$$

for any solution v of (1.3), see [7, 52].

Before state our results, we introduce some definitions and notations.

For $K \in C^2(\mathbb{S}^n)$, we introduce the following notation:

$$\begin{aligned} \mathscr{K} &= \{q \in \mathbb{S}^n : \nabla_{g_0} K(q) = 0\}, \\ \mathscr{K}^+ &= \{q \in \mathbb{S}^n : \nabla_{g_0} K(q) = 0,\ \Delta_{g_0} K(q) > 0\}, \\ \mathscr{K}^- &= \{q \in \mathbb{S}^n : \nabla_{g_0} K(q) = 0,\ \Delta_{g_0} K(q) < 0\}, \\ \mathscr{M}_K &= \{v \in C^{2\sigma}(\mathbb{S}^n) : v \text{ satisfies } (1.3)\}. \end{aligned} \tag{1.5}$$

For any k ($k \geq 1$) distinct points $q^{(1)}, \cdots, q^{(k)} \in \mathscr{K} \backslash \mathscr{K}^+$, the $k \times k$ symmetric matrix $M = (M(q^{(1)}, \cdots, q^{(k)}))$ is defined by

$$\begin{aligned} M_{ii} &= -\frac{\Delta_{g_0} K(q^{(i)})}{K(q^{(i)})^{n/2\sigma}}, \\ M_{ij} &= -n(n-1)\frac{G_{q^{(i)}}(q^{(j)})}{(K(q^{(i)})K(q^{(j)}))^{1/2\sigma}}, \quad i \neq j, \end{aligned} \tag{1.6}$$

where

$$G_{q^{(i)}}(q^{(j)}) = \frac{1}{1 - \cos d(q^{(i)}, q^{(j)})} \tag{1.7}$$

is the Green's function of P_σ on $\mathbb{S}^n$, and $d(\cdot, \cdot)$ denotes the geodesic distance. Let $\mu(M)$ denote the smallest eigenvalue of M, and when $k = 1$,

$$\mu(M) = M = -\frac{\Delta_{g_0} K(q^{(1)})}{K(q^{(1)})^{n/2\sigma}}.$$

In what follows, we define

$$\begin{aligned} C^2(\mathbb{S}^n)^* :=& \{K \in C^2(\mathbb{S}^n) : K > 0 \text{ on } \mathbb{S}^n, \text{ and} \\ & K \text{ has only non-degenerate critical points}\}, \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} \mathscr{A} = \Big\{ & K \in C^2(\mathbb{S}^n)^* : \Delta_{g_0} K \neq 0 \text{ on } \mathscr{K}, \text{ and} \\ & \mu(M(q^{(1)}, \cdots, q^{(k)})) \neq 0, \forall q^{(1)}, \cdots, q^{(k)} \in \mathscr{K}^-, k \geq 2 \Big\}. \end{aligned} \tag{1.9}$$

We can observe that $\mathscr{A}$ is open in $C^2(\mathbb{S}^n)$ and $\mathscr{A}$ is dense in $C^2(\mathbb{S}^n)^*$ with respect to the $C^2(\mathbb{S}^n)$ norm.

Remark 1.1 In this paper, we mainly establish the compactness and existence results for (1.3) when $K \in \mathscr{A}$. It is worth noting that the sign of the smallest eigenvalue of $M(q^{(1)}, \cdots, q^{(k)})$ plays a key role in counting formula of all sloutions and compactness results.

We will introduce an integer-valued continuous function Index: $\mathscr{A} \to \mathbb{Z}$, which has an explicit formula for $K \in \mathscr{A}$ being a Morse function.

Definition 1.1 We define Index: $\mathscr{A} \to \mathbb{Z}$ by the following properties:

(i) For any Morse function $K \in \mathscr{A}$ with $\mathscr{K}^- = \{q^{(1)}, \cdots, q^{(s)}\}$, we define

$$\mathrm{Index}(K) = -1 + \sum_{k=1}^{s} \sum_{\substack{\mu(M(q^{(i_1)},\cdots,q^{(i_k)}))>0,\\ 1\le i_1<\cdots<i_k\le s}} (-1)^{k-1+\sum_{j=1}^{k} i(q^{(i_j)})}, \tag{1.10}$$

where $i(q^{(i_j)})$ denotes the Morse index of K at $q^{(i_j)}$.

(ii) Index : $\mathscr{A} \to \mathbb{Z}$ is continuous with respect to the $C^2(\mathbb{S}^n)$ norm of $\mathscr{A}$ and hence is locally constant.

Remark 1.2 The existence and uniqueness of the Index mapping follows from Theorem 1.1 and the proof of Theorem 1.2 below.

Our first result is about the compactness of the solutions when $K \in \mathscr{A}$, which is:

Theorem 1.1 *Let $\sigma = 1 + m/2$, $m \in \mathbb{N}_+$ and $n = 2\sigma + 2$. Let $\mathscr{A}$ be as in (1.9) and $K \in \mathscr{A}$. Then for any $\alpha \in (0, 1)$, there exists constants $\delta = \delta(K) > 0$ and $C = C(K) > 0$, such that for any $\mathcal{K} \in C^2(\mathbb{S}^n)$ satisfying $\|\mathcal{K} - K\|_{C^2(\mathbb{S}^n)} < \delta$, and any $v \in \mathscr{M}_{\mathcal{K}}$, we have*

$$v \in C^{2\sigma,\alpha}(\mathbb{S}^n) : 1/C < v < C, \ \|v\|_{C^{2\sigma,\alpha}(\mathbb{S}^n)} < C, \tag{1.11}$$

where $\mathscr{M}_{\mathcal{K}}$ is as in (1.5).

For any given $\sigma = \frac{n-2}{2}$, $0 < \alpha < 1$, $R > 0$, we define

$$\mathscr{O}_R := \{v \in C^{2\sigma,\alpha}(\mathbb{S}^n) : 1/R < v < R, \ \|v\|_{C^{2\sigma,\alpha}(\mathbb{S}^n)} < R\}. \tag{1.12}$$

Our second result is about degree-counting formula and the existence of the solutions to (1.3), which is:

Theorem 1.2 *Let $\sigma = 1 + m/2$, $m \in \mathbb{N}_+$ and $n = 2\sigma + 2$. Let $\mathscr{A}$ be as in (1.9), $K \in \mathscr{A}$ and* Index(K) *be as in Definition 1.1. Then for any $\alpha \in (0, 1)$, there exists a constant $R_0 = R_0(K, \alpha)$, such that for all $R > R_0$, we have*

$$\deg_{C^{2\sigma,\alpha}}(v - P_\sigma^{-1}(c(n,\sigma)K v^{\frac{n+2\sigma}{n-2\sigma}}), \mathscr{O}_R, 0) = \mathrm{Index}(K), \tag{1.13}$$

where $\deg_{C^{2\sigma,\alpha}}$ denotes the Leray-Schauder degree in $C^{2\sigma,\alpha}(\mathbb{S}^n)$.

Furthermore, if Index$(K) \neq 0$, *then (1.3) has at least one solution.*

Remark 1.3 It follows from Theorem 2.1 that when $K \in \mathscr{A}$, the solutions to (1.3) belong to $\mathscr{O}_R$ for some $R > 0$. We call the left-hand side of (1.13) the total degree of the solutions to the conformally invariant equation. From Theorem 1.2, the total degree is Index(K).

For any finite subset $\mathcal{R} \subset \mathbb{S}^n$, we use $\sharp\mathcal{R}$ to denote the number of elements in the set $\mathcal{R}$. Let us now state a corollary of Theorem 1.2, which is:

Corollary 1.1 *Let $\sigma = 1 + m/2$, $m \in \mathbb{N}_+$ and $n = 2\sigma + 2$. Let $\mathscr{A}$ be as in (1.9) and $K \in \mathscr{A}$ be a Morse function satisfying $\sharp\mathscr{K}^- \le 1$ or for any distinct $P, Q \in \mathscr{K}^-$,*

$$\Delta_{g_0} K(P) \Delta_{g_0} K(Q) < \frac{n^2(n-1)^2}{4} K(P)K(Q). \tag{1.14}$$

Then for any $\alpha \in (0,1)$, there exists a constant $C = C(K, \alpha) > 0$, such that for all solutions v to (1.3), we have $v \in \mathscr{O}_C$, and for all $R \ge C$,

$$\deg_{C^{2\sigma,\alpha}}(v - P_\sigma^{-1}(c(n,\sigma)K v^{\frac{n+2\sigma}{n-2\sigma}}), \mathscr{O}_R, 0) = -1 + \sum_{\substack{\nabla_{g_0} K(q_0)=0,\\ \Delta_{g_0} K(q_0)<0}} (-1)^{i(q_0)},$$

where $\mathscr{O}_C$ is as in (1.12) and $i(q_0)$ denotes the Morse index of K at q_0.

Furthermore, if

$$\sum_{\substack{\nabla_{g_0} K(q_0)=0,\\ \Delta_{g_0} K(q_0)<0}} (-1)^{i(q_0)} \ne 1,$$

then (1.3) has at least one solution.

Our third result is about the blow up behavior of the solutions when the prescribing fractional Q-curvature function $K \in C^2(\mathbb{S}^n)^* \backslash \mathscr{A} = \partial\mathscr{A}$, which is:

Theorem 1.3 *Let $\sigma = 1 + m/2$, $m \in \mathbb{N}_+$ and $n = 2\sigma + 2$. Let $\mathscr{A}$ be as in (1.9) and $C^2(\mathbb{S}^n)^*$ be as in (1.8). Then for any $K \in C^2(\mathbb{S}^n)^* \backslash \mathscr{A} = \partial\mathscr{A}$, there exists $K_i \to K$ in $C^2(\mathbb{S}^n)$ and $v_i \in \mathscr{M}_{K_i}$, such that*

$$\lim_{i\to\infty} (\max_{\mathbb{S}^n} v_i) = \infty, \quad \lim_{i\to\infty} (\min_{\mathbb{S}^n} v_i) = 0, \tag{1.15}$$

where $\mathscr{M}_{K_i}$ is as in (1.5).

From Theorems 1.1, 1.2, and 1.3, we can know that the total degree of solutions to (1.3) strongly depend on the sign of the smallest eigenvalue of $M(q^{(1)}, \cdots, q^{(k)})$. In fact, the points $q^{(1)}, \cdots, q^{(k)}$ for which $\mu(M(q^{(1)}, \cdots, q^{(k)}))$ is positive characterize the so-called asymptotic in the theory of critical points at infinity developed by Bahri [4, 6]. For instance, considering a continuous family of functions K_t $(0 \le t \le 1)$, the total degree changes when the smallest eigenvalue of $M(K_t; (q^{(1)}, \cdots, q^{(k)}))$ crosses zero while it remains unchanged when other eigenvalues cross zero.

It follows from Theorem 1.3 that when $K \in C^2(\mathbb{S}^n)^* \backslash \mathscr{A}$, the solutions to (1.3) may blow up. A natural question is where the blow up occur? The following results present the accurate location of the blow up.

For any $K \in C^2(\mathbb{S}^n)$, we first define

$$\begin{aligned} \mathscr{H}(K) = \{(q^{(1)}, \cdots, q^{(k)}) : k \ge 1,\ q^{(j)} \in \mathscr{K}\backslash\mathscr{K}^+,\ \forall\, j : 1 \le j \le k, \\ q^{(j)} \ne q^{(\ell)},\ \forall\, j \ne \ell,\ \mu(M(q^{(1)}, \cdots, q^{(k)})) = 0\}. \end{aligned} \tag{1.16}$$

Combined with Theorem 2.1, we give the fourth result in this paper, which is about the location of blowing up when $K \in C^2(\mathbb{S}^n)^* \backslash \mathscr{A}$:

Theorem 1.4 *Let $\sigma = 1 + m/2$, $m \in \mathbb{N}_+$ and $n = 2\sigma + 2$. Let $\mathscr{A}$ be as in (1.9) and $C^2(\mathbb{S}^n)^*$ be as in (1.8). For a given function $K \in C^2(\mathbb{S}^n)^* \backslash \mathscr{A}$, we have the following results:*

(i) *For any $K_i \to K$ in $C^2(\mathbb{S}^n)$, and $v_i \in \mathscr{M}_{K_i}$ with $\max_{\mathbb{S}^n} v_i \to \infty$, then for some $(q^{(1)}, \cdots, q^{(k)}) \in \mathscr{H}(K)$, $\{v_i\}$ (after passing to a subsequence) blows up at precisely the k points.*

(ii) *For any* $(q^{(1)}, \cdots, q^{(k)}) \in \mathscr{H}(K)$, *there exists* $K_i \to K$ *in* $C^2(\mathbb{S}^n)$, $v_i \in \mathscr{M}_{K_i}$, *such that* $\{v_i\}$ *blows up at precisely the k points.*

Corollary 1.2 *For any* $k \in \mathbb{N}_+$ *distinct points* $q^{(1)}, \cdots, q^{(k)} \in \mathbb{S}^n$, *there exists a sequence of Morse functions* $\{K_i\} \subset \mathscr{A}$, *such that for some* $v_i \in \mathscr{M}_{K_i}$, $\{v_i\}$ *blows up at precisely the k points.*

In order to obtain the compactness results, we need to further characterize the behavior of the blow up point of the solutions to (1.3) (see Theorem 2.1 below). More precisely, we will use the Pohozaev type identity (see Proposition A.3 below) to judge the sign of the Laplacian of the prescribing curvature function at these isolated simple blow up point (see Definition 2.3 below). Due to the limit of the form of the Pohozaev type identity, the method of our proof is only effective for the case $n - 2\sigma = 2$. In addition, when proving the existence results, we transform the conclusion to be proved into solving the Brouwer degree of the operator on finite dimensional manifolds through the homotopy invariance of the Leray-Schauder degree. In the process of obtaining the degree-counting formula, we need to get a strictly convex function according to the form of the operator, and the condition "$n - 2\sigma = 2$" just ensures the existence of the form of strictly convex function. For $n = 2\sigma + 2$, $0 < \sigma < 1$, we obtain the corresponding compactness and existence results with $n = 3, \sigma = 1/2$, see [39].

The paper is organized as follows:

In Sect. 2, our main task is to prove Theorem 1.1. Before that, we should further characterizes the behavior of blow up points for solutions to (1.3) (see Theorem 2.1 below), we mainly consider the subcritical equation with $\tau > 0$ small:

$$P_\sigma v_i = c(n,\sigma) K v_i^{n-1-\tau}, \quad v_i > 0 \quad \text{on } \mathbb{S}^n. \tag{1.17}$$

In proving Theorem 2.1, we first use the Green's representation (1.4) to transform (1.17) into

$$v_i(\xi) = \frac{\Gamma(\frac{n+2\sigma}{2})}{2^{2\sigma}\pi^{n/2}\Gamma(\sigma)} \int_{\mathbb{S}^n} \frac{K_i(\eta) v_i(\eta)^{n-1-\tau_i}}{|\xi - \eta|^2} \, d\eta \quad \text{on } \mathbb{S}^n,$$

and then use some results of blow up analysis given in Appendix A to complete the proof. By using Theorem 2.1, integral representation, Harnack inequality and Schauder type estimates, we have completed the proof of Theorem 1.1.

Section 3 is devoted to proving the Theorems 1.2, 1.3, and 1.4. Firstly, recall the classification of solutions for integral equation [14] and optimal representation in small tubular neighborhood [6], we give the definition of $\Sigma_\tau = \Sigma_\tau(\overline{P}_1, \cdots, \overline{P}_k)$ for $\overline{P}_1, \cdots, \overline{P}_k \in \mathscr{K}^-$ with $\mu(M(\overline{P}_1, \cdots, \overline{P}_k)) > 0$. Then by using Theorem 2.1 and some results in [31], we obtain that for $\tau > 0$ very small, the solutions to (1.17) either stay bounded or stay in one of the Σ_τ (see Proposition 3.1 below). Furthermore, we obtain the H^σ topological degree of the solutions to (1.17) on Σ_τ (see Theorem 3.1 below). It follows from the above results that for all $0 < \tau < 2$, the H^σ total degree of the solutions to (1.17) is equal to -1 (see Proposition A.7 below). Then we can conclude that H^σ topological degree of those solutions to (1.17) which remain bounded as τ tends to zero is equal to Index(K). Some well-known results in degree theory imply that the H^σ degree contribution above is equal to the $C^{2\sigma,\alpha}$ topological degree of those bounded solutions to (1.17). Thus, we proved Theorem 1.2. Furthermore, we complete the proof of Theorem 1.3 by using the degree-counting formula and perturbing the function K near its critical point. In the end, using Theorem 2.1 and the idea of the proof of Theorem 1.3, we prove Theorem 1.4.

In Appendix A, by the Green's representation (1.4) and the stereographic projection, we can write Eq. (1.3) as the form

$$u(x)=\int_{\mathbb{R}^n}\frac{K(y)u(y)^{\frac{n+2\sigma}{n-2\sigma}}}{|x-y|^{n-2\sigma}}\,\mathrm{d}y \quad \text{on } \mathbb{R}^n, \tag{1.18}$$

we first review the Hölder estimates, Schauder type estimates, blow up profile for nonlinear integral equations (1.18) established by Jin-Li-Xiong [31].

In Appendix B, we provide some useful technical results and elementary estimates.

2 The characterization of blow up behavior and compactness result

In this section, our main task is to prove Theorem 1.1. Before that, we need further characterizes the blow up points for solutions to (1.3) by using integral representation and some estimates in the Appendix A (see Theorem 2.1 below), which plays a key role in proving main result concerning compactness and existence. We first review some definitions of blow up points.

Let Ω be a domain in $\mathbb{R}^n$ and K_i are nonnegative bounded functions in $\mathbb{R}^n$. Let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence of nonnegative constants satisfying $\lim_{i\to\infty}\tau_i=0$, and set

$$p_i=\frac{n+2\sigma}{n-2\sigma}-\tau_i.$$

Suppose that $0\le u_i\in L^{\infty}_{loc}(\mathbb{R}^n)$ satisfies the nonlinear integral equation

$$u_i(x)=\int_{\mathbb{R}^n}\frac{K_i(y)u_i(y)^{p_i}}{|x-y|^{n-2\sigma}}\,\mathrm{d}y \quad \text{in } \Omega. \tag{2.1}$$

We assume that $K_i\in C^1(\Omega)$ ($K_i\in C^{1,1}(\mathbb{S}^n)$ if $\sigma\le 1/2$) and, for some positive constants A_1 and A_2,

$$1/A_1\le K_i, \quad \text{and} \quad \|K_i\|_{C^1(\Omega)}\le A_2,\ (\|K_i\|_{C^{1,1}(\Omega)}\le A_2 \text{ if } \sigma\le\frac{1}{2}). \tag{2.2}$$

Definition 2.1 Suppose that $\{K_i\}$ satisfies (2.2) and $\{u_i\}$ satisfies (2.1). A point $\overline{y}\in\Omega$ is called a blow up point of $\{u_i\}$ if there exists a sequence y_i tending to $\overline{y}$ such that $u_i(y_i)\to\infty$.

Definition 2.2 A blow up point $\overline{y}\in\Omega$ is called an isolated blow up point of $\{u_i\}$ if there exists $0<\overline{r}<\mathrm{dist}(\overline{y},\Omega)$, $\overline{C}>0$, and *a* sequence y_i tending to $\overline{y}$, such that y_i is a local maximum point of u_i, $u_i(y_i)\to\infty$ and

$$u_i(y)\le\overline{C}|y-y_i|^{-2\sigma/(p_i-1)} \quad \text{for all } y\in B_{\overline{r}}(y_i). \tag{2.3}$$

Let $y_i\to\overline{y}$ be an isolated blow up point of $\{u_i\}$, and define, for $r>0$,

$$\overline{u}_i(r):=\frac{1}{|\partial B_r(y_i)|}\int_{\partial B_r(y_i)}u_i \quad \text{and} \quad \overline{w}_i(r):=r^{2\sigma/(p_i-1)}\overline{u}_i(r).$$

Definition 2.3 A point $y_i\to\overline{y}\in\Omega$ is called an isolated simple blow up point if $y_i\to\overline{y}$ is an isolated blow up point such that for some $\rho>0$ (independent of i), $\overline{w}_i$ has precisely one critical point in $(0,\rho)$ for large i.

2.1 Characterization of blow up behavior

Recall the definitions of the matrix M given in (1.6) and its smallest eigenvalue $\mu(M)$. The result about characterization of blow up behavior of the solutions to (1.3) is:

Theorem 2.1 *Let $\sigma = 1 + m/2$, $m \in \mathbb{N}_+$ and $n = 2\sigma + 2$. Let $K \in C^2(\mathbb{S}^n)$ be a positive function and $\mathscr{K}, \mathscr{K}^-, \mathscr{K}^+$ be as in (1.5). Let p_i satisfy $p_i \leq \frac{n+2\sigma}{n-2\sigma} = \frac{n+2\sigma}{2} = n-1$, $p_i \to n-1$, $K_i \in C^2(\mathbb{S}^n)$ satisfy $K_i \to K$ in $C^2(\mathbb{S}^n)$, and $v_i \in C^{2\sigma}(\mathbb{S}^n)$ satisfy*

$$P_\sigma v_i = c(n,\sigma) K_i v_i^{p_i} \tag{2.4}$$

and

$$\lim_{i\to\infty} \max_{\mathbb{S}^n} v_i = \infty.$$

Then there exists a constant $\delta^* > 0$ depending only on $\min_{\mathbb{S}^n} K$, $\|K\|_{C^2(\mathbb{S}^n)}$, and the modulus of continuity of $\nabla_{g_0} K$ if $\sigma > 1/2$ such that after passing to a subsequence, we have:

(i) *$\{v_i\}$ (still denote the subsequence by $\{v_i\}$) has only isolated simple blow up points $q^{(1)}, \cdots, q^{(k)} \in \mathscr{K}\backslash\mathscr{K}^+$ $(k \geq 1)$ with $|q^{(j)} - q^{(\ell)}| \geq \delta^*$, $\forall j \neq \ell$, and $\mu(M(q^{(1)}, \cdots, q^{(k)})) \geq 0$. Furthermore, $q^{(1)}, \cdots, q^{(k)} \in \mathscr{K}^-$ if $k \geq 2$.*

(ii) *Let $q^{(1)}, \cdots, q^{(k)}$ be as in (i), and $q_i^{(j)}$ be the local maximum of v_i with $q_i^{(j)} \to q^{(j)}$, we have*

$$\lambda_j := K(q^{(j)})^{-1/2\sigma} \lim_{i\to\infty} v_i(q_i^{(1)})(v_i(q_i^{(j)}))^{-1} \in (0,\infty), \tag{2.5}$$

$$\mu^{(j)} := \lim_{i\to\infty} \tau_i v_i(q_i^{(j)})^2 \in [0,\infty). \tag{2.6}$$

(iii) *Let $\lambda_j, \mu^{(j)}$, $j = 1, \cdots, k$ be as in (ii), then when $k = 1$,*

$$\mu^{(1)} = -\frac{2}{\sigma} \frac{\Delta_{g_0} K(q^{(1)})}{K(q^{(1)})^{n/2\sigma}}, \tag{2.7}$$

when $k \geq 2$,

$$\sum_{\ell=1}^{k} M_{\ell j}(q^{(1)}, \cdots, q^{(k)})\lambda_\ell = \frac{\sigma}{2}\lambda_j \mu^{(j)}, \quad \forall j : 1 \leq j \leq k. \tag{2.8}$$

(iv) *$\mu^{(j)} \in (0,\infty)$, $\forall j = 1, \cdots, k$, if and only if $\mu(M(q^{(1)}, \cdots, q^{(k)})) > 0$.*

We first give the following proposition:

Proposition 2.1 *Let $K \in C^2(\mathbb{S}^n)$, $n \geq 2$, be a positive function and $\mathscr{K}, \mathscr{K}^-, \mathscr{K}^+$ be as in (1.5). Let p_i satisfy $p_i \leq \frac{n+2\sigma}{n-2\sigma}$, $p_i \to \frac{n+2\sigma}{n-2\sigma}$, $K_i \in C^2(\mathbb{S}^n)$ satisfy $K_i \to K$ in $C^2(\mathbb{S}^n)$, and v_i satisfy*

$$P_\sigma v_i = c(n,\sigma) K_i v_i^{p_i}.$$

Then exists a constant $\delta^* > 0$ depending only on $\min_{\mathbb{S}^n} K$, $\|K\|_{C^2(\mathbb{S}^n)}$, and the modulus of continuity of $\nabla_{g_0} K$ if $\sigma > 1/2$ such that, after passing to a subsequence, either $\{v_i\}$ stays bounded in $L^\infty(\mathbb{S}^n)$ or $\{v_i\}$ has only isolated simple blow up points and the distance between any two blow up points is bounded blow by δ^*.

Proof The proof follows from the same arguments used to prove Theorem 3.3 in [31], so we omit it. ◻

Proof of Theorem 2.1 From Proposition 2.1 and $\lim_{i\to\infty}\max_{\mathbb{S}^n} v_i = \infty$, there exists a constant $\delta^* > 0$ depending only on $\min_{\mathbb{S}^n} K$, $\|K\|_{C^2(\mathbb{S}^n)}$, and the modulus of continuity of $\nabla_{g_0} K$ if $\sigma > 1/2$ such that $\{v_i\}$ has only isolated simple blow up points $q^{(1)}, \cdots, q^{(k)} \in \mathscr{K}$ $(k \geq 1)$ with $|q^{(j)} - q^{(\ell)}| \geq \delta^*$ $(j \neq \ell)$.

By (1.4), (2.4) is equivalent to

$$v_i(\xi) = \frac{\Gamma(\frac{n+2\sigma}{2})}{2^{2\sigma}\pi^{n/2}\Gamma(\sigma)} \int_{\mathbb{S}^n} \frac{K_i(\eta) v_i(\eta)^{p_i}}{|\xi - \eta|^2} \, d\eta \quad \text{on } \mathbb{S}^n. \tag{2.9}$$

Let F be the stereographic projection with with $q^{(j)}$ being the south pole:

$$F : \mathbb{R}^n \to \mathbb{S}^n \backslash \{-q^{(j)}\},$$
$$x \mapsto \Big(\frac{2x}{1+|x|^2}, \frac{|x|^2 - 1}{|x|^2 + 1}\Big).$$

Let $\tau_i = n - 1 - p_i$, via the stereographic projection, the equation (2.9) is translated to

$$u_i(x) = \frac{\Gamma(\frac{n+2\sigma}{2})}{2^{2\sigma}\pi^{n/2}\Gamma(\sigma)} \int_{\mathbb{R}^n} \frac{\widetilde{K}_i(y) H(y)^{\tau_i} u_i(y)^{p_i}}{|x - y|^2} \, dy \quad \text{on } \mathbb{R}^n,$$

where

$$H(x) = \frac{2}{1+|x|^2}, \quad u_i(x) = H(x) v_i(F(x)), \quad \widetilde{K}_i(x) = K_i(F(x)). \tag{2.10}$$

Let $x_i^{(j)}$ be the local maximum of u_i and $x_i^{(j)} \to 0$. It follows from Propositions A.7 and A.8 that

$$u_i(x_i^{(j)}) u_i(x) \to h^{(j)}(x) := a K(q^{(j)})^{-1/\sigma} |x|^{-2} + b^{(j)}(x) \quad \text{in } C^2_{loc}(\mathbb{R}^n \backslash \{\cup_{\ell=1}^k x^{(\ell)}\}), \tag{2.11}$$

where

$$\begin{aligned} a &= 2^{2+2\sigma} c_{n,\sigma} c(n,\sigma) \int_{\mathbb{R}^n} \Big(\frac{1}{1+|y|^2}\Big)^{n-1} dy \\ &= 2^{1+2\sigma} c_{n,\sigma} c(n,\sigma) |\mathbb{S}^{n-1}| \mathrm{B}(\sigma, n/2), \end{aligned} \tag{2.12}$$

$\mathrm{B}(\sigma, n/2)$ is the Beta function, and $c_{n,\sigma}$ is as in (1.4). From the properties of integral equation and the same proof as Proposition A.8, we can obtain that $b^{(1)} \equiv constant$. In addition, it is easy to see that $b^{(j)}(x)$ satisfies

$$b^{(j)}(x) > 0 \quad \text{if } k \geq 2. \tag{2.13}$$

By (2.10) and $y_i^{(j)} \to 0$ as $i \to \infty$, we have

$$\lim_{i\to\infty} v_i(q_i^{(j)}) v_i(q) = \frac{1}{4} \lim_{i\to\infty} (1+|x|^2) u_i(x_i^{(j)}) u_i(x),$$

combining with (2.11), it easy to see that for $q \neq q^{(j)}$ and close to $q^{(j)}$,

$$\lim_{i\to\infty} v_i(q_i^{(j)}) v_i(q) = \frac{a G_{q^{(j)}}(q)}{2K(q^{(j)})^{1/\sigma}} + \widetilde{b}^{(j)}(q) \quad \text{in } C^2_{loc}(\mathbb{S}^n \backslash \{\cup_{\ell=1}^k q^{(\ell)}\}), \tag{2.14}$$

where a is as in (2.12), and $\widetilde{b}^{(j)}(q)$ is some regular function on $\mathbb{S}^n \backslash \cup_{\ell\neq j} \{q^{(\ell)}\}$ satisfying $P_\sigma \widetilde{b}^{(j)} = 0$, and $G_{q^{(j)}}(q)$ is the Green function defined as in (1.7).

When $k \geq 2$, taking into account the contribution of all the poles, we deduce

$$\lim_{i\to\infty} v_i(q_i^{(j)})v_i(q) = \frac{aG_{q^{(j)}}(q)}{2K(q^{(j)})^{1/\sigma}} + \frac{a}{2}\sum_{\ell\neq j}\lim_{i\to\infty}\frac{v_i(q_i^{(j)})}{v_i(q_i^{(\ell)})}\frac{G_{q^{(\ell)}}(q)}{K(q^{(\ell)})^{1/\sigma}} \quad \text{in } C^2_{loc}(\mathbb{S}^n\backslash\{\cup_{\ell=1}^k q^{(\ell)}\}). \tag{2.15}$$

In fact, subtracting all the poles from the limit function, we obtain a regular function $\widetilde{b}_0 : \mathbb{S}^n \to \mathbb{R}$ such that $P_\sigma \widetilde{b}_0 = 0$ on $\mathbb{S}^n$, so it must be $\widetilde{b}_0 \equiv 0$. Using (2.15), we have, for $|y| > 0$ small,

$$h^{(j)}(y) = \frac{a}{K(q^{(j)})^{1/\sigma}|y|^2} + 2a\sum_{\ell\neq j}\lim_{i\to\infty}\frac{v_i(q_i^{(j)})}{v_i(q_i^{(\ell)})}\frac{G_{q^{(\ell)}}(q^{(j)})}{K(q^{(\ell)})^{1/\sigma}} + O(|y|), \tag{2.16}$$

where a is as in (2.12). The conclusion obtained from the above is easy to see that (2.5) is true.

Before stating the result to be proved, we give the following estimates (2.17) and (2.18). Using Proposition A.10, we obtain

$$|\nabla K_i(y_i^{(j)})| = O(u_i(y_i^{(j)})^{-1}), \quad \tau_i = O(u_i(y_i^{(j)})^{-2}). \tag{2.17}$$

It is obvious that (2.6) can be proved by (2.17). We have proved Part (ii).

Let $y = (y_{(1)}, \cdots, y_{(n)}) \in \mathbb{R}^n$. It follows from Propositions A.5, A.7, and A.9, that for sufficiently small $\delta > 0$,

$$\begin{aligned}
&\sum_{j=1}^n \left|\int_{B_\delta} y_{(j)} u_i(y + x_i^{(j)})^{p_i+1}\right| = o(u_i(x_i^{(j)})^{-1}),\\
&\sum_{j\neq\ell} \left|\int_{B_\delta} y_{(j)} y_{(\ell)} u_i(y + x_i^{(j)})^{p_i+1}\right| = o(u_i(x_i^{(j)})^{-2}),\\
&\int_{\partial B_\delta} u_i(y + x_i^{(j)})^{p_i+1} = O(u_i(x_i^{(j)})^{-p_i-1}),\\
&\lim_{i\to\infty} u_i(x_i^{(j)})^2 \int_{B_\delta} |y|^2 u_i(y + x_i^{(j)})^{p_i+1} = \frac{n2^{1+n}|\mathbb{S}^{n-1}|}{n+2\sigma}\frac{\mathrm{B}(\sigma, n/2)}{K(q^{(j)})^{1+2/\sigma}}.
\end{aligned} \tag{2.18}$$

In fact, the first three formulas in (2.18) can be easily obtained from Proposition A.9. For the last formula in (2.18), let R_i be as in Proposition A.5 and

$$m_{ij} := u_i(x_i^{(j)}), \quad r_{ij} := R_i m_{ij}^{-(p_i-1)/2\sigma}, \quad k_{ij} := 2^{-2}\widetilde{K}_i(x_i^{(j)})^{1/\sigma}. \tag{2.19}$$

Using Proposition A.5 again, we have

$$\begin{aligned}
&m_{ij}^2 \int_{|y|\leq r_{ij}} |y|^2 u_i(y + x_i^{(j)})^{p_i+1}\,\mathrm{d}y\\
&= m_{ij}^2 \int_{|x|\leq R_i} m_{ij}^{\frac{-(2+n)(p_i-1)}{2\sigma}+p_i+1} |x|^2 (m_{ij}^{-1} u_i(m_{ij}^{-(p_i-1)/2\sigma}x + x_i^{(j)}))^{p_i+1}\,\mathrm{d}x\\
&= m_{ij}^{\frac{-(2+n)(p_i-1)}{2\sigma}+p_i+3} \int_{|x|\leq R_i} |x|^2 \Big(\frac{1}{1+k_{ij}|x|^2}\Big)^{p_i+1}\,\mathrm{d}x + o(1)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} \frac{|x|^2}{(1+k_{ij}|x|^2)^n} \, \mathrm{d}x + o(1) \\
&= \frac{n 2^{1+n} |\mathbb{S}^{n-1}|}{n+2\sigma} K(q^{(j)})^{-1-2/\sigma} \mathrm{B}(\sigma, n/2) + o(1).
\end{aligned}
$$

We have completed the proof of (2.18).

By $n - 2\sigma = 2$ and $\tau_i = (n+2\sigma)/(n-2\sigma) - p_i$, it is easy to see that

$$
\frac{1}{p_i+1} = \frac{1}{2\sigma + 2 - \tau_i} = \frac{1}{n}\Big(1 + \frac{\tau_i}{n} + O(\tau_i^2)\Big). \tag{2.20}
$$

For sufficiently small $\delta > 0$, u_i satisfy

$$
u_i(x) = c_{n,\sigma} c(n,\sigma) \int_{B_\delta(x_i^{(j)})} \frac{\widetilde{K}_i(y) H(y)^{\tau_i} u_i(y)^{p_i}}{|x-y|^2} \, \mathrm{d}y + h_\delta(x),
$$

where

$$
h_\delta(x) = c_{n,\sigma} c(n,\sigma) \int_{\mathbb{R}^n \setminus B_\delta(x_i^{(j)})} \frac{\widetilde{K}_i(y) H(y)^{\tau_i} u_i(y)^{p_i}}{|x-y|^2} \, \mathrm{d}y. \tag{2.21}
$$

By Proposition A.3, we have

$$
\begin{aligned}
&\Big(\frac{n-2\sigma}{2} - \frac{n}{p_i+1}\Big) \int_{B_\delta(x_i^{(j)})} \widetilde{K}_i(x) H(x)^{\tau_i} u_i(x)^{p_i+1} \, \mathrm{d}x \\
&\quad - \frac{1}{p_i+1} \int_{B_\delta(x_i^{(j)})} (x - x_i^{(j)}) \cdot \nabla (\widetilde{K}_i(x) H(x)^{\tau_i}) u(x)^{p_i+1} \, \mathrm{d}x \\
&= \frac{n-2\sigma}{2} \int_{B_\delta(x_i^{(j)})} \widetilde{K}_i(x) H(x)^{\tau_i} u_i(x)^{p_i} h_\delta(x) \, \mathrm{d}x \\
&\quad + \int_{B_\delta(x_i^{(j)})} (x - x_i^{(j)}) \cdot \nabla h_\delta(x) \widetilde{K}_i(x) H(x)^{\tau_i} u_i(x)^{p_i} \, \mathrm{d}x \\
&\quad - \frac{\delta}{p_i+1} \int_{\partial B_\delta(x_i^{(j)})} \widetilde{K}_i(x) H(x)^{\tau_i} u_i(x)^{p_i+1} \, \mathrm{d}s.
\end{aligned} \tag{2.22}
$$

Let $x = (x_{(1)}, \cdots, x_{(n)}) \in \mathbb{R}^n$, by (2.20) we have

$$
\begin{aligned}
&- \frac{1}{p_i+1} \int_{B_\delta} x \cdot \nabla (\widetilde{K}_i(x + x_i^{(j)}) H(x + x_i^{(j)})^{\tau_i}) u_i(x + x_i^{(j)})^{p_i+1} \, \mathrm{d}x \\
&= -\frac{1}{n} \sum_{\ell=1}^n \int_{B_\delta} x_{(\ell)} \frac{\partial \widetilde{K}_i}{\partial x_{(\ell)}} (x + x_i^{(j)}) u_i(x + x_i^{(j)})^{p_i+1} \, \mathrm{d}x + o(m_{ij}^{-2}) \\
&= -\frac{1}{n} \int_{B_\delta} x \cdot \nabla \widetilde{K}(x_i^{(j)}) u_i(x + x_i^{(j)})^{p_i+1} \, \mathrm{d}x \\
&\quad - \frac{1}{n} \sum_{\ell,m} \int_{B_\delta} x_{(\ell)} x_{(m)} \frac{\partial^2 \widetilde{K}}{\partial x_{(\ell)} \partial x_{(m)}} (x_i^{(j)}) u_i(x + x_i^{(j)})^{p_i+1} \, \mathrm{d}x + o(m_{ij}^{-2}) \\
&= -\frac{1}{n^2} \Delta \widetilde{K}(0) \int_{B_\delta} |x|^2 u_i(x + x_i^{(j)})^{p_i+1} \, \mathrm{d}x + o(m_{ij}^{-2}) \\
&= -\frac{4}{n^2} \Delta_{g_0} K(q^{(j)}) \int_{B_\delta} |x|^2 u_i(x + x_i^{(j)})^{p_i+1} \, \mathrm{d}x + o(m_{ij}^{-2}).
\end{aligned} \tag{2.23}
$$

Then, by (2.18) and (2.23),

$$\begin{aligned}\lim_{i\to\infty} -\frac{m_{ij}^2}{p_i+1}\int_{B_\delta} x\cdot\nabla\widetilde{K}_i(x+x_i^{(j)})u_i(x+x_i^{(j)})^{p_i+1}\,\mathrm{d}x\\ = -\frac{2^{3+n}|\mathbb{S}^{n-1}|\mathrm{B}(\sigma,n/2)}{n(n+2\sigma)}\frac{\Delta_{g_0}K(q^{(j)})}{K(q^{(j)})^{1+2/\sigma}}.\end{aligned}\tag{2.24}$$

Let r_i be as in (2.19) and by (2.20), we have

$$\begin{aligned}&\Big(\frac{n-2\sigma}{2}-\frac{n}{p_i+1}\Big)\int_{B_\delta}\widetilde{K}_i(x+x_i^{(j)})H(x+x_i^{(j)})^{\tau_i}u_i(x+x_i^{(j)})^{p_i+1}\,\mathrm{d}x\\ &=-\frac{\tau_i}{n}\int_{B_\delta}\widetilde{K}_i(x+x_i^{(j)})u_i(x+x_i^{(j)})^{p_i+1}\,\mathrm{d}x+o(m_{ij}^{-2})\\ &=-\frac{\tau_i}{n}\int_{B_\delta}\widetilde{K}_i(x_i^{(j)})u_i(x+x_i^{(j)})^{p_i+1}\,\mathrm{d}x\\ &\quad+O\Big(\Big|\int_{B_\delta}x\cdot\nabla\widetilde{K}_i(x_i^{(j)})u_i(x+x_i^{(j)})^{p_i+1}\,\mathrm{d}x\Big|\Big)\\ &\quad+O\Big(\int_{B_\delta}|x|^2u_i(x+x_i^{(j)})^{p_i+1}\,\mathrm{d}x\Big)+o(m_{ij}^{-2})\\ &=-\frac{\tau_i}{n}\widetilde{K}(x_i^{(j)})\int_{|x|<r_{ij}}u_i(x+x_i^{(j)})^{p_i+1}\,\mathrm{d}x+o(m_{ij}^{-2})\\ &=-\frac{\tau_i2^n}{n}K(q^{(j)})^{-1/\sigma}\int_{\mathbb{R}^n}\frac{1}{(1+|x|^2)^n}\,\mathrm{d}x+o(m_{ij}^{-2})\\ &=-\frac{\tau_i2^n|\mathbb{S}^{n-1}|}{n}\Big(\frac{\sigma}{n+2\sigma}\Big)\mathrm{B}(n/2,\sigma)K(q^{(j)})^{-1/\sigma}+o(m_{ij}^{-2}).\end{aligned}\tag{2.25}$$

It follows from (2.6) and (2.25) that

$$\begin{aligned}\lim_{i\to\infty} -m_{ij}^2\Big(1-\frac{n}{p_i+1}\Big)\int_{B_\delta}\widetilde{K}_i(x+x_i^{(j)})H(x+x_i^{(j)})^{\tau_i}u_i(x+x_i^{(j)})^{p_i+1}\,\mathrm{d}x\\ =-\frac{2^{n+2}|\mathbb{S}^{n-1}|\sigma\mathrm{B}(\sigma,n/2)}{n(n+2\sigma)}\frac{\mu^{(j)}}{K(q^{(j)})^{1/\sigma}}.\end{aligned}\tag{2.26}$$

In view of (2.18), we obtain

$$\begin{aligned}&\lim_{i\to\infty} -m_{ij}^2\frac{\delta}{p_i+1}\int_{\partial B_\delta}\widetilde{K}_i(x+x_i^{(j)})H(x+x_i^{(j)})^{\tau_i}u_i(x+x_i^{(j)})^{p_i+1}\,\mathrm{d}x\\ &=\lim_{i\to\infty} -m_{ij}^2\frac{\delta}{n}\Big(1+\frac{\tau_i}{n}\Big)\int_{\partial B_\delta}\widetilde{K}_i(x+x_i^{(j)})u_i(x+x_i^{(j)})^{p_i+1}\,\mathrm{d}x\\ &=0.\end{aligned}\tag{2.27}$$

Using (2.21) and Proposition A.5, we have

$$\begin{aligned}&m_{ij}^2\frac{n-2\sigma}{2}\int_{B_\delta(x_i^{(j)})}\widetilde{K}_i(x)u_i(x)^{p_i}H(x)^{\tau_i}h_\delta(x)\,\mathrm{d}x\\ &=m_{ij}^2\int_{B_\delta(x_i^{(j)})}(\widetilde{K}_i(x_i^{(j)})+(x-x_i^{(j)})\cdot\nabla\widetilde{K}(x_i^{(j)})+O(|x-x_i^{(j)}|^2))u_i(x)^{p_i}h_\delta(x)\,\mathrm{d}x\\ &=m_{ij}^{2-\frac{n(p_i-1)}{2\sigma}+p_i}\widetilde{K}_i(x_i^{(j)})\int_{|y|<R_i}(m_{ij}^{-1}u_i(m_{ij}^{-\frac{p_i-1}{2\sigma}}y+x_i^{(j)}))^{p_i}\end{aligned}$$

$$
\begin{aligned}
&h_\delta(m_{ij}^{-\frac{p_i-1}{2\sigma}} y + x_i^{(j)})\,\mathrm{d}y + o(1) \\
&= \widetilde{K}_i(x_i^{(j)}) \int_{\mathbb{R}^n} \frac{1}{(1+k_{ij}|y|^2)^{p_i}} h_\delta(m_{ij}^{-\frac{p_i-1}{2\sigma}} y + x_i^{(j)})\,\mathrm{d}y + o(1) \\
&= 2^{n-1} K(q^{(j)})^{-1/\sigma} |\mathbb{S}^{n-1}| \mathrm{B}(\sigma, n/2) b^{(j)}(0) + o(1),
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&\lim_{i\to\infty} m_{ij}^2 \frac{n-2\sigma}{2} \int_{B_\delta(x_i^{(j)})} \widetilde{K}_i(x) u_i(x)^{p_i} H(x)^{\tau_i} h_\delta(x)\,\mathrm{d}x \\
&\quad = 2^{n-1} |\mathbb{S}^{n-1}| \mathrm{B}(\sigma, n/2) \frac{1}{K(q^{(j)})^{1/\sigma}} b^{(j)}(0).
\end{aligned} \tag{2.28}
$$

When $|x - x_i^{(j)}| < \delta$, a direct calculation gives

$$
|\nabla h_\delta(x)| \le \begin{cases} C \dfrac{|\delta^{2\sigma-1} - (\delta - |x - x_i^{(j)}|)^{2\sigma-1}|}{2\sigma - 1} m_{ij}^{-1} & \text{if } \sigma \ne 1/2, \\ C|\log\delta - \log(\delta - |x - x_i^{(j)}|)| m_{ij}^{-1} & \text{if } \sigma = 1/2. \end{cases} \tag{2.29}
$$

The detailed proof of (2.29) can refer to [31]. Using Proposition A.5 and (2.29), we can obtain

$$
\begin{aligned}
&\Big| \int_{B_\delta(x_i^{(j)})} (x - x_i^{(j)}) \nabla h_\delta(x) \widetilde{K}_i(x) H(x)^{\tau_i} u_i(x)^{p_i}\,\mathrm{d}x \Big| \\
&\le C m_{ij}^{-1} \int_{|x - x_i^{(j)}|<\delta} |x - x_i^{(j)}| u_i(x)^{p_i}\,\mathrm{d}x \\
&\le C m_{ij}^{-1-\frac{(n+1)(p_i-1)}{2\sigma}+p_i} \int_{|y|<R_i} |y| (m_{ij}^{-1} u_i(m_{ij}^{-\frac{(p_i-1)}{2\sigma}} y + x_i^{(j)}))^{p_i}\,\mathrm{d}y \\
&= o(m_{ij}^{-2}).
\end{aligned} \tag{2.30}
$$

By (2.22), (2.24), (2.26), (2.27), (2.28), and (2.30), we have

$$
\frac{8\sigma\mu^{(j)}}{n(n+2\sigma)} \frac{1}{K(q^{(j)})^{1/\sigma}} + \frac{16}{n(n+2\sigma)} \frac{\Delta_{g_0} K(q^{(j)})}{K(q^{(j)})^{1+2/\sigma}} = -\frac{1}{K(q^{(j)})^{1/\sigma}} b^{(j)}(0). \tag{2.31}
$$

Consequently, $q^{(j)} \in \mathscr{K} \backslash \mathscr{K}^+$, $1 \le j \le k$, and when $k \ge 2$, $q^{(j)} \in \mathscr{K}^-$, $1 \le j \le k$. It is easy to see that (2.7) follows from (2.13) and (2.31) when $k = 1$.

From (2.16), (2.12), (2.10), and (2.5), we can obtain

$$
\begin{aligned}
b^{(j)}(0) &= \frac{2^2 \Gamma(n/2) |\mathbb{S}^{n-1}|}{\pi^{n/2}} \sum_{\ell \ne j} \frac{\lambda_\ell}{\lambda_j} \frac{G_{q^{(\ell)}}(q^{(j)})}{(K(q^{(j)}) K(q^{(\ell)}))^{1/2\sigma}} \\
&= 8 \sum_{\ell \ne j} \frac{\lambda_\ell}{\lambda_j} \frac{G_{q^{(\ell)}}(q^{(j)})}{(K(q^{(j)}) K(q^{(\ell)}))^{1/2\sigma}}.
\end{aligned} \tag{2.32}
$$

Substituting (2.32) into (2.31) to get

$$
-n(n-1) \sum_{\ell \ne j} \frac{G_{q^{(\ell)}}(q^{(j)})}{(K(q^{(j)}) K(q^{(\ell)}))^{1/2\sigma}} \lambda_\ell - \frac{\Delta_{g_0} K(q^{(j)})}{K(q^{(j)})^{n/2\sigma}} \lambda_j = \frac{\sigma}{2} \lambda_j \mu^{(j)}.
$$

We have established (2.8) and thus verified Part (iii).

We claim that there exists some

$$\eta = (\eta_1, \cdots, \eta_k) \neq 0 \quad \text{with} \quad \eta_\ell \geq 0, \forall\, \ell = 1, \cdots, k, \tag{2.33}$$

such that

$$\sum_{\ell=1}^{k} M_{\ell j}(q^{(1)}, \cdots, q^{(k)})\eta_\ell = \mu(M)\eta_j, \quad \forall\, j = 1, \cdots, k.$$

Indeed, choose $\Lambda > \max_i M_{ii}$, then the matrix $\Lambda I - M$ is a positive matrix (see [27] for the definition), where I denotes the unit matrix. The claim can follows from [27, Theorem 8.2.2].

Multiplying (2.8) by η_j and summing over j, then using Part (ii) and (2.33), we have

$$\mu(M)\sum_j \lambda_j\eta_j = \sum_{\ell,j} M_{\ell j}\lambda_\ell\eta_j = \frac{1}{4}\sum_j \lambda_j\eta_j\mu^{(j)} \geq 0. \tag{2.34}$$

It follows that $\mu(M) \geq 0$. We have verified part (i) of Theorem 2.1. Part (iv) follows from (i)–(iii). The proof is completed. □

2.2 Proof of Theorem 1.1

Using some results of blow up analysis in Appendix A and Theorem 2.1, we are going to prove Theorem 1.1.

Proof of Theorem 1.1 We first prove the upper bounds. Suppose the assertion of the theorem is false. Then we can find that there exists $K_i \to K$ in $C^2(\mathbb{S}^n)$ such that $\max_{\mathbb{S}^n} v_i \to \infty$ for some $v_i \in \mathscr{M}_{K_i}$. Theorem 2.1 shows that $\{v_i\}$ has only isolated simple blow up points $\{q^{(1)}, \cdots, q^{(k)}\} \subset \mathscr{K}\backslash\mathscr{K}^+$.

Next, we prove that $k > 1$. Let q_0 be the isolated simple blow up point of v_i. It follows from Proposition A.10 and $K \in \mathscr{A}$ that q_0 is a non-degenerate critical point of K. Let F be the stereographic projection with q_0 being the south pole, and $\widetilde{K} := K(F(y))$.

We assert that for any $\widehat{y} \in \mathbb{R}^n$,

$$\begin{pmatrix} \int_{\mathbb{R}^n} \nabla^2\widetilde{K}(0)(y+\widehat{y})(1+|y|^2)^{-n} \\ \int_{\mathbb{R}^n} \frac{1}{2}\langle (y+\widehat{y}), \nabla^2\widetilde{K}(0)(y+\widehat{y})\rangle(1+|y|^2)^{-n} \end{pmatrix} \neq 0. \tag{2.35}$$

In fact, if there exists some $\widehat{y} \in \mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} \nabla^2\widetilde{K}(0)(y+\widehat{y})(1+|y|^2)^{-n} = 0,$$

then by the property of odd function, the non degeneracy of $\nabla^2\widetilde{K}(0)$, and $\Delta\widetilde{K}(0) \neq 0$, we can obtain that

$$\int_{\mathbb{R}^n} \frac{1}{2}\langle (y+\widehat{y}), \nabla^2\widetilde{K}(0)(y+\widehat{y})\rangle(1+|y|^2)^{-n} \neq 0.$$

Thus (2.35) is proved.

Suppose the contrary that q_0 is the only blow up of v_i. We are going to find some $\widehat{y}$ such that (2.35) fails. By (1.4), we know that (1.3) is equivalent to

$$v_i(\xi) = \frac{\Gamma(\frac{n+2\sigma}{2})}{2^{2\sigma}\pi^{n/2}\Gamma(\sigma)}\int_{\mathbb{S}^n} \frac{K_i(\eta)v_i(\eta)^{n-1}}{|\xi-\eta|^2}\,\mathrm{d}\eta \quad \text{on } \mathbb{S}^n. \tag{2.36}$$

Under the stereographic projection F, the equation (2.36) is transformed to

$$u_i(x) = \frac{\Gamma(\frac{n+2\sigma}{2})}{2^{2\sigma}\pi^{n/2}\Gamma(\sigma)} \int_{\mathbb{R}^n} \frac{\widetilde{K}_i(y)u_i(y)^{n-1}}{|x-y|^2}\,\mathrm{d}y \quad \text{on } \mathbb{R}^n,$$

where

$$H(x) = \frac{2}{1+|x|^2}, \quad u_i(x) = H(x)v_i(F(x)), \quad \widetilde{K}_i(x) = K_i(F(x)). \tag{2.37}$$

Let y_i be the local maximum point of $u_i(y)$ and $m_i =: u_i(y_i)$. First, we establish

$$|y_i| = O(m_i^{-1}). \tag{2.38}$$

Since we have assumed that v_i has no blow up point other than q_0, it follows from Proposition A.7 and the Harnack inequality that $u_i(y) \le C(\varepsilon)|y|^{-2}m_i^{-1}$ for $|y| \ge \varepsilon > 0$.

By the Kazdan-Warner condition, we have

$$\int_{\mathbb{R}^n} \nabla\widetilde{K}_i u_i^n = 0. \tag{2.39}$$

It follows that for $\varepsilon > 0$ small we have

$$\Big|\int_{B_\varepsilon} \nabla\widetilde{K}_i(y+y_i)u_i(y+y_i)^n\Big| \le C(\varepsilon)m_i^{-n}. \tag{2.40}$$

For $|y| \le \varepsilon$,

$$\widetilde{K}_i(y) = \widetilde{K}_i(0) + \frac{1}{2}\langle y, \nabla^2\widetilde{K}_i(0)y\rangle + o(|y|^2), \tag{2.41}$$

it follows that

$$\lim_{|y|\to 0} \nabla\big(\widetilde{K}_i(y) - \langle y, \nabla^2\widetilde{K}_i(0)y\rangle\big)|y|^{-1} = 0, \tag{2.42}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^n$. Since $\det(\nabla^2\widetilde{K}(0)) \ne 0$ and $\widetilde{K}_i \to \widetilde{K}$, there exists a constant $C > 0$ such that

$$\Big|\frac{1}{2}\nabla\langle y, \nabla^2\widetilde{K}_i(0)y\rangle\Big| = |\nabla^2\widetilde{K}_i(0)y| \ge C|y|, \quad \forall\, |y| \le \varepsilon. \tag{2.43}$$

By (2.40), (2.42) and (2.43), we can obtain

$$\Big|\int_{B_\varepsilon} (1+o_\varepsilon(1))\nabla^2\widetilde{K}_i(0)(y+y_i)u_i(y+y_i)^n\Big| \le C(\varepsilon)m_i^{-n}.$$

Multiplying the above by m_i, and let $\widetilde{y}_i := m_i y_i$, we have

$$\Big|\int_{B_\varepsilon} (1+o_\varepsilon(1))\nabla^2\widetilde{K}_i(0)(m_i y+\widetilde{y}_i)u_i(y+y_i)^n\Big| \le C(\varepsilon)m_i^{1-n}.$$

Suppose (2.38) is false, namely $\widetilde{y}_i \to \infty$ along a subsequence. From Proposition A.5, we can choose $R_i \le |\widetilde{y}_i|/4$ such that

$$\begin{aligned}
&\Big|\int_{|y|\le R_i m_i^{-1}} (1+o_\varepsilon(1))\nabla^2\widetilde{K}_i(0)(m_i y+\widetilde{y}_i)u_i(y+y_i)^n\Big| \\
&= \Big|\int_{|z|\le R_i} (1+o_\varepsilon(1))\nabla^2\widetilde{K}_i(0)(z+\widetilde{y}_i)(m_i^{-1}u_i(m_i^{-1}z+y_i))^n\Big| \sim |\widetilde{y}_i|.
\end{aligned}$$

On the other hand, it follows from Proposition A.9 that

$$\Big|\int_{R_i m_i^{-1}\le|y|\le\varepsilon}(1+o_\varepsilon(1))\nabla^2\widetilde{K}_i(0)(m_i y+\widetilde{y}_i)u_i(y+y_i)^n\Big|$$
$$\le C\Big|\int_{R_i m_i^{-1}\le|y|\le\varepsilon}(|m_i y|+|\widetilde{y}_i|)u_i(y+y_i)^n\Big|\le o(1)|\widetilde{y}_i|.$$

It follows that $|\widetilde{y}_i|\le C(\varepsilon)m_i^{1-n}$. This contradicts to $\widetilde{y}_i\to\infty$. Thus (2.38) is proved.

It follows from the Kazdan-Warner condition that

$$\int_{\mathbb{R}^n}\langle y,\nabla\widetilde{K}_i(y+y_i)\rangle u_i(y+y_i)^n=0.$$

Similar to (2.40), we have for any $\varepsilon>0$,

$$\Big|\int_{B_\varepsilon}\langle y,\nabla\widetilde{K}_i(y+y_i)\rangle u_i(y+y_i)^n\Big|\le C(\varepsilon)m_i^{-n}.$$

By (2.41), (2.42), (2.43), and Proposition A.9, we have

$$\Big|\int_{B_\varepsilon}\langle y,\nabla^2\widetilde{K}_i(0)(y+y_i)\rangle u_i(y+y_i)^n\Big|$$
$$\le C(\varepsilon)m_i^{-n}+o_\varepsilon(1)\int_{B_\varepsilon}(|y|^2+|y||y_i|)u_i(y+y_i)^n$$
$$\le C(\varepsilon)m_i^{-n}+o_\varepsilon(1)m_i^{-2}.$$

Multiplying the above by m_i^2, due to $n-2=2\sigma$, we have

$$\lim_{i\to\infty}m_i^2\Big|\int_{B_\varepsilon}\langle y,\nabla^2\widetilde{K}_i(0)(y+y_i)\rangle u_i(y+y_i)^n\Big|=o_\varepsilon(1).$$

Let $R_i\to\infty$ as $i\to\infty$, and $r_i:=R_i m_i^{-1}$. By Proposition A.9, we have

$$m_i^2\Big|\int_{r_i\le|y|\le\varepsilon}\langle y,\nabla^2\widetilde{K}_i(y+y_i)\rangle u_i(y+y_i)^n\Big|$$
$$\le Cm_i^2\Big|\int_{r_i\le|y|\le\varepsilon}(|y|^2+|y||y_i|)u_i(y+y_i)^n\Big|\to 0\quad\text{as }i\to\infty.$$

Using Proposition A.5, making a change of variable $z=m_i y$, and then letting $\varepsilon\to 0$, we have,

$$\int_{\mathbb{R}^n}\langle z,\nabla^2\widetilde{K}(0)(z+z_0)\rangle(1+k|z|^2)^{-n}=0,\tag{2.44}$$

where $z_0=\lim_{i\to\infty}m_i y_i$ and $k=\lim_{i\to\infty}\widetilde{K}_i(y_i)^{1/\sigma}/4$.

It follows from (2.39) that

$$\int_{\mathbb{R}^n}\nabla\widetilde{K}_i(y+y_i)u_i(y+y_i)^n=0.$$

Using the same method above, we obtain

$$\int_{\mathbb{R}^n}\nabla^2\widetilde{K}(0)(z+z_0)(1+k|z|^2)^{-n}=0.\tag{2.45}$$

It follows from (2.44) and (2.45) that

$$\int_{\mathbb{R}^n} \frac{1}{2}\langle z+z_0, \nabla^2 \widetilde{K}(0)(z+z_0)\rangle(1+k|z|^2)^{-n} = 0. \tag{2.46}$$

From (2.45) and (2.46) we can see that (2.35) does not hold for $\widehat{y} = k^{1/2}z_0$. Therefore, we proved that $k > 1$.

By Part (i) of Theorem 2.1, we have $\{q^{(1)}, \cdots, q^{(k)}\} \subset \mathscr{K}^-$ and $\mu(M(q^{(1)}, \cdots, q^{(k)})) \geq 0$. It follows from $v_i \in \mathscr{M}_{K_i}$ that $\tau_i = 0$. Applying Part (iv) of Theorem 2.1, we deduce that $\mu(M(q^{(1)}, \cdots, q^{(k)})) = 0$. This leads to a contradiction with $K \in \mathscr{A}$. From the Harnack inequality and Schauder type estimates, we complete the proof of Theorem 1.1. □

3 The degree-counting formula and existence results

This section is devoted to the proof of Theorems 1.2, 1.3, and 1.4. It is worth noting that due to Theorem 1.1, homotopy invariance of Leray-Schauder degree and the properties of "Index", we only need to prove Theorem 1.2 for $K \in \mathscr{A}$ being a Morse function. Once this is achieved, we also prove that the Index as in Definition 1.1 is well defined on $\mathscr{A}$. Therefore, we always assume that $K \in \mathscr{A}$ is a Morse function in this section.

3.1 On the case of subcritical equations

Let $\sigma = 1 + m/2$, $m \in \mathbb{N}_+$, and $n = 2\sigma + 2$. In this subsection, we consider the following subcritical equation:

$$P_\sigma v = c(n,\sigma) K v^{n-1-\tau} \quad \text{on } \mathbb{S}^n, \tag{3.1}$$

where $c(n,\sigma) = \Gamma(n-1)$, $K \in C^2(\mathbb{S}^n)$, and $\tau > 0$.

We will soon prove that when $K \in \mathscr{A}$, the solutions to (3.1) either stay bounded and converge to the solutions to critical equations (1.3) in $C^{2\sigma}$ norm or become unbounded and blow up at finite points as $\tau \to 0^+$.

Denote the $H^\sigma(\mathbb{S}^n)$ inner product and norm by

$$\langle u, v\rangle = \int_{\mathbb{S}^n} (P_\sigma u)v, \quad \|u\|_\sigma = \sqrt{\langle u, u\rangle}.$$

The Euler-Lagrange functional associated with (3.1) is

$$I_\tau(u) = \frac{1}{2}\int_{\mathbb{S}^n} (P_\sigma u)u - \frac{\Gamma(n-1)}{n-\tau}\int_{\mathbb{S}^n} K|u|^{n-\tau}, \quad \forall u \in H^\sigma(\mathbb{S}^n). \tag{3.2}$$

Definition 3.1 Let $K \in C^2(\mathbb{S}^n)$, $\mathscr{K}^-$ be as in (1.5) and $k \in \mathbb{N}_+$. Let $\overline{P}_1, \cdots, \overline{P}_k \in \mathscr{K}^-$ be the critical points of K with $\mu(M(\overline{P}_1, \cdots, \overline{P}_k)) > 0$, and $\varepsilon_0 > 0$ be sufficiently small. Define

$$\begin{aligned}\Omega_{\varepsilon_0} =& \Omega_{\varepsilon_0}(\overline{P}_1, \cdots, \overline{P}_k)\\ =& \{(\alpha, t, P) \in \mathbb{R}^k_+ \times \mathbb{R}^k_+ \times (\mathbb{S}^n)^k : |\alpha_i - (K(P_i))^{-1/2\sigma}| < \varepsilon_0,\\ & t_i > 1/\varepsilon_0,\ |P_i - \overline{P}_i| < \varepsilon_0,\ 1 \leq i \leq k\}.\end{aligned}$$

For $P \in \mathbb{S}^n$ and $t > 0$,

$$\delta_{P,t}(x) = \frac{t}{1 + \frac{t^2-1}{2}(1 - \cos d(x,P))}, \quad x \in \mathbb{S}^n \tag{3.3}$$

is the family of the solutions for

$$P_\sigma v = \Gamma(n-1)v^{n-1}, \quad v > 0 \quad \text{on } \mathbb{S}^n. \tag{3.4}$$

We have the following lemma based on the ideas provided by Bahri in [5]:

Lemma 3.1 *Let ε_0 be sufficiently small and $\Omega_{\varepsilon_0} = \Omega_{\varepsilon_0}(\overline{P}_1, \cdots, \overline{P}_k)$ be as in Definition 3.1. For any $u \in H^\sigma(\mathbb{S}^n)$ satisfying*

$$\Big\| u - \sum_{i=1}^{k} \widetilde{\alpha}_i \delta_{\widetilde{P}_i, \widetilde{t}_i} \Big\|_\sigma < \frac{\varepsilon_0}{2}$$

for some $(\widetilde{\alpha}, \widetilde{t}, \widetilde{P}) \in \Omega_{\varepsilon_0/2}$, then there exists a unique $(\alpha, t, P) \in \Omega_{\varepsilon_0}$ such that

$$u = \sum_{i=1}^{k} \alpha_i \delta_{P_i, t_i} + v,$$

with v satisfies

$$\langle v, \delta_{P_i,t_i} \rangle = \Big\langle v, \frac{\partial \delta_{P_i,t_i}}{\partial P_i^{(\ell)}} \Big\rangle = \Big\langle v, \frac{\partial \delta_{P_i,t_i}}{\partial t_i} \Big\rangle = 0, \tag{3.5}$$

where $\frac{\partial}{\partial P_i^{(\ell)}}$ denotes the corresponding derivatives.

In what follows, we say that $v \in E_{P,t}$ if v satisfies (3.5) and we work in some orthonormal basis near $\{\overline{P}_1, \cdots, \overline{P}_k\}$.

Definition 3.2 Let τ, ε_0, $\nu_0 > 0$ be sufficiently small, $A > 0$ be sufficiently large, and $\Omega_{\varepsilon_0/2} = \Omega_{\varepsilon_0/2}(\overline{P}_1, \cdots, \overline{P}_k)$ be as in Definition 3.1. Define

$$\begin{aligned} &\Sigma_\tau(\overline{P}_1, \cdots, \overline{P}_k) \\ =&\{(\alpha, t, P, v) \in \Omega_{\varepsilon_0/2} \times H^\sigma(\mathbb{S}^n) : \\ &|P_i - \overline{P}_i| < \tau^{1/2}|\log\tau|,\ A^{-1}\tau^{-1/2} < t_i < A\tau^{-1/2},\ v \in E_{P,t},\ \|v\|_\sigma < \nu_0\}. \end{aligned} \tag{3.6}$$

Without confusion we use the same notation for

$$\Sigma_\tau(\overline{P}_1, \cdots, \overline{P}_k) = \Big\{ u = \sum_{i=1}^{k} \alpha_i \delta_{P_i,t_i} + v : (\alpha, t, P, v) \in \Sigma_\tau \Big\} \subset H^\sigma(\mathbb{S}^n).$$

Combined with Theorem 2.1, we can obtain the necessary conditions on blowing up solutions to (3.1) when $K \in \mathscr{A}$ as τ tends to 0^+.

Proposition 3.1 *Let $\sigma = 1 + m/2$, $m \in \mathbb{N}_+$, and $n = 2\sigma + 2$. Let $K \in \mathscr{A}$ be a Morse function and $\mathscr{K}^-$ be as in (1.5). Then for any $\alpha \in (0,1)$, there exist some positive constants $\varepsilon_0, \nu_0 \ll 1$, and $A, R \gg 1$ depending only on K, such that when $\tau > 0$ is sufficiently small, for all u satisfying $u \in H^\sigma(\mathbb{S}^n)$, $u > 0$, $I_\tau'(u) = 0$, we have*

$$u \in \mathscr{O}_R \cup \{\cup_{k \geq 1} \cup_{\overline{P}_1, \cdots, \overline{P}_k \in \mathscr{K}^-, \mu(M(\overline{P}_1, \cdots, \overline{P}_k)) > 0} \Sigma_\tau(\overline{P}_1, \cdots, \overline{P}_k)\},$$

where $I_\tau'(u)$ is as in (3.1), $\mathscr{O}_R$ is as in (1.12) and $\Sigma_\tau(\overline{P}_1, \cdots, \overline{P}_k)$ is as in (3.6).

Proof For any $\tau > 0$ sufficiently small, let $u_\tau \in H^\sigma(\mathbb{S}^n)$, $u_\tau > 0$ be a critiacl point of $I_\tau(u)$. If u_τ is uniformly bounded, then by the Schauder type estimates we know that there exists a $R > 0$ such that $u_\tau \in \mathscr{O}_R$. The proof is now completed. If not, there exists $\tau_i \to 0$ such that $\max_{\mathbb{S}^n} u_{\tau_i} \to \infty$. It follows from Theorem 2.1 and $K \in \mathscr{A}$ that there exists a constant $\delta^* > 0$ such that $\{u_{\tau_i}\}$ has only isolated simple blow up points $q^{(1)}, \cdots, q^{(k)} \in \mathscr{K}^-$, with $|q^{(j)} - q^{(\ell)}| \geq \delta^*$, $\forall\, j \neq \ell$, and $\mu(M(q^{(1)}, \cdots, q^{(k)})) > 0$. Then Proposition 3.1 can be deduced from Propositions A.6, A.7, A.8, and Lemma 3.1. $\square$

Now we are going to show that if $K \in \mathscr{A}$ is a Morse function, one can construct solutions highly concentrating at arbitrary points $q^{(1)}, \cdots, q^{(k)} \in \mathscr{K}^-$ provided $\mu(M(q^{(1)}, \cdots, q^{(k)})) > 0$.

Theorem 3.1 *Let $\sigma = 1 + m/2$, $m \in \mathbb{N}_+$, and $n = 2\sigma + 2$. Let $K \in \mathscr{A}$ be a Morse function and $\mathscr{K}^-$ be as in (1.5). Let $\tau, \varepsilon_0, \nu_0 > 0$ be sufficiently small, $A > 0$ be sufficiently large and $k \in \mathbb{N}_+$. Then for any $\overline{P}_1, \cdots, \overline{P}_k \in \mathscr{K}^-$ satisfying $\mu(M(\overline{P}_1, \cdots, \overline{P}_k)) > 0$, we have*

$$\deg_{H^\sigma}(u - P_\sigma^{-1}(c(n,\sigma)K|u|^{2\sigma-\tau}u), \Sigma_\tau(\overline{P}_1, \cdots, \overline{P}_k), 0) = (-1)^{k + \sum_{j=1}^k i(\overline{P}_j)}, \tag{3.7}$$

where $\deg_{H^\sigma}$ denotes the Leray-Schauder degree in $H^\sigma(\mathbb{S}^n)$, and $i(\overline{P}_j)$ is the Morse index of K at $\overline{P}_j$.

In order to prove Theorem 3.1, we need the following Lemmas 3.2, 3.3 and Propositions 3.2, 3.3, 3.4, 3.5, whose proofs mainly uses the estimates in the appendix.

Lemma 3.2 *Under the hypotheses of Theorem 3.1, in addition that $\Sigma_\tau = \Sigma_\tau(\overline{P}_1, \cdots, \overline{P}_k)$ is as in Definition 3.2 for the given $\tau, \varepsilon_0, \nu_0, A$, and $\overline{P}_1, \cdots, \overline{P}_k \in \mathscr{K}^-$. Then for any $(\alpha, t, P, v) \in \Sigma_\tau$, we have:*

$$\begin{aligned} & I_\tau\Big(\sum_{i=1}^k \alpha_i \delta_{P_i,t_i} + v\Big) \\ &= \frac{\Gamma(n-1)}{2}\Big(\sum_{i=1}^k \alpha_i^2 \int_{\mathbb{S}^n} \delta_{P_i,t_i}^n + \sum_{i\neq j} \alpha_i\alpha_j \int_{\mathbb{S}^n} \delta_{P_i,t_i}^{n-1}\delta_{P_j,t_j}\Big) \\ &\quad - \frac{\Gamma(n-1)}{n-\tau}\int_{\mathbb{S}^n} K\Big(\sum_{i=1}^k \alpha_i\delta_{P_i,t_i}\Big)^{n-\tau} + f_\tau(v) + Q_\tau(v,v) + V(\tau,\alpha,t,P,v), \end{aligned}$$

where

$$f_\tau(v) := -\Gamma(n-1)\int_{\mathbb{S}^n} K\Big(\sum_{i=1}^k \alpha_i\delta_{P_i,t_i}\Big)^{n-1-\tau} v, \tag{3.8}$$

$$Q_\tau(v,v) := \frac{1}{2}\int_{\mathbb{S}^n}(P_\sigma v)v - (n-1-\tau)\frac{\Gamma(n-1)}{2}\int_{\mathbb{S}^n} K\Big(\sum_{i=1}^k \alpha_i\delta_{P_i,t_i}\Big)^{2\sigma-\tau} v^2, \tag{3.9}$$

and there exists a constant $C > 0$ depends only on K, ν_0, and A such that

$$|V(\tau,\alpha,t,P,v)| \leq C\|v\|_\sigma^3.$$

Proof By (3.2) and (3.5), we have

$$
\begin{aligned}
&I_\tau\Big(\sum_{i=1}^k \alpha_i\delta_{P_i,t_i}+v\Big)\\
&=\frac{\Gamma(n-1)}{2}\Big(\sum_{i=1}^k \alpha_i^2\int_{\mathbb{S}^n}\delta_{P_i,t_i}^n+\sum_{j\neq i}\alpha_i\alpha_j\int_{\mathbb{S}^n}\delta_{P_i,t_i}^{n-1}\delta_{P_j,t_j}\Big)+\frac{1}{2}\int_{\mathbb{S}^n}(P_\sigma v)v \qquad (3.10)\\
&\quad-\frac{\Gamma(n-1)}{n-\tau}\int_{\mathbb{S}^n}K\Big|\sum_{i=1}^k\alpha_i\delta_{P_i,t_i}+v\Big|^{n-\tau}.
\end{aligned}
$$

Then, it follows from Lemma B.1 and (B.6) that Lemma 3.2 holds. □

Lemma 3.3 *Under the hypotheses of Lemma 3.2, in addition that $E_{P,t}$ is as in (3.5). Then for any $(\alpha, t, P, v)\in\Sigma_\tau$, there exists some function V_v and a constant $C>0$ depending only on K, ν_0, and A such that*

$$
I'_\tau\Big(\sum_{i=1}^k\alpha_i\delta_{P_i,t_i}+v\Big)\varphi=f_\tau(\varphi)+2Q_\tau(v,\varphi)+\langle V_v(\tau,\alpha,t,P,v),\varphi\rangle,
$$

and

$$
\|V_v(\tau,\alpha,t,P,v)\|_\sigma\le\|v\|_\sigma^2,
$$

where $f_\tau(v)$ is as in (3.8) and $Q_\tau(v,\varphi)$ is as in (3.9).

Proof For any $\varphi\in E_{P,t}$, by using (3.10), Lemma B.1, and (3.5), we have

$$
\begin{aligned}
&I'_\tau\Big(\sum_{i=1}^k\alpha_i\delta_{P_i,t_i}+v\Big)\varphi\\
&=\int_{\mathbb{S}^n}P_\sigma(v)\varphi-\Gamma(n-1)\int_{\mathbb{S}^n}K\Big|\sum_{i=1}^k\alpha_i\delta_{P_i,t_i}+v\Big|^{2\sigma-\tau}\Big(\sum_{i=1}^k\alpha_i\delta_{P_i,t_i}+v\Big)\varphi\\
&=\int_{\mathbb{S}^n}P_\sigma(v)\varphi-\Gamma(n-1)\int_{\mathbb{S}^n}K\Big(\sum_{i=1}^k\alpha_i\delta_{P_i,t_i}\Big)^{n-1-\tau}\varphi\\
&\quad-\Gamma(n-1)(n-1-\tau)\int_{\mathbb{S}^n}K\Big(\sum_{i=1}^k\alpha_i\delta_{P_i,t_i}\Big)^{2\sigma-\tau}v\varphi\\
&\quad+\langle V_v(\tau,\alpha,t,P,v),\varphi\rangle.
\end{aligned}
$$

Then, the estimates of $V_v(\tau,\alpha,t,P,v)$ can be can be obtained by Sobolev imbedding and (B.6). □

Proposition 3.2 *Under the hypotheses of the Theorem 3.1, in addition that $\Sigma_\tau(\overline{P}_1,\cdots,\overline{P}_k)$ is as in (3.7) and $E_{P,t}$ is as in (3.5) for the given (α,t,P). Then there exists a unique minimizer $\overline{v}=\overline{v}_\tau(\alpha,t,P)\in E_{P,t}$ of $I_\tau(\sum_{i=1}^k\alpha_i\delta_{P_i,t_i}+v)$ with respect to $\{v\in E_{P,t}:\|v\|_\sigma<\nu_0\}$. Furthermore, there exists a constant C independent of τ such that*

$$
\|\overline{v}\|_\sigma\le C\sum_{i=1}^k|\nabla K(P_i)|\tau^{1/2}+C\tau|\log\tau|\le C\tau|\log\tau|. \qquad (3.11)
$$

Proof From Lemma 3.3, we have, for all $\varphi \in E_{P,t}$,

$$I'_\tau\Big(\sum_{i=1}^k \alpha_i \delta_{P_i,t_i} + v\Big)\varphi = f_\tau(\varphi) + 2Q_\tau(v,\varphi) + \langle V_v(\tau,\alpha,t,P,v),\varphi\rangle, \tag{3.12}$$

where

$$f_\tau(\varphi) := -\Gamma(n-1)\int_{\mathbb{S}^n} K\Big(\sum_{i=1}^k \alpha_i \delta_{P_i,t_i}\Big)^{n-1-\tau}\varphi,$$

and

$$Q_\tau(v,\varphi) := \frac{1}{2}\int_{\mathbb{S}^n}(P_\sigma v)\varphi - (n-1-\tau)\frac{\Gamma(n-1)}{2}\int_{\mathbb{S}^n} K\Big(\sum_{i=1}^k \alpha_i \delta_{P_i,t_i}\Big)^{2\sigma-\tau} v\varphi.$$

It is obviously that f_τ is a continuous linear functional over $E_{P,t}$, there exists a unique $\widetilde{f_\tau} \in E_{P,t}$ such that

$$f_\tau(\varphi) = \langle \widetilde{f_\tau},\varphi\rangle, \quad \forall \varphi \in E_{P,t}. \tag{3.13}$$

By the same method of proving the coercivity of the quadratic form Q_τ in [1, 15], it follows that there exists a constant $\delta_0 > 0$ (independent of τ) such that

$$Q_\tau(v,v) \geq \frac{\delta_0}{2}\|v\|_\sigma^2, \quad \forall (\alpha,t,P,v) \in \Sigma_\tau, \tag{3.14}$$

thus, there exists a unique symmetric continuous and coercive operator $\widetilde{Q}_\tau$ from $E_{P,t}$ onto itself such that,

$$Q_\tau(v,\varphi) = \langle \widetilde{Q}_\tau v,\varphi\rangle, \quad \forall \varphi \in E_{P,t}. \tag{3.15}$$

Using these notations, (3.12), (3.13), and (3.15), we have

$$I'_\tau\Big(\sum_{i=1}^k \alpha_i \delta_{P_i,t_i} + v\Big) = \widetilde{f_\tau} + 2\widetilde{Q}_\tau v + V_v(\tau,\alpha,t,P,v). \tag{3.16}$$

There is an equivalence between the existence of minimizer $\overline{v}_\tau$ and

$$\widetilde{f_\tau} + 2\widetilde{Q}_\tau v + V_v(\tau,\alpha,t,P,v) = 0, \quad v \in E_{P,t}. \tag{3.17}$$

As in [44, 47], by the implicit function theorem, there exist a C^1-map $\overline{v} : (\alpha,t,P) \mapsto E_{P,t}$ satisfying (3.17) and

$$\|\overline{v}\|_\sigma \leq C\|\widetilde{f_\tau}\|_\sigma. \tag{3.18}$$

Therefore, in order to prove (3.11), we only need to estimate $\|\widetilde{f_\tau}\|_\sigma$.

Applying Lemma B.2, (B.10), (B.11), (3.6), and (B.13), we can obtain

$$\begin{aligned}
f_\tau(v) &= -\Gamma(n-1)\int_{\mathbb{S}^n} K\Big(\sum_{i=1}^k (\alpha_i \delta_{P_i,t_i})^{n-1-\tau}\Big)v + O\Big(\sum_{i\neq j}\int_{\mathbb{S}^n}\delta_{P_i,t_i}^{n-2-\tau}\delta_{P_j,t_j}|v|\Big)\\
&= -\Gamma(n-1)\int_{\mathbb{S}^n}(K - K(P_i))\sum_{i=1}^k \alpha_i^{n-1-\tau}\delta_{P_i,t_i}^{n-1} v\\
&\quad + O\Big(\sum_{i=1}^k\int_{\mathbb{S}^n}|\delta_{P_i,t_i}^{n-1-\tau} - \delta_{P_i,t_i}^{n-1}||v|\Big) + O\Big(\sum_{i\neq j}\|\delta_{P_i,t_i}^{n-2-\tau}\delta_{P_j,t_j}\|_{L^{n/(n-1)}(\mathbb{S}^n)}\|v\|_\sigma\Big)
\end{aligned}$$

$$
= O\Big(\sum_{i=1}^{k}|\nabla_{g_0}K(P_i)|\int_{\mathbb{S}^n}|P-P_i|\delta_{P_i,t_i}^{n-1}|v|\Big)+O\Big(\sum_{i=1}^{k}\int_{\mathbb{S}^n}|P-P_i|^2\delta_{P_i,t_i}^{n-1}|v|\Big)
+O(\tau|\log\tau|\|v\|_\sigma),
$$

where $|P-P_i|$ represents the distance between two points P and P_i after through a stereographic projection with P_i as the south pole of $\mathbb{S}^n$.

From (3.6) and (B.13), we have, for all $(\alpha,t,P,v)\in\Sigma_\tau(\overline{P}_1,\cdots,\overline{P}_k)$,

$$
\begin{aligned}
|f_\tau(v)| \le & C\Big\{\tau^{1/2}\sum_{i=1}^{k}|\nabla K(P_i)|+\tau+\tau|\log\tau|\Big\}\|v\|_\sigma \\
\le & C\tau|\log\tau|\|v\|_\sigma,
\end{aligned}
\tag{3.19}
$$

this, combining (3.13) and (3.18), we obtain (3.11). □

Proposition 3.3 *Under the hypotheses of Theorem 3.1, then for any* $(\alpha,t,P,v)\in\Sigma_\tau(\overline{P}_1,\cdots,\overline{P}_k)$, *we have*

$$
\frac{\partial}{\partial\alpha_i}I_\tau\Big(\sum_{j=1}^{k}\alpha_j\delta_{P_j,t_j}+v\Big)=-2\sigma\|\delta_{P_i,t_i}\|_\sigma^2\beta_i+V_{\alpha_i}(\tau,\alpha,t,P,v),
$$

where $\beta=(\beta_1,\cdots,\beta_k)$, $\beta_i:=\alpha_i-K(P_i)^{-1/2\sigma}$, $i=1,\cdots,k$, *and*

$$
V_{\alpha_i}(\tau,\alpha,t,P,v)=O(|\beta|^2)+O(\tau|\log\tau|)+O(\|v\|_\sigma^2).
$$

Furthermore, let $\overline{v}$ *be as in Proposition 3.2, then we have*

$$
\frac{\partial}{\partial\alpha_i}I_\tau\Big(\sum_{j=1}^{k}\alpha_j\delta_{P_j,t_j}+\overline{v}\Big)=-2\sigma\|\delta_{P_i,t_i}\|_\sigma^2\beta_i+O(|\beta|^2+\tau|\log\tau|).
$$

Proof Using Lemma B.1, (B.7), (B.10), and Lemma B.2 we have

$$
\begin{aligned}
&\frac{\partial}{\partial\alpha_i}I_\tau\Big(\sum_{j=1}^{k}\alpha_j\delta_{P_j,t_j}+v\Big)\\
&=\Gamma(n-1)\Big(\alpha_i\int_{\mathbb{S}^n}\delta_{P_i,t_i}^n+\sum_{j\neq i}\alpha_j\int_{\mathbb{S}^n}\delta_{P_i,t_i}^{n-1}\delta_{P_j,t_j}\Big)\\
&\quad-\Gamma(n-1)\int_{\mathbb{S}^n}K\Big|\sum_{i=1}^{k}\alpha_i\delta_{P_i,t_i}+v\Big|^{n-2-\tau}\Big(\sum_{j=1}^{k}\alpha_j\delta_{P_j,t_j}+v\Big)\delta_{P_i,t_i}\\
&=\Gamma(n-1)\Big(\alpha_i\int_{\mathbb{S}^n}\delta_{P_i,t_i}^n-\int_{\mathbb{S}^n}K\Big|\sum_{j=1}^{k}\alpha_j\delta_{P_j,t_j}\Big|^{n-1-\tau}\delta_{P_i,t_i}\Big)\\
&\quad-\Gamma(n-1)(n-\tau-1)\int_{\mathbb{S}^n}K\Big|\sum_{j=1}^{k}\alpha_j\delta_{P_j,t_j}\Big|^{n-2-\tau}v\delta_{P_i,t_i}+O(\tau)+O(\|v\|_\sigma^2)
\end{aligned}
$$

$$= \Gamma(n-1)\Big(\alpha_i \int_{\mathbb{S}^n} \delta_{P_i,t_i}^n - \int_{\mathbb{S}^n} K\Big(\sum_{j=1}^k (\alpha_j \delta_{P_j,t_j})^{n-1-\tau}\Big)\delta_{P_i,t_i}\Big)$$
$$- \Gamma(n-1) \int_{\mathbb{S}^n} K\Big(\sum_{j=1}^k (\alpha_j \delta_{P_j,t_j})^{n-2-\tau}\Big) v\delta_{P_i,t_i} + O(\tau) + O(\|v\|_\sigma^2).$$

It follows from (B.13) and (3.6) that

$$\begin{aligned}\int_{\mathbb{S}^n} K\alpha_i^{n-1}\delta_{P_i,t_i}^{n-\tau} &= \int_{\mathbb{S}^n} K(P_i)\alpha_i^{n-1}\delta_{P_i,t_i}^{n-\tau} - \int_{\mathbb{S}^n} (K(P)-K(P_i))\alpha_i^{n-1}\delta_{P_i,t_i}^{n-\tau} \\ &= \int_{\mathbb{S}^n} K(P_i)\alpha_i^{n-1}\delta_{P_i,t_i}^{n-\tau} + O(\tau).\end{aligned} \tag{3.20}$$

Similarly, by (3.5), (B.11), (3.6), and (B.13), we have

$$\begin{aligned}&\int_{\mathbb{S}^n} K\alpha_i^{n-2}\delta_{P_i,t_i}^{n-1-\tau} v \\ &= \int_{\mathbb{S}^n} K(P_i)\alpha_i^{n-2}\delta_{P_i,t_i}^{n-1} v + \int_{\mathbb{S}^n} (K(P)-K(P_i))\alpha_i^{n-2}\delta_{P_i,t_i}^{n-1} v + O(\tau|\log\tau|\|v\|_\sigma) \\ &= O(\tau|\log\tau|) + O(\|v\|_\sigma^2).\end{aligned} \tag{3.21}$$

By using the fact $|\alpha_i^{n-1-\tau} - \alpha_i^{n-1}| = O(\tau)$, (3.20), (3.21), (B.2), and (B.12) that

$$\begin{aligned}&\frac{\partial}{\partial\alpha_i} I_\tau\Big(\sum_{j=1}^k \alpha_j\delta_{P_j,t_j} + v\Big) \\ &= \Gamma(n-1)\Big(\alpha_i \int_{\mathbb{S}^n} \delta_{P_i,t_i}^n - K(P_i)\int_{\mathbb{S}^n} \alpha_i^{n-1}\delta_{P_i,t_i}^{n-\tau} - \int_{\mathbb{S}^n} K\alpha_i^{n-2}\delta_{P_i,t_i}^{n-1-\tau} v\Big) \\ &\quad + O(\tau|\log\tau|) + O(\|v\|_\sigma^2) \\ &= -2\sigma\|\delta_{P_i,t_i}\|_\sigma^2\beta_i + O(|\beta|^2) + O(\tau|\log\tau|) + O(\|v\|_\sigma^2).\end{aligned}$$

Hence

$$\frac{\partial}{\partial\alpha_i} I_\tau\Big(\sum_{j=1}^k \alpha_j\delta_{P_j,t_j} + v\Big) = -2\sigma\|\delta_{P_i,t_i}\|_\sigma^2\beta_i + V_{\alpha_i}(\tau,\alpha,t,P,v),$$

where

$$V_{\alpha_i}(\tau,\alpha,t,P,v) = O(|\beta|^2) + O(\tau|\log\tau|) + O(\|v\|_\sigma^2).$$

Combining with (3.19), we obtain

$$\begin{aligned}\frac{\partial}{\partial\alpha_i} I_\tau\Big(\sum_{j=1}^k \alpha_j\delta_{P_j,t_j} + \overline{v}\Big) &= -2\sigma\|\delta_{P_i,t_i}\|_\sigma^2\beta_i + V_{\alpha_i}(\tau,\alpha,t,P,\overline{v}) \\ &= -2\sigma\|\delta_{P_i,t_i}\|_\sigma^2\beta_i + O(|\beta|^2 + \tau|\log\tau|).\end{aligned} \tag{3.22}$$

Proposition 3.3 follows from the above. □

Proposition 3.4 *Under the hypotheses of Proposition 3.3, then for any* $(\alpha,t,P,v) \in \Sigma_\tau(\overline{P}_1,\cdots,\overline{P}_k)$, *we have*

$$\frac{\partial}{\partial t_i} I_\tau\Big(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v\Big) = \Theta_1 \frac{1}{K(P_i)^{1/\sigma}} \frac{\tau}{t_i} + \Theta_2 \frac{\Delta_{g_0} K(P_i)}{K(P_i)^{n/2\sigma}} \frac{1}{t_i^3} + \Theta_3 \sum_{j\neq i} \frac{G_{P_i}(P_j)}{(K(P_i)K(P_j))^{1/2\sigma}} \frac{1}{t_i^2 t_j} + V_{t_i}(\tau,\alpha,t,P,v),$$

where $\Theta_1, \Theta_2, \Theta_3$ *are positive constants,* $G_{P_i}(P_j)$ *is as in* (1.7)*, and*

$$V_{t_i}(\tau,\alpha,t,P,v) = O(\tau\|v\|_\sigma) + O(\tau^{1/2}\|v\|_\sigma^2) + O(|\beta|\tau^{3/2}) + o(\tau^{3/2}).$$

Proof By (3.10), Lemma B.1, Hölder inequality, and Sobolev embedding, we have

$$\begin{aligned} &\frac{\partial}{\partial t_i} I_\tau\Big(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v\Big) \\ &= \Gamma(n-1)\Big(\sum_{j\neq i} \alpha_i\alpha_j \frac{\partial}{\partial t_i}\int_{\mathbb{S}^n} \delta_{P_i,t_i}^{n-1}\delta_{P_j,t_j} - \int_{\mathbb{S}^n} K\Big(\sum_{j=1}^k \alpha_j\delta_{P_j,t_j}\Big)^{n-1-\tau}\alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial t_i}\Big) \\ &\quad - \Gamma(n-1)(n-1-\tau)\int_{\mathbb{S}^n} K\Big(\sum_{j=1}^k \alpha_j\delta_{P_j,t_j}\Big)^{n-2-\tau} v\alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial t_i} \\ &\quad + O\Big(\|v\|_\sigma^2 \Big\|\frac{\partial \delta_{P_i,t_i}}{\partial t_i}\Big\|_\sigma\Big). \end{aligned} \tag{3.23}$$

From (3.5), we can obtain

$$\begin{aligned} \int_{\mathbb{S}^n} \delta_{P_i,t_i}^{n-2}\frac{\partial \delta_{P_i,t_i}}{\partial t_i} v &= \frac{2}{n+2\sigma}\int_{\mathbb{S}^n} v \frac{\partial}{\partial t_i}(\delta_{P_i,t_i}^{n-1}) \\ &= \frac{2}{(n+2\sigma)c(n,\sigma)}\frac{\partial}{\partial t_i}\langle v, \delta_{P_i,t_i}\rangle \\ &= \frac{2}{(n+2\sigma)c(n,\sigma)}\Big\langle v, \frac{\partial \delta_{P_i,t_i}}{\partial t_i}\Big\rangle \\ &= 0. \end{aligned} \tag{3.24}$$

It follows from (3.24), (3.6), (B.11), (B.8), and (B.14) that

$$\begin{aligned} &\Big|\int_{\mathbb{S}^n} K\delta_{P_i,t_i}^{n-2-\tau}\frac{\partial \delta_{P_i,t_i}}{\partial t_i} v\Big| \\ &= \Big|\int_{\mathbb{S}^n} (K - K(P_i))\delta_{P_i,t_i}^{n-2}\frac{\partial \delta_{P_i,t_i}}{\partial t_i} v + \int_{\mathbb{S}^n} K(\delta_{P_i,t_i}^{n-2-\tau} - \delta_{P_i,t_i}^{n-2})\frac{\partial \delta_{P_i,t_i}}{\partial t_i} v\Big| \\ &\le C(\tau^{1/2}|\log\tau|)\int_{\mathbb{S}^n} |P - P_i|\delta_{P_i,t_i}^{n-2}\frac{\partial \delta_{P_i,t_i}}{\partial t_i} v \\ &\quad + O\Big(\|\delta_{P_i,t_i}^{n-2-\tau} - \delta_{P_i,t_i}^{n-2}\|_{L^{n/(n-2)}(\mathbb{S}^n)}\Big\|\frac{\partial \delta_{P_i,t_i}}{\partial t_i}\Big\|_\sigma \|v\|_\sigma\Big) \\ &\le (\tau^{1/2}|\log\tau|)O\Big(\||\cdot - P_i|\delta_{P_i,t_i}^{n-2}\|_{L^{n/(n-2)}(\mathbb{S}^n)}\Big\|\frac{\partial \delta_{P_i,t_i}}{\partial t_i}\Big\|_\sigma \|v\|_\sigma\Big) + O\big(\tau^{3/2}|\log\tau|\|v\|_\sigma\big) \\ &\le C\tau^{3/2}|\log\tau|\|v\|_\sigma, \end{aligned}$$

this, Lemma B.2, (B.16), and (B.17) yields,

$$
\begin{aligned}
&\Big|\int_{\mathbb{S}^n} K\Big(\sum_{j=1}^{k}\alpha_j\delta_{P_j,t_j}\Big)^{n-2-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}v\Big|\\
&\le \Big|\int_{\mathbb{S}^n} K\alpha_i^{n-2-\tau}\delta_{P_i,t_i}^{n-2-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}v\Big| + C\sum_{j\neq i}\int_{\mathbb{S}^n}\delta_{P_i,t_i}\delta_{P_j,t_j}^{n-3-\tau}\Big|\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\Big||v|\\
&+ C\sum_{j\neq i}\int_{\mathbb{S}^n}\delta_{P_j,t_j}^{n-2-\tau}\Big|\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\Big||v|\\
&\le C(\tau^{3/2}|\log\tau|\|v\|_\sigma + \tau^{3/2}\|v\|_\sigma)\\
&\le C(\tau^{3/2}|\log\tau|\|v\|_\sigma).
\end{aligned}
\tag{3.25}
$$

Using (3.23), (3.25), Lemma B.2, and (B.15), we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial t_i}I_\tau\Big(\sum_{j=1}^{k}\alpha_j\delta_{P_j,t_j}+v\Big)\\
&=\Gamma(n-1)\Big(\sum_{j\neq i}\alpha_i\alpha_j\frac{\partial}{\partial t_i}\int_{\mathbb{S}^n}\delta_{P_j,t_j}\delta_{P_i,t_i}^{n-1} - \int_{\mathbb{S}^n}K(\alpha_i\delta_{P_i,t_i})^{n-1-\tau}\alpha_i\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\Big)\\
&-\Gamma(n-1)\int_{\mathbb{S}^n}K\Big(\sum_{j\neq i}\alpha_j\delta_{P_j,t_j}\Big)^{n-1-\tau}\alpha_i\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\\
&-\Gamma(n-1)(n-1-\tau)\int_{\mathbb{S}^n}K(\alpha_i\delta_{P_i,t_i})^{n-2-\tau}\Big(\sum_{j\neq i}\alpha_j\delta_{P_j,t_j}\Big)\alpha_i\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\\
&+o(\tau^{3/2}) + O(\tau^{3/2}|\log\tau|\|v\|_\sigma) + O(\tau^{1/2}\|v\|_\sigma^2)\\
&=\Gamma(n-1)\Big(\sum_{j\neq i}\alpha_i\alpha_j\frac{\partial}{\partial t_i}\int_{\mathbb{S}^n}\delta_{P_j,t_j}\delta_{P_i,t_i}^{n-1} - \int_{\mathbb{S}^n}K\alpha_i^{n-\tau}\delta_{P_i,t_i}^{n-1-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\Big)\\
&-\Gamma(n-1)\int_{\mathbb{S}^n}\alpha_i K\Big(\sum_{j\neq i}\alpha_j\delta_{P_j,t_j}\Big)^{n-1-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\\
&-\Gamma(n-1)(n-1)\int_{\mathbb{S}^n}\alpha_i^{n-1}K\Big(\sum_{j\neq i}\alpha_j\delta_{P_j,t_j}\Big)\delta_{P_i,t_i}^{n-2-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\\
&+o(\tau^{3/2}) + O(\tau^{3/2}|\log\tau|\|v\|_\sigma) + O(\tau^{1/2}\|v\|_\sigma^2).
\end{aligned}
\tag{3.26}
$$

It follows from Lemma B.2 that

$$
\Big(\sum_{j\neq i}\alpha_j\delta_{P_j,t_j}\Big)^{n-1-\tau} = \sum_{j\neq i}(\alpha_j\delta_{P_j,t_j})^{n-1-\tau} + O\Big(\sum_{j\neq i,\ell\neq i,j\neq\ell}\delta_{P_j,t_j}^{n-2-\tau}\delta_{P_\ell,t_\ell}\Big).
\tag{3.27}
$$

By (3.26), (3.27), (B.8), (B.10), and (B.15), we can obtain

$$
\begin{aligned}
&\frac{\partial}{\partial t_i}I_\tau\Big(\sum_{j=1}^{k}\alpha_j\delta_{P_j,t_j}+v\Big)\\
&\quad=\Gamma(n-1)\Big(\sum_{j\neq i}\alpha_i\alpha_j\frac{\partial}{\partial t_i}\int_{\mathbb{S}^n}\delta_{P_j,t_j}\delta_{P_i,t_i}^{n-1} - \int_{\mathbb{S}^n}K\alpha_i^{n}\delta_{P_i,t_i}^{n-1-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&- \Gamma(n-1) \int_{\mathbb{S}^n} \alpha_i K \sum_{j\neq i} (\alpha_j \delta_{P_j,t_j})^{n-1-\tau} \frac{\partial \delta_{P_i,t_i}}{\partial t_i} \\
&- \Gamma(n-1)\alpha_i^{n-1} \sum_{j\neq i} \int_{\mathbb{S}^n} K \alpha_j \delta_{P_j,t_j} \frac{\partial}{\partial t_i} (\delta_{P_i,t_i})^{n-1-\tau} \\
&+ o(\tau^{3/2}) + O(\tau^{3/2} |\log \tau| \|v\|_\sigma) + O(\tau^{1/2} \|v\|_\sigma^2).
\end{aligned}
$$

By (B.18), we have

$$
\begin{aligned}
&\int_{\mathbb{S}^n} K \delta_{P_j,t_j} \frac{\partial}{\partial t_i} (\delta_{P_i,t_i})^{n-1-\tau} \\
&= \frac{\partial}{\partial t_i} \int_{\mathbb{S}^n} K \delta_{P_j,t_j} \delta_{P_i,t_i}^{n-1-\tau} \\
&= K(P_i) \frac{\partial}{\partial t_i} \int_{\mathbb{S}^n} \delta_{P_j,t_j} \delta_{P_i,t_i}^{n-1-\tau} + \frac{\partial}{\partial t_i} \int_{\mathbb{S}^n} (K - K(P_i)) \delta_{P_j,t_j} \delta_{P_i,t_i}^{n-1-\tau} \\
&= K(P_i) \frac{\partial}{\partial t_i} \int_{\mathbb{S}^n} \delta_{P_j,t_j} \delta_{P_i,t_i}^{n-1-\tau} + O(\tau^2),
\end{aligned}
\tag{3.28}
$$

and by (B.19),

$$
\begin{aligned}
&\int_{\mathbb{S}^n} K \delta_{P_j,t_j}^{n-1-\tau} \frac{\partial \delta_{P_i,t_i}}{\partial t_i} \\
&= \frac{\partial}{\partial t_i} \int_{\mathbb{S}^n} K \delta_{P_i,t_i} \delta_{P_j,t_j}^{n-1-\tau} \\
&= K(P_j) \frac{\partial}{\partial t_i} \int_{\mathbb{S}^n} \delta_{P_i,t_i} \delta_{P_j,t_j}^{n-1-\tau} + \frac{\partial}{\partial t_i} \int_{\mathbb{S}^n} (K - K(P_j)) \delta_{P_i,t_i} \delta_{P_j,t_j}^{n-1-\tau} \\
&= K(P_j) \frac{\partial}{\partial t_i} \int_{\mathbb{S}^n} \delta_{P_i,t_i} \delta_{P_j,t_j}^{n-1-\tau} + O(\tau^2).
\end{aligned}
\tag{3.29}
$$

Thus, from (3.28), (3.29), (B.3), (B.4), and (B.5), we get

$$
\begin{aligned}
&\frac{\partial}{\partial t_i} I_\tau \Big(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v \Big) \\
&= \Gamma(n-1) \Big(\sum_{j\neq i} \alpha_i \alpha_j \frac{\partial}{\partial t_i} \int_{\mathbb{S}^n} \delta_{P_j,t_j} \delta_{P_i,t_i}^{n-1} - \int_{\mathbb{S}^n} K \alpha_i^n \delta_{P_i,t_i}^{n-1-\tau} \frac{\partial \delta_{P_i,t_i}}{\partial t_i} \Big) \\
&\quad - \Gamma(n-1) K(P_i) \alpha_i^{n-1} \sum_{j\neq i} \alpha_j \frac{\partial}{\partial t_i} \int_{\mathbb{S}^n} \delta_{P_j,t_j} \delta_{P_i,t_i}^{n-1} \\
&\quad - \Gamma(n-1) \alpha_i \sum_{j\neq i} K(P_j) \alpha_j^{n-1} \frac{\partial}{\partial t_i} \int_{\mathbb{S}^n} \delta_{P_i,t_i} \delta_{P_j,t_j}^{n-1} \\
&\quad + o(\tau^{3/2}) + O(\tau^{3/2} |\log \tau| \|v\|_\sigma) + O(\tau^{1/2} \|v\|_\sigma^2) \\
&= \Gamma(n-1) \Big(\sum_{j\neq i} \alpha_i \alpha_j \frac{\partial}{\partial t_i} \int_{\mathbb{S}^n} \delta_{P_j,t_j} \delta_{P_i,t_i}^{n-1} - \frac{1}{n-\tau} \int_{\mathbb{S}^n} K(P_i) \alpha_i^n \frac{\partial \delta_{P_i,t_i}^{n-\tau}}{\partial t_i} \Big) \\
&\quad - \Gamma(n-1) \frac{2\Delta_{g_0} K(P_i)}{n(n-\tau)} \int_{\mathbb{S}^n} |P - P_i|^2 \alpha_i^n \frac{\partial \delta_{P_i,t_i}^{n-\tau}}{\partial t_i}
\end{aligned}
$$

$$
\begin{aligned}
&- \Gamma(n-1)\sum_{j\neq i}\{\alpha_i^{n-1}\alpha_j K(P_i)+\alpha_i\alpha_j^{n-1}K(P_j)\}\frac{\partial}{\partial t_i}\int_{\mathbb{S}^n}\delta_{P_j,t_j}\delta_{P_i,t_i}^{n-1}\\
&+o(\tau^{3/2})+O(\tau^{3/2}|\log\tau|\|v\|_\sigma)+O(\tau^{1/2}\|v\|_\sigma^2)\\
=&\ \Gamma(n-1)\sum_{j\neq i}\{\alpha_i\alpha_j-\alpha_i^{n-1}\alpha_j K(P_i)-\alpha_i\alpha_j^{n-1}K(P_j)\}\frac{\partial}{\partial t_i}\int_{\mathbb{S}^n}\delta_{P_j,t_j}\delta_{P_i,t_i}^{n-1}\\
&-\frac{\Gamma(n-1)}{n-\tau}\alpha_i^n K(P_i)\frac{\partial}{\partial t_i}\int_{\mathbb{S}^n}\delta_{P_i,t_i}^{n-\tau}\\
&-\frac{2\Gamma(n-1)}{n(n-\tau)}\Delta_{g_0}K(P_i)\alpha_i^n\frac{\partial}{\partial t_i}\int_{\mathbb{S}^n}|P-P_i|^2\delta_{P_i,t_i}^{n-\tau}\\
&+o(\tau^{3/2})+O(\tau^{3/2}|\log\tau|\|v\|_\sigma)+O(\tau^{1/2}\|v\|_\sigma^2)\\
=&-\Gamma(n-1)\sum_{j\neq i}\frac{1}{(K(P_i)K(P_j))^{1/2\sigma}}\frac{\partial}{\partial t_i}\int_{\mathbb{S}^n}\delta_{P_j,t_j}\delta_{P_i,t_i}^{n-1}\\
&-\frac{\Gamma(n-1)}{n}\frac{1}{K(P_i)^{1/\sigma}}\frac{\partial}{\partial t_i}\int_{\mathbb{S}^n}\delta_{P_i,t_i}^{n-\tau}\\
&-\frac{2\Gamma(n-1)}{n^2}\frac{\Delta_{g_0}K(P_i)}{K(P_i)^{n/2\sigma}}\frac{\partial}{\partial t_i}\int_{\mathbb{S}^n}|P-P_i|^2\delta_{P_i,t_i}^{n-\tau}\\
&+o(\tau^{3/2})+O(\tau^{3/2}|\log\tau|\|v\|_\sigma)+O(\tau^{1/2}\|v\|_\sigma^2)+O(|\beta|\tau^{3/2}),
\end{aligned}
$$

where $|P-P_i|$ represents the distance between two points P and P_i after through a stereographic projection with P_i as the south pole of $\mathbb{S}^n$.

It follows that

$$
\begin{aligned}
&\frac{\partial}{\partial t_i}I_\tau\Big(\sum_{j=1}^k\alpha_j\delta_{P_j,t_j}+v\Big)\\
&=\Theta_1\frac{1}{K(P_i)^{1/\sigma}}\frac{\tau}{t_i}+\Theta_2\frac{\Delta_{g_0}K(P_i)}{K(P_i)^{n/2\sigma}}\frac{1}{t_i^3}\\
&+\Theta_3\sum_{j\neq i}\frac{G_{P_i}(P_j)}{(K(P_i)K(P_j))^{1/2\sigma}}\frac{1}{t_i^2t_j}+V_{t_i}(\tau,\alpha,t,P,v),
\end{aligned}
\tag{3.30}
$$

where

$$
\begin{aligned}
\Theta_1&=2^{n-2}\Gamma(n-1)|\mathbb{S}^{n-1}|\frac{n-2}{n(n-1)}\mathrm{B}(\frac{n}{2},\frac{n}{2}-1),\\
\Theta_2&=2^n\Gamma(n-1)|\mathbb{S}^{n-1}|\frac{1}{n(n-1)}\mathrm{B}(\frac{n}{2},\frac{n}{2}-1),\\
\Theta_3&=2^n\Gamma(n-1)|\mathbb{S}^{n-1}|\mathrm{B}(\frac{n}{2},\frac{n}{2}-1),
\end{aligned}
$$

and

$$V_{t_i}(\tau,\alpha,t,P,v)=o(\tau^{3/2})+O(\tau\|v\|_\sigma)+O(\tau^{1/2}\|v\|_\sigma^2)+O(|\beta|\tau^{3/2}).$$

Proposition 3.4 follows from the above. □

Proposition 3.5 *Under the hypotheses of Proposition 3.3, then for any* $(\alpha, t, P, v) \in \Sigma_\tau(\overline{P}_1, \cdots, \overline{P}_k)$, *we have*

$$\frac{\partial}{\partial P_i} I_\tau\Big(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v\Big) = -\Theta_4 \nabla_{g_0} K(P_i) + V_{P_i}(\tau, \alpha, t, P, v),$$

where $\Theta_4 \geq \nu_1 > 0$ *is a constant, and*

$$V_{P_i}(\tau, \alpha, t, P, v) = O(\tau^{1/2}) + O(\|v\|_\sigma) + O(\tau^{-1/2}\|v\|_\sigma^2).$$

Proof Using (3.10) and Lemma B.1, we have

$$\begin{aligned}
&\frac{\partial}{\partial P_i} I_\tau\Big(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v\Big)\\
&= \Gamma(n-1)\sum_{j\neq i}\alpha_i\alpha_j \frac{\partial}{\partial P_i}\int_{\mathbb{S}^n} \delta_{P_j,t_j}^{n-1}\delta_{P_i,t_i}\\
&\quad - \Gamma(n-1)\int_{\mathbb{S}^n} K\Big|\sum_{j=1}^k \alpha_j\delta_{P_j,t_j} + v\Big|^{n-2-\tau}\Big(\sum_{j=1}^k \alpha_j\delta_{P_j,t_j} + v\Big)\alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i}\\
&= \Gamma(n-1)\sum_{j\neq i}\alpha_i\alpha_j \frac{\partial}{\partial P_i}\int_{\mathbb{S}^n} \delta_{P_j,t_j}^{n-1}\delta_{P_i,t_i}\\
&\quad - \Gamma(n-1)\int_{\mathbb{S}^n} K\Big(\sum_{j=1}^k \alpha_j\delta_{P_j,t_j}\Big)^{n-1-\tau}\alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i}\\
&\quad - \Gamma(n-1)(n-1-\tau)\int_{\mathbb{S}^n} K\Big(\sum_{j=1}^k \alpha_j\delta_{P_j,t_j}\Big)^{n-2-\tau} v\alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i}\\
&\quad + O\Big(\|v\|_\sigma^2\Big\|\frac{\partial \delta_{P_i,t_i}}{\partial P_i}\Big\|_\sigma\Big).
\end{aligned}\tag{3.31}$$

It follows from Lemma B.2 that

$$\begin{aligned}
\Big(\sum_{j=1}^k \alpha_j\delta_{P_j,t_j}\Big)^{n-2-\tau} &= \Big(\alpha_i\delta_{P_i,t_i} + \sum_{j\neq i}\alpha_j\delta_{P_j,t_j}\Big)^{n-2-\tau}\\
&= (\alpha_i\delta_{P_i,t_i})^{n-2-\tau} + O\Big(\sum_{j\neq i}\delta_{P_i,t_i}^{n-3-\tau}\delta_{P_j,t_j} + \sum_{j\neq i}\delta_{P_j,t_j}^{n-2-\tau}\Big).
\end{aligned}$$

By (B.22), (B.9), (B.10), (B.14), and (B.11), we have

$$\begin{aligned}
&\int_{\mathbb{S}^n} K\Big(\sum_{j=1}^k \alpha_j\delta_{P_j,t_j}\Big)^{n-2-\tau} v\alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i}\\
&= \int_{\mathbb{S}^n} K(\alpha_j\delta_{P_j,t_j})^{n-2-\tau} v\alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i} + O\Big(\sum_{j\neq i}\int_{\mathbb{S}^n} \delta_{P_i,t_i}^{n-3-\tau}\delta_{P_j,t_j}\Big|\frac{\partial \delta_{P_i,t_i}}{\partial P_i}\Big||v|\Big)\\
&\quad + O\Big(\sum_{j\neq i}\int_{\mathbb{S}^n} \delta_{P_j,t_j}^{n-2-\tau}\Big|\frac{\partial \delta_{P_i,t_i}}{\partial P_i}\Big||v|\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{S}^n} K(P_i)(\alpha_j \delta_{P_j,t_j})^{n-2-\tau} v\alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i} + O\Big(\int_{\mathbb{S}^n} |P - P_i| \delta_{P_i,t_i}^{n-2-\tau} \Big| \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \Big| |v| \Big) \\
&+ O(\tau^{1/2} \|v\|_\sigma) + O(\tau \|v\|_\sigma) \\
&= O\Big(\int_{\mathbb{S}^n} |\delta_{P_i,t_i}^{n-2-\tau} - \delta_{P_i,t_i}^{n-2}| \Big| \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \Big| |v| \Big) + O(\|v\|_\sigma) \\
&= O(\|v\|_\sigma).
\end{aligned}
$$

From Lemma B.2, we have

$$
\begin{aligned}
&\Big(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} \Big)^{n-1-\tau} \\
&= \Big(\alpha_i \delta_{P_i,t_i} + \sum_{j \neq i} \alpha_j \delta_{P_j,t_j} \Big)^{n-1-\tau} \\
&= (\alpha_i \delta_{P_i,t_i})^{n-1-\tau} + \Big(\sum_{j \neq i} \alpha_j \delta_{P_j,t_j} \Big)^{n-1-\tau} + (n-1-\tau)\alpha_i \delta_{P_i,t_i}^{n-2-\tau} \Big(\sum_{j \neq i} \alpha_j \delta_{P_j,t_j} \Big) \\
&+ O\Big(\sum_{j \neq i} \delta_{P_i,t_i}^{n-3-\tau} \delta_{P_j t_j}^2 \Big),
\end{aligned}
$$

then, by using (B.9) and (B.23), (B.21), (B.24), we can obtain

$$
\begin{aligned}
&\frac{\partial}{\partial P_i} I_\tau \Big(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v \Big) \\
&= \Gamma(n-1) \sum_{j \neq i} \alpha_i \alpha_j \frac{\partial}{\partial P_i} \int_{\mathbb{S}^n} \delta_{P_j,t_j}^{n-1} \delta_{P_i,t_i} \\
&\quad - \Gamma(n-1)\alpha_i \int_{\mathbb{S}^n} K (\alpha_i \delta_{P_i,t_i})^{n-1-\tau} \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \\
&\quad - \Gamma(n-1)\alpha_i (n-1-\tau) \int_{\mathbb{S}^n} K \Big(\sum_{j \neq i} \alpha_j \delta_{P_j,t_j} \Big) (\alpha_i \delta_{P_i,t_i})^{n-2-\tau} \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \\
&\quad - \Gamma(n-1)\alpha_i \int_{\mathbb{S}^n} \Big(\sum_{j \neq i} \alpha_j \delta_{P_j,t_j} \Big)^{n-1-\tau} \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \\
&\quad + O(\|v\|_\sigma) + O(\tau^{-1/2} \|v\|_\sigma^2) + O(\tau^{3/2}) \\
&= -\Gamma(n-1)\alpha_i^n \int_{\mathbb{S}^n} K \delta_{P_i,t_i}^{n-1-\tau} \frac{\partial \delta_{P_i,t_i}}{\partial P_i} + O(\tau^{1/2}) + O(\|v\|_\sigma) + O(\tau^{-1/2} \|v\|_\sigma^2) \\
&= -\Theta_4(\tau, \alpha, t, P, v) \nabla K(P_i) + V_{P_i}(\tau, \alpha, t, P, v),
\end{aligned}
$$

where

$$
\Theta_4(\tau, \alpha, t, P, v) \geq \nu_1 > 0 \quad \text{with } \nu_1 \text{ independent of } \tau,
$$

and

$$
V_{P_i}(\tau, \alpha, t, P, v) = O(\tau^{1/2}) + O(\|v\|_\sigma) + O(\tau^{-1/2} \|v\|_\sigma^2). \tag{3.32}
$$

We now prove that the existence of ν_1. Let P_i be the south pole and make a stereographic projection F to the equatorial plane of $\mathbb{S}^n$ with $y = (y_{(1)}, \cdots, y_{(n)})$ as the stereographic projection coordinates, let $\widetilde{K} = K(F(y))$ and $|J_F| := (2/(1+|y|^2))^n$. Then we have $F(0) = P_i$ and

$$\begin{aligned}&\int_{\mathbb{S}^n} K\delta_{P_i,t_i}^{n-1-\tau}\frac{\partial \delta_{P_i,t_i}}{\partial P_i}\\ &\quad= \int_{\mathbb{R}^n} \omega_{y_i,t_i}^{n-1}(\nabla\widetilde{K}(0)\cdot y + O(|y|^2))g_\tau(y)\frac{\partial \omega_{y_i,t_i}}{\partial y_i}\\ &\quad=: \mathcal{L} = (\mathcal{L}^{(1)},\cdots,\mathcal{L}^{(n)}),\end{aligned}$$

where $\omega_{y_i,t_i}(y) = \frac{2t_i}{1+t_i^2|y|^2}$, and $g_\tau(y) := (\omega_{y_1,0}^{-1}|J_F|^{1/n})^\tau$. For $\ell = 1,\cdots,n$, we have

$$\begin{aligned}\mathcal{L}^{(\ell)} &= \int_{\mathbb{R}^n} \omega_{y_i,t_i}^{n-1}(\nabla\widetilde{K}(0)\cdot y + O(|y|^2))g_\tau(y)\frac{\partial \omega_{y_i,t_i}}{\partial y_i}\\ &= \int_{\mathbb{R}^n} t_i y_{(\ell)}\omega_{y_i,t_i}^{n+1}(\nabla\widetilde{K}(0) + O(|y|^2))g_\tau(y)\\ &=\frac{1}{n}\frac{\partial \widetilde{K}}{\partial y_{(\ell)}}(0)\int_{\mathbb{R}^n} t_i|y|^2\omega_{y_i,t_i}^{n+1}g_\tau(y) + O(\tau^{1/2}),\end{aligned}$$

thus,

$$\mathcal{L} = \nabla_{g_0}K(P_i)\frac{2}{n}\int_{\mathbb{S}^n} t_i|y|^2\omega_{y_i,t_i}^{n+1}g_\tau(y) + O(\tau^{1/2}).$$

It follows from $t_i^{-\tau} \le g_\tau(y) \le t_i^\tau$ that

$$\int_{\mathbb{S}^n} t_i|y|^2\omega_{y_i,t_i}^{n+1}g_\tau(y) \ge t_i^{-\tau}\int_{\mathbb{S}^n} t_i|y|^2\omega_{y_i,t_i}^{n+1} \to \int_{\mathbb{R}^n}\frac{2^{n+1}}{(1+|x|^2)^n}$$

as $\tau \to 0$. This ensures the existence of ν_1. We have proved Proposition 3.5. □

By using Propositions 3.2, 3.3, 3.4, 3.5, and constructing a family of homotopy Id+compact operators, we will obtain the degree-counting formula of the solutions to the subcritical equation (3.1) on $\Sigma_\tau(\overline{P}_1,\cdots,\overline{P}_k)$.

Proof of Theorem 3.1 The $\mathscr{K}^-$ be as in (1.5) for the given K and $\Sigma_\tau(\overline{P}_1,\cdots,\overline{P}_k)$ be as in (3.6) for the given $\overline{P}_1,\cdots,\overline{P}_k \in \mathscr{K}^-$. For $u = \sum_{i=1}^k \alpha_i\delta_{P_i,t_i} + v \in \Sigma_\tau(\overline{P}_1,\cdots,\overline{P}_k)$, we have

$$T_uH^\sigma(\mathbb{S}^n) = E_{P,t}\bigoplus \mathrm{span}\{\delta_{P_i,t_i},\frac{\partial\delta_{P_i,t_i}}{\partial t_i},\frac{\partial\delta_{P_i,t_i}}{\partial P_i}\}.$$

Since $I_\tau'(u) \in T_uH^\sigma(\mathbb{S}^n)$, there exist $\xi \in E_{P,t}$, $\eta \in \mathrm{span}\{\delta_{P_i,t_i},\frac{\partial\delta_{P_i,t_i}}{\partial t_i},\frac{\partial\delta_{P_i,t_i}}{\partial P_i}\}$ such that

$$I_\tau'(u) = \xi + \eta.$$

By Lemma 3.3, we have

$$\langle\xi,\varphi\rangle = I_\tau'(u)\varphi = f_\tau(\varphi) + 2Q_\tau(v,\varphi) + \langle V_v(\tau,\alpha,t,P,v),\varphi\rangle,\quad \forall\varphi \in E_{P,t}, \tag{3.33}$$

where $\|V_v(\tau,\alpha,t,P,v)\|_\sigma \le C\|v\|_\sigma^2$. Replacing φ by v in (3.33) and using (3.14), we have

$$\|\xi\|_\sigma \ge \delta_0\|v\|_\sigma - \|f_\tau\| - O(\|v\|_\sigma^2) \ge \frac{\delta_0}{2}\|v\|_\sigma - \|f_\tau\|, \tag{3.34}$$

where δ_0 is as in (3.14). Let $\beta = (\beta_1, \cdots, \beta_k)$, $\beta_i = \alpha_i - K(P_i)^{-1/2\sigma}$ be as in Proposition 3.3, we define

$$\widehat{\Sigma}_\tau = \Big\{u = \sum_{i=1}^k \alpha_i \delta_{P_i,t_i} + v \in \Sigma_\tau(\overline{P}_1, \cdots, \overline{P}_k) : \|v\|_\sigma < \tau|\log\tau|^3,\ |\beta| < \tau|\log\tau|^2\Big\}.$$

It follows from Proposition 3.2 and (3.22) that

$$I_\tau'(u) \neq 0, \quad \forall\, u \in \Sigma_\tau(\overline{P}_1, \cdots, \overline{P}_k)\backslash\widehat{\Sigma}_\tau.$$

For any $u = \sum_{i=1}^k \alpha_i \delta_{P_i,t_i} + v \in \widehat{\Sigma}_\tau$, by (3.10) and Proposition 3.3, we have

$$\begin{aligned}
\langle \eta, \delta_{P_i,t_i}\rangle &= I_\tau'(u)\delta_{P_i,t_i}\\
&= \Gamma(n-1)\Big(\alpha_i \int_{\mathbb{S}^n} \delta_{P_i,t_i}^n + \sum_{j=1}^k \alpha_j \int_{\mathbb{S}^n} \delta_{P_i,t_i}^{n-1}\delta_{P_j,t_j}\Big)\\
&\quad - \Gamma(n-1)\int_{\mathbb{S}^n} K\Big|\sum_{i=1}^k \alpha_i\delta_{P_i,t_i} + v\Big|^{n-2-\tau}\Big(\sum_{j=1}^k \alpha_j\delta_{P_j,t_j} + v\Big)\delta_{P_i,t_i}\\
&= \frac{\partial}{\partial\alpha_i} I_\tau\Big(\sum_{j=1}^k \alpha_j\delta_{P_j,t_j} + v\Big)\\
&= -2\sigma\|\delta_{P_i,t_i}\|_\sigma^2\beta_i + V_{\alpha_i}(\tau, \alpha, t, P, v),
\end{aligned}$$

and

$$V_{\alpha_i}(\tau, \alpha, t, P, v) = O(\tau|\log\tau|). \tag{3.35}$$

It follows from (3.23) and (3.30) that

$$\begin{aligned}
\Big\langle \eta, \frac{\partial\delta_{P_i,t_i}}{\partial t_i}\Big\rangle &= I_\tau'(u)\,\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\\
&= \frac{1}{\alpha_i}\frac{\partial}{\partial t_i} I_\tau\Big(\sum_{j=1}^k \alpha_j\delta_{P_j,t_j} + v\Big)\\
&= \frac{1}{\alpha_i}\Big\{\Theta_1 \frac{1}{K(P_i)^{1/\sigma}}\frac{\tau}{t_i} + \Theta_2 \frac{\Delta_{g_0}K(P_i)}{K(P_i)^{n/2\sigma}}\frac{1}{t_i^3}\\
&\quad + \Theta_3 \sum_{j\neq i} \frac{G_{P_i}(P_j)}{(K(P_i)K(P_j))^{1/2\sigma}}\frac{1}{t_i^2 t_j} + V_{t_i}(\tau, \alpha, t, P, v)\Big\},
\end{aligned}$$

where

$$|V_{t_i}(\tau, \alpha, t, P, v)| = o(\tau^{3/2}). \tag{3.36}$$

By (3.31) and (3.32), we obtain

$$\begin{aligned}
\Big\langle \eta, \frac{\partial\delta_{P_i,t_i}}{\partial P_i}\Big\rangle &= I_\tau'(u)\frac{\partial\delta_{P_i,t_i}}{\partial P_i}\\
&= \frac{1}{\alpha_i}\frac{\partial}{\partial P_i} I_\tau\Big(\sum_{i=1}^k \alpha_i\delta_{P_i,t_i} + v\Big)\\
&= \frac{1}{\alpha_i}\{-\Theta_4\nabla_{g_0}K(P_i) + V_{P_i}(\tau, \alpha, t, P, v)\},
\end{aligned}$$

with V_{P_i} satisfying

$$|V_{P_i}(\tau, \alpha, t, P, v)| = O(\tau^{1/2}). \tag{3.37}$$

Using the estimates stated above, we define a family of operators on $\widehat{\Sigma}_\tau$ as follows: for any $u = \sum_{i=1}^k \alpha_i \delta_{P_i,t_i} + v \in \widehat{\Sigma}_\tau$,

$$X_\theta(u) := \xi_\theta(u) + \eta_\theta(u), \quad 0 \le \theta \le 1,$$

where, for any $\varphi \in E_{P,t}$,

$$\langle \xi_\theta, \varphi \rangle := \theta f_\tau(\varphi) + (1-\theta)\langle v, \phi \rangle + 2\theta Q_\tau(\varphi, v) + \theta \langle V_v(\tau, \alpha, t, P, v), \varphi \rangle, \tag{3.38}$$

and

$$\begin{aligned}
\langle \eta_\theta, \delta_{P_i,t_i} \rangle &:= -2\sigma \|\delta_{P_i,t_i}\|_\sigma^2 \left\{ \alpha_i - \frac{\theta}{K(P_i)^{1/2\sigma}} - \frac{1-\theta}{K(\overline{P}_i)^{1/2\sigma}} \right\} \\
&\quad + \theta V_{\alpha_i}(\tau, \alpha, t, P, v), \\
\left\langle \eta_\theta, \frac{\partial \delta_{P_i,t_i}}{\partial t_i} \right\rangle &:= \left\{ \frac{\theta}{\alpha_i} + (1-\theta) \right\} \left\{ \frac{\Theta_1}{K(P_i(\theta))^{1/\sigma}} \frac{\tau}{t_i} + \frac{\Theta_2 \Delta_{g_0} K(P_i(\theta))}{K(P_i(\theta))^{n/2\sigma}} \frac{1}{t_i^3} \right. \\
&\quad \left. + \sum_{j \ne i} \frac{\Theta_3 G_{P_i(\theta)}(P_j(\theta))}{(K(P_i(\theta))K(P_j(\theta)))^{1/2\sigma}} \frac{1}{t_i^2 t_j} \right\} + \frac{\theta}{\alpha_i} V_{t_i}(\tau, \alpha, t, P, v), \\
\left\langle \eta_\theta, \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \right\rangle &:= -\left\{ (1-\theta) + \frac{\theta}{\alpha_i}\Theta_4 \right\} \nabla_{g_0} K(P_i) + \frac{\theta}{\alpha_i} V_{P_i}(\tau, \alpha, t, P, v),
\end{aligned} \tag{3.39}$$

where $P_i(\theta)$ is the short geodesic trajectory on $\mathbb{S}^n$ with $P_i(0) = \overline{P}_i$, $P_i(1) = P_i$.

Obviously, $X_1 = I'_\tau(u) = \xi + \eta$. It is well known from (3.2) that $I'_\tau(u)$ is of the form Id+compact on $H^\sigma(\mathbb{S}^n)$. From Sobolev compact imbedding theorem, the explicit forms of V_v, V_{α_i}, V_{t_i}, V_{P_i}, $A^{-2} < t_i^2 \tau < A^2$, (3.35), (3.36), (3.37), and $\Omega_{\varepsilon_0/2}$ in the definition of $\widehat{\Sigma}_\tau$ is a finite dimensional submanifold of $H^\sigma(\mathbb{S}^n)$, we can conclude that X_θ $(0 \le \theta \le 1)$ is the form Id+compact. Furthermore, we have $X_\theta \ne 0$ on $\partial \widehat{\Sigma}_\tau$, $\forall\, 0 \le \theta \le 1$. In fact, for a given $u = \sum_{i=1}^k \alpha_i \delta_{P_i,t_i} + v \in \partial \widehat{\Sigma}_\tau$, we obtain $\xi \ne 0$ by using (3.34) and (3.19). When $\theta = 0$, $\xi_0 = v \ne 0$. It follows from (3.38) that $\xi_\theta \ne 0$, $\forall\, 0 < \theta < 1$. By the homotopy invariance of the Leray-Schauder degree, we have

$$\deg_{H^\sigma}(X_1, \widehat{\Sigma}_\tau, 0) = \deg_{H^\sigma}(X_0, \widehat{\Sigma}_\tau, 0). \tag{3.40}$$

It is easily seen from (3.38) and (3.39) that for any $u = \sum_{i=1}^k \alpha_i \delta_{P_i,t_i} + v \in \widehat{\Sigma}_\tau$,

$$X_0(u) = \xi_0(u) + \eta_0(u),$$

where $\xi_0 \in E_{P,t}$, $\eta_0 \in \text{span}\{\delta_{P_i,t_i}, \frac{\partial \delta_{P_i,t_i}}{\partial t_i}, \frac{\partial \delta_{P_i,t_i}}{\partial P_i}\}$ satisfy

$$\begin{aligned}
\langle \xi_0, \varphi \rangle &= \langle v, \varphi \rangle, \\
\langle \eta_0, \delta_{P_i,t_i} \rangle &= -\beta_i 2\sigma \|\delta_{P_i,t_i}\|_\sigma^2 (\alpha_i - K(\overline{P}_i)^{-1/2\sigma}), \\
\left\langle \eta_0, \frac{\partial \delta_{P_i,t_i}}{\partial t_i} \right\rangle &= \frac{\Theta_1}{K(\overline{P}_i)^{1/\sigma}} \frac{\tau}{t_i} + \frac{\Theta_2 \Delta_{g_0} K(\overline{P}_i)}{K(\overline{P}_i)^{n/2\sigma}} \frac{1}{t_i^3} + \sum_{j \ne i} \frac{\Theta_3 G_{\overline{P}_i}(\overline{P}_j)}{(K(\overline{P}_i)K(\overline{P}_j))^{1/2\sigma}} \frac{1}{t_i^2 t_j}, \\
\left\langle \eta_0, \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \right\rangle &= -\nabla_{g_0} K(P_i).
\end{aligned} \tag{3.41}$$

Recalling the definition of $M(\overline{P}_1, \cdots, \overline{P}_k)$. From the above, we can easily get

$$X_0(u) = 0 \quad \text{on } \widehat{\Sigma}_\tau,$$

if and only if

$$\begin{aligned} &\alpha_i = K(\overline{P}_i)^{-1/2\sigma}, \quad P_i = \overline{P}_i, \quad v = 0, \\ &\frac{\sigma}{4K(\overline{P}_i)^{1/\sigma}} \frac{\tau}{t_i} - \sum_{j=1}^{k} M_{ij}(\overline{P}_1, \cdots, \overline{P}_k) \frac{1}{t_i^2 t_j} = 0. \end{aligned} \tag{3.42}$$

For any $(s_1, \cdots, s_k) \in \mathbb{R}^k$, $s_i > 0$, $i = 1, \cdots, k$, we define

$$F(s_1, \cdots, s_k) := -\frac{\sigma\tau}{4} \sum_{j=1}^{k} \frac{1}{K(\overline{P}_j)^{1/\sigma}} \log s_j + \frac{1}{2} \sum_{i,j=1}^{k} M_{ij}(\overline{P}_1, \cdots, \overline{P}_k) s_i s_j,$$

and for $t_i = s_i^{-1}$,

$$\widehat{F}(t_1, \cdots, t_k) := F(s_1, \cdots, s_k).$$

The derivative with respect to t_i is

$$\frac{\partial \widehat{F}}{\partial t_i}(t_1, \cdots, t_k) = \frac{\sigma\tau}{4K(\overline{P}_i)^{1/\sigma}} \frac{\tau}{t_i} - \sum_{j=1}^{k} M_{ij}(\overline{P}_1, \cdots, \overline{P}_k) \frac{1}{t_i^2 t_j},$$

combining this and (3.41), we have

$$\left\langle \eta_0, \frac{\partial \delta_{P_i, t_i}}{\partial t_i} \right\rangle = \frac{\partial \widehat{F}}{\partial t_i}(t_1, \cdots, t_k).$$

It is obvious that $\nabla \widehat{F}(t_1, \cdots, t_k) = 0$ if and only if $\nabla F(s_1, \cdots, s_k) = 0$. Since $\mu(M(\overline{P_1}, \cdots, \overline{P}_k)) > 0$, a trivial verification shows that $F(s_1, \cdots, s_k)$ is a strictly convex function, and having a unique critical point in the first quadrant. It follows that $\widehat{F}(t_1, \cdots, t_k)$ has unique critical point in the first quadrant with Morse index zero. Hence X_0 has precisely one non-degenerate zero in $\widehat{\Sigma}_\tau$. Furthermore, by (3.42) we can easily obtain

$$\deg_{H^\sigma}(X_0, \widehat{\Sigma}_\tau, 0) = (-1)^{k + \sum_{i=1}^{k} i(\overline{P}_i)}. \tag{3.43}$$

Combining (3.43) and (3.40), we complete the proof of Theorem 3.1. □

Recall the definition of $\mathscr{O}_R$ in (1.12). For $\delta > 0$ suitably small, define

$$\mathscr{O}_{R,\delta} := \{u \in H^\sigma(\mathbb{S}^n) : \inf_{\omega \in \mathscr{O}_R} \|u - \omega\|_\sigma < \delta\}. \tag{3.44}$$

Proposition 3.6 *Let $\sigma = 1 + m/2$, $m \in \mathbb{N}_+$, and $n = 2\sigma + 2$. Let $K \in \mathscr{A}$ be a Morse function and $0 < \tau_0 \le \tau \le 4/(n - 2\sigma) - \tau_0$. Then there exists some constants $C_0 > 0$, $\delta_0 > 0$ depending only on τ_0 and K, such that*

$$\{u \in H^\sigma(\mathbb{S}^n) : u > 0 \ \ a.e., \ I_\tau'(u) = 0\} \subset \mathscr{O}_{C_0, \delta_0}. \tag{3.45}$$

Furthermore, we have $I_\tau'(u) \neq 0$ on $\partial \mathscr{O}_{C_0, \delta_0}$ and

$$\deg_{H^\sigma}(u - P_\sigma^{-1}(\Gamma(n-1) K |u|^{\frac{4\sigma}{n-2\sigma} - \tau} u), \mathscr{O}_{C_0, \delta_0}, 0) = -1. \tag{3.46}$$

Proof From Proposition 3.1, we know that for $\tau > 0$ small there exists some suitable value of ν_0, A, R such that u satisfying $u \in H^\sigma(\mathbb{S}^n)$, $u > 0$, a.e., $I_\tau'(u) = 0$ are either in $\mathscr{O}_R$ or in some $\Sigma_\tau(q^{(1)}, \cdots, q^{(k)})$. Combining (3.6), (3.5), (B.1), and (B.6), we conclude that there exists some positive constants C_0 and δ_0 such that (3.45) holds.

For $K^*(x) = x_{(n+1)} + 2$, $x = (x_{(1)}, \cdots, x_{(n+1)}) \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$ and $t \in (0,1)$, we consider $K_t = tK + (1-t)K^*$. By the homotopy invariance of the Leray-Schauder degree, we only need to establish (3.46) for K^* and τ very small. It is easy to see that $K^* \in \mathscr{A}$ is a Morse function. The proof of (3.46) is straightforward by the Kazdan-Warner condition, Theorem 3.1, and a homotopy argument. □

3.2 Proof of Theorems 1.2, 1.3 and 1.4

Using Theorem 3.1 and Proposition 3.6, we next prove Theorem 1.2.

Proof of Theorem 1.2 The existence of R_0 can be easily obtained from Theorem 1.1. For all $R \geq R_0$, using Theorem 1.1, Proposition 3.1, and the homotopy invariance of the Leray-Schauder degree, we have

$$\begin{aligned}&\deg_{C^{2\sigma,\alpha}}(u - P_\sigma^{-1}(\Gamma(n-1)Ku^{n-1}), \mathscr{O}_R, 0)\\ &\quad= \deg_{C^{2\sigma,\alpha}}(u - P_\sigma^{-1}(\Gamma(n-1)K|u|^{2\sigma-\tau}u), \mathscr{O}_R, 0)\end{aligned} \tag{3.47}$$

for $\tau > 0$ sufficiently small.

Let $C_0 \gg R$, $0 < \delta_1 \ll \delta_0$, and τ_0 be given by Proposition 3.6. Using (3.46), Proposition 3.1, (3.7), (1.10), and the excision property of the degree, we have

$$\deg_{H^\sigma}(u - P_\sigma^{-1}(\Gamma(n-1)K|u|^{2\sigma-\tau}u), \mathscr{O}_{R,\delta_1}, 0) = \mathrm{Index}(K). \tag{3.48}$$

As in the proof of Proposition 3.6, one can check that there are no critical points of I_τ in $\overline{\mathscr{O}_{R,\delta_1}} \backslash \mathscr{O}_R$. Using the same proof idea as Li [40, Theorem B.2] and [31, Theorems 2.4 and 2.5], we can easily get

$$\begin{aligned}&\deg_{C^{2\sigma,\alpha}}(u - P_\sigma^{-1}(\Gamma(n-1)K|u|^{2\sigma-\tau}u), \mathscr{O}_R, 0)\\ &= \deg_{H^\sigma}(u - P_\sigma^{-1}(\Gamma(n-1)K|u|^{2\sigma-\tau}u), \mathscr{O}_{R,\delta_1}, 0).\end{aligned} \tag{3.49}$$

It follows from (3.47)–(3.49) that for $R \geq R_0$, (1.13) is proved. Theorem 1.2 follows from the above. □

Using Theorem 1.2 and perturbing the prescribing function near its critical point, we can know exactly where the blow up occur when $K \notin \mathscr{A}$.

Proof of the Theorem 1.3 Since the Morse functions in $C^2(\mathbb{S}^n)^* \backslash \mathscr{A} = \partial\mathscr{A}$ are dense in $\partial\mathscr{A}$, without loss of generality we consider the case that $K \in \partial\mathscr{A}$ is a Morse function. First recall the definition of $\mathscr{K}$ and $\mathscr{K}^+$, we can assume here $\mathscr{K} \backslash \mathscr{K}^+ = \{q^{(1)}, \cdots, q^{(m)}\}$, $m \in \mathbb{N}_+$. From the definition of $\mathscr{A}$ and $K \in \partial\mathscr{A}$, we know that there exists $1 \leq i_1 < \cdots < i_k \leq m$, $k \geq 1$, such that

$$\mu(M(q^{(i_1)}, \cdots, q^{(i_k)})) = 0. \tag{3.50}$$

Case 1: There is only one such $\{q^{(i_1)}, \cdots, q^{(i_k)}\}$ satisfying (3.50). Using the same C^2 perturbation method as in Li [41, 42], we can obtain a smooth, one-parameter family of Morse functions $\{K_t\}$ $(-1 \leq t \leq 1)$ with the following properties:

(a) K_t $(-1 \leq t \leq 1)$ are identically the same as K except in some small balls around $q^{(i_1)}, \cdots, q^{(i_k)}$ and $K_0 = K$. K_t have the same critical points with the same Morse index for any $-1 \leq t \leq 1$.

(b) $\mu(M(K_t; q^{(j_1)}, \cdots, q^{(j_s)}))$ have the same sign for $-1 < t < 1$ for any $1 \le j_1 < \cdots < j_s \le m$, $(j_1, \cdots, j_s) \neq (i_1, \cdots, i_k)$. Furthermore,

$$\mu(M(K_t; q^{(i_1)}, \cdots, q^{(i_k)})) \begin{cases} < 0, & \text{if } -1 < t < 0, \\ = 0, & \text{if } t = 0, \\ > 0, & \text{if } 0 < t < 1. \end{cases}$$

It is easily seen that $K_t \in \mathscr{A}$ when $t \neq 0$. From the definition of Index, we have

$$\mathrm{Index}(K_1) = \mathrm{Index}(K_{-1}) + (-1)^{k-1+\sum_{j=1}^{k} i(q^{(i_j)})},$$

evidently, $\mathrm{Index}(K_1) \neq \mathrm{Index}(K_{-1})$.

By the homotopy invariance of the Leray-Schauder degree and Theorem 1.2, there exists t_i and $v_i \in \mathscr{M}_{K_{t_i}}$, such that

$$\lim_{i\to\infty} \|v_i\|_{C^{2\sigma,\alpha}(\mathbb{S}^n)} = \infty \quad \text{or} \quad \lim_{i\to\infty} (\min_{\mathbb{S}^n} v_i) = 0.$$

In fact, we can prove that if $\lim_{i\to\infty}(\min_{\mathbb{S}^n} v_i) = 0$, then $\lim_{i\to\infty} \|v\|_{C^{2\sigma,\alpha}} = \infty$. If not, it means that $\{v_i\}$ has no blow up point, then $v_i \equiv 0$ on $\mathbb{S}^n$ can be obtained from $\lim_{i\to\infty}(\min_{\mathbb{S}^n} v_i) = 0$ and Hanarck inequality. This leads to contradictions and we deduce that (1.15)holds.

It follows from $K_t \in \mathscr{A}$ $(t \neq 0)$ and Theorem 1.1 that $t_i \to 0$, namely, $K_{t_i} \to K$. Then by Theorem 2.1, we can know that $\{v_i\}$ blows up exactly at k points $q^{(i_1)}, \cdots, q^{(i_k)}$.

Case 2: If $\{q^{(i_1)}, \cdots, q^{(i_k)}\}$ satisfying (3.50) is not unique, we can perturb as described above the function K near its some critical points to change the Hessian matrix of K at these points, such that there exists a sequence of Morse functions K_ℓ satisfying: $K_\ell \to K$, K_ℓ are identically the same as K except in some small balls and have the same critical points with the same Morse index; there is only one such $(i_1, \cdots, i_k)$ such that (3.50) is true for any ℓ. From Case 1, we know that there exists a sequence of $K_i \to K$ in $C^2(\mathbb{S}^n)$, $v_i \in \mathscr{M}_{K_i}$ such that $\{v_i\}$ blows up at precisely the k points $q^{(i_1)}, \cdots, q^{(i_k)}$. We have thus proved Theorem 1.3. □

Using Theorem 2.1 and the proof method of Theorem 1.3, we show Theorem 1.4 holds.

Proof of Theorem 1.4 By using Theorem 2.1 we can prove the Part (i) of Theorem 1.4. The Part (ii) of Theorem 1.4 is similar to the proof of Theorem 1.3, we omit it here. □

A Appendix

In this section, we review some results about the local analysis and blow up profiles for integral equations obtained in Jin-Li-Xiong [31]. For any $x \in \mathbb{R}^n$ and $r > 0$, the symbol $B_r(x)$ denotes the ball in $\mathbb{R}^n$ with radius r and center x, and $B_r := B_r(0)$.

A.1 Hölder estimates and Schauder type estimates

Consider nonnegative solutions of the integral equation

$$u(x) = \int_{\mathbb{R}^n} \frac{V(y)u(y)}{|x-y|^{n-2\sigma}} \mathrm{d}y \quad \text{a.e in } B_3, \tag{A.1}$$

where $0 < \sigma < n/2$.

The Hölder estimates for solutions to (A.1) is following:

Proposition A.1 *For $n \geq 1$, $0 < \sigma < n/2$, $r > n/(n-2\sigma)$ and $p > n/2\sigma$, let $0 \leq V \in L^p(B_3)$, $0 \leq u \in L^r(B_3)$ and $0 \leq Vu \in L^1_{loc}(\mathbb{R}^n)$. If u satisfies (A.1), then $u \in C^\alpha(B_1)$,*

$$\|u\|_{C^\alpha(B_1)} \leq C\|u\|_{L^r(B_3)},$$

and u satisfies the Harnack inequality

$$\max_{\bar{B}_1} u \leq C \min_{\bar{B}_1} u,$$

where $C > 0$ and $\alpha \in (0, 1)$ depend only on n, σ, p, and an upper bound of $\|V\|_{L^p(B_3)}$.

The Schauder type estimates for solutions u to (A.1) is following:

Proposition A.2 *In addition to the assumptions in Proposition A.1, we assume that $V \in C^\alpha(B_3)$ for some $\alpha > 0$ but not an integer, then $u \in C^{2\sigma+\alpha'}(B_1)$ and*

$$\|u\|_{C^{2\sigma+\alpha'}(B_1)} \leq C\|u\|_{L^r(B_3)},$$

where $\alpha' = \alpha$ if $2\sigma + \alpha \notin \mathbb{N}_+$, otherwise α' can be any positive constant less than α. Here $C > 0$ depends only on n, σ, α and an upper bound of $\|V\|_{C^\alpha(B_3)}$.

A.2 Blow up profiles for nonlinear integral equations

Proposition A.3 (Pohozaev type identity) *Let $u \geq 0$ in $\mathbb{R}^n$, and $u \in C(\overline{B}_R)$ be a solution of*

$$u(x) = \int_{B_R} \frac{K(y)u(y)^p}{|x-y|^{n-2\sigma}}\,\mathrm{d}y + h_R(x),$$

where $1 < p \leq \frac{n+2\sigma}{n-2\sigma}$, and $h_R(x) \in C^1(B_R)$, $\nabla h_R \in L^1(B_R)$. Then

$$\begin{aligned}
&\Big(\frac{n-2\sigma}{2} - \frac{n}{p+1}\Big)\int_{B_R} K(x)u(x)^{p+1}\,\mathrm{d}x - \frac{1}{p+1}\int_{B_R} x\nabla K(x)u(x)^{p+1}\,\mathrm{d}x\\
&= \frac{n-2\sigma}{2}\int_{B_R} K(x)u(x)^p h_R(x)\,\mathrm{d}x + \int_{B_R} x\nabla h_R(x)K(x)u(x)^p\,\mathrm{d}x\\
&\quad - \frac{R}{p+1}\int_{\partial B_R} K(x)u(x)^{p+1}\,\mathrm{d}s.
\end{aligned}$$

Proposition A.4 *Suppose that $0 \leq u_i \in L^\infty_{loc}(\mathbb{R}^n)$ satisfies (2.1) with K_i satisfying (2.2). Suppose that $x_i \to 0$ is an isolated blow up point of $\{u_i\}$, i.e., for some positive constants A_3 and $\bar{r}$ independent of i,*

$$|x - x_i|^{2\sigma/(p_i-1)}u_i(x) \leq A_3 \quad \text{for all } x \in B_{\bar{r}} \subset \Omega.$$

Then for any $0 < r < \bar{r}/3$, we have the following Harnack inequality

$$\sup_{B_{2r}(x_i)\setminus\overline{B_{r/2}(x_i)}} u_i \leq C \inf_{B_{2r}(x_i)\setminus\overline{B_{r/2}(x_i)}} u_i,$$

where C is a positive constant depending only on $\sup_i \|K_i\|_{L^\infty(B_{\bar{r}}(x_i))}$, $n, \sigma, \bar{r}$ and A_3.

Proposition A.5 *Assume the hypotheses in Proposition A.4. Then for every $R_i \to \infty$, $\varepsilon_i \to 0^+$, we have, after passing to a subsequence (still denoted as $\{u_i\}$, $\{x_i\}$, etc.), that*

$$\|m_i^{-1} u_i(m_i^{-(p_i-1)/2\sigma} \cdot + x_i) - (1 + k_i|\cdot|^2)^{(2\sigma-n)/2}\|_{C^2(B_{2R_i}(0))} \le \varepsilon_i,$$
$$r_i := R_i m_i^{-(p_i-1)/2\sigma} \to 0 \quad as\ i \to \infty,$$

where $m_i := u_i(x_i)$ and $k_i := (K_i(x_i)\pi^{n/2}\Gamma(\sigma)/\Gamma(\frac{n}{2}+\sigma))^{1/\sigma}$.

Proposition A.6 *Under the hypotheses of Proposition A.5, there exists a positive constant $C = C(n, \sigma, A_1, A_2, A_3)$ such that,*

$$u_i(x) \ge C^{-1} m_i (1 + k_i m_i^{(p_i-1)/\sigma} |x - x_i|^2)^{(2\sigma-n)/2} \quad \text{for all} \quad |x - x_i| \le 1.$$

In particular, for any $e \in \mathbb{R}^n$, $|e| = 1$, we have

$$u_i(x_i + e) \ge C^{-1} m_i^{-1+((n-2\sigma)/2\sigma)\tau_i}$$

where $\tau_i = (n + 2\sigma)/(n - 2\sigma) - p_i$.

Proposition A.7 *Under the hypotheses of Proposition A.4 with $\bar{r} = 2$, and in addition that $x_i \to 0$ is also an isolated simple blow up point with constant ρ, we have*

$$\tau_i = O(u_i(x_i)^{-c_1+o(1)}) \quad \text{and} \quad u_i(x_i)^{\tau_i} = 1 + o(1),$$

where $c_1 = \min\{2, 2/(n - 2\sigma)\}$. Moreover,

$$u_i(x) \le C u_i^{-1}(x_i)|x - x_i|^{2\sigma-n} \quad \text{for all } |x - x_i| \le 1.$$

Proposition A.8 *Under the hypotheses of Proposition A.7, let*

$$\begin{aligned} T_i(x) :=& u_i(x_i) \int_{B_1(x_i)} \frac{K_i(y)u_i(y)^{p_i}}{|x-y|^{n-2\sigma}} \mathrm{d}y + u_i(x_i) \int_{\mathbb{R}^n \setminus B_1(x_i)} \frac{K_i(y)u_i(y)^{p_i}}{|x-y|^{n-2\sigma}} \mathrm{d}y \\ =: & T_i'(x) + T_i''(x). \end{aligned}$$

Then, after passing a subsequence,

$$T_i'(x) \to a|x|^{2\sigma-n} \quad in\ C^2_{loc}(B_1 \setminus \{0\})$$

and

$$T_i''(x) \to h(x) \quad in\ C^2_{loc}(B_1)$$

for some $h(x) \in C^2(B_2)$, where

$$a = \left(\frac{\pi^{n/2}\Gamma(\sigma)}{\Gamma(\frac{n}{2}+\sigma)}\right)^{-\frac{n}{2\sigma}} \int_{\mathbb{R}^n} \left(\frac{1}{1+|y|^2}\right)^{\frac{n+2\sigma}{2}} \mathrm{d}y \lim_{i\to\infty} K_i(0)^{\frac{2\sigma-n}{2\sigma}}.$$

Consequently, we have

$$u_i(x_i)u_i(x) \to a|x|^{2\sigma-n} + h(x) \quad in\ C^2_{loc}(B_1 \setminus \{0\}).$$

Proposition A.9 *Under the hypotheses of Proposition A.7, we have*

$$\int_{|x-x_i|\le r_i} |x - x_i|^s u_i(x)^{p_i+1} \, \mathrm{d}x = \begin{cases} O(u_i(x_i)^{-2s/(n-2\sigma)}), & -n < s < n, \\ O(u_i(x_i)^{-2n/(n-2\sigma)} \log u_i(x_i)), & s = n, \\ o(u_i(x_i)^{-2n/(n-2\sigma)}), & s > n, \end{cases}$$

and

$$\int_{r_i<|x-x_i|\le 1}|x-x_i|^s u_i(x)^{p_i+1}\,\mathrm{d}x=\begin{cases} o(u_i(x_i)^{-2s/(n-2\sigma)}), & -n<s<n,\\ O(u_i(x_i)^{-2n/(n-2\sigma)}\log u_i(x_i)), & s=n,\\ O(u_i(x_i)^{-2n/(n-2\sigma)}), & s>n,\end{cases}$$

where r_i is as in Proposition A.5.

Proposition A.10 *Let $\sigma=1+m/2$, $m\in\mathbb{N}_+$, $n=2\sigma+2$, and $K_i\to K$ in $C^2(B_3)$. Let $p_i\le\frac{n+2\sigma}{n-2\sigma}=n-1$, $p_i\to n-1$, and $\tau_i=n-1-p_i$. Let $u_i(x)$ satisfy*

$$u_i(x)=\int_{\mathbb{R}^n}\frac{K_i(y)H(y)^{\tau_i}(y)u_i(y)^{p_i}}{|x-y|^2}\mathrm{d}y\quad \text{for}\quad x\in B_3,$$

where $H(y)=2/(1+|y|^2)$. Let $x_i\to 0$ is an isolated simple blow up point of $\{u_i\}$ with constant A_3 and ρ, i.e., $|x-x_i|^{(p_i-1)/2\sigma}u_i(x)\le A_3$, and $r^{\frac{p_i-1}{2\sigma}}\bar{u}_i(r)$ has precisely one critical point in $(0,\rho)$ for large i, where $\bar{u}_i(r)=⨍_{\partial B_r(x_i)}u_i\,\mathrm{d}s$.

Then there exists some constants C_1, C_2 depending only on n, A_3, $\|K\|_{C^2(B_3)}$, ρ, such that

$$|\nabla K_i(x_i)|\le C_1u_i(x_i)^{-1},\quad \tau_i\le C_2u_i(x_i)^{-2}.$$

B Appendix

In this appendix, we provide some estimates that can be verified by elementary calculations which have been used in the proof of Theorem 1.2.

For $P\in\mathbb{S}^n$ and $t>0$, let

$$\delta_{P,t}(x)=\frac{t}{1+\frac{t^2-1}{2}(1-\cos d(x,P))},\quad x\in\mathbb{S}^n,$$

where $d(\cdot,\cdot)$ is the distance induced by the standard metric of $\mathbb{S}^n$. Let P be the south pole of $\mathbb{S}^n$ and make a stereographic projection with respect to the equatorial plane, we then have

$$\delta_{P,t}(y)=\frac{t(1+|y|^2)}{1+t^2|y|^2},\quad \forall\, y\in\mathbb{R}^n.$$

Lemma B.1 *Let $2\le\alpha\le\beta$, there exists a positive constant C depending only on β such that, for any $a\ge 0$, $b\in\mathbb{R}$,*

$$\left||a+b|^{\alpha-1}(a+b)-a^\alpha-\alpha a^{\alpha-1}b-\frac{\alpha(\alpha-1)}{2}a^{\alpha-2}b^2\right|\le C(|b|^\alpha+a^\gamma|b|^{\alpha-\gamma}),$$

where $\gamma=\max\{0,\alpha-3\}$.

Lemma B.2 *For any $2\le\alpha\le 3$ and any $a,b\ge 0$, there exists some universal constant $C>0$ such that for any $a,b\ge 0$, we have*

$$|(a+b)^\alpha-a^\alpha-b^\alpha-\alpha a^{\alpha-1}b|\le Ca^{\alpha-2}b^2,$$
$$|(a+b)^\alpha-a^\alpha-b^\alpha|\le C|a^{\alpha-1}b+ab^{\alpha-1}|.$$

For any $1\le\alpha\le 2$, there exists some universal constant $C>0$ such that for any $a,b\ge 0$, we have

$$|(a+b)^\alpha-a^\alpha|\le C(a^{\alpha-1}b+b^\alpha).$$

Lemma B.3 *We have*

$$\int_{\mathbb{R}^n}\frac{1}{(1+|x|^2)^n}=\frac{(n-2)|\mathbb{S}^{n-1}|}{4(n-1)}\mathrm{B}(\frac{n}{2},\frac{n}{2}-1),$$
$$\int_{\mathbb{R}^n}\frac{|x|^2}{(1+|x|^2)^n}=\frac{n|\mathbb{S}^{n-1}|}{4(n-1)}\mathrm{B}(\frac{n}{2},\frac{n}{2}-1),$$
$$\int_{\mathbb{R}^n}\frac{|x|^2-1}{(1+|x|^2)^n}=\frac{|\mathbb{S}^{n-1}|}{2(n-1)}\mathrm{B}(\frac{n}{2},\frac{n}{2}-1),$$

where $\mathrm{B}(\frac{n}{2},\frac{n}{2}-1)$ *is the Beta function.*

Lemma B.4 *Let* $\varepsilon_0,\tau>0$ *be suitably small and* $A>0$ *be suitably large. Let* $A^{-1}\tau^{-1/2}<t_1,t_2<A\tau^{-1/2}$, $P_1,P_2\in\mathbb{S}^n$, $|P_1-P_2|\geq\varepsilon_0$, δ_{P_i,t_i} *be as in* (3.3) *and* $G_{P_1}(P_2)$ *be as in* (1.7)*, where* $|P_1-P_2|$ *represents the distance between two points* P_1 *and* P_2 *after through a stereographic projection. Then, we have,*

$$\int_{\mathbb{S}^n}\delta^2_{P_1,t_1}\delta_{P_2,t_2}=2^{n+1}\Big(\int_{\mathbb{R}^n}\frac{1}{(1+|x|^2)^{n-1}}\Big)\frac{G_{P_1}(P_2)}{t_1t_2}+O(\tau^2), \tag{B.1}$$
$$\int_{\mathbb{S}^n}\delta^{n-1-\tau}_{P_1,t_1}\delta_{P_2,t_2}=O(\tau), \tag{B.2}$$
$$\frac{\partial}{\partial t_1}\int_{\mathbb{S}^n}\delta^{n-1}_{P_1,t_1}\delta_{P_2,t_2}=-(n-1)2^{n+1}\frac{G_{P_1}(P_2)}{t_1^2t_2}\Big(\int_{\mathbb{R}^n}\frac{|x|^2-1}{(1+|x|^2)^n}\Big)+O(\tau^2), \tag{B.3}$$
$$\frac{\partial}{\partial t_1}\int_{\mathbb{S}^n}\delta^{n-\tau}_{P_1,t_1}=-\frac{\tau}{t_1}\int_{\mathbb{R}^n}\frac{2^n}{(1+|x|^2)^n}+O(\tau^{\frac{5}{2}}|\log\tau|), \tag{B.4}$$
$$\frac{\partial}{\partial t_1}\int_{\mathbb{S}^n}|P-P_1|^2\delta^{n-\tau}_{P_1,t_1}=-\frac{2^{n+1}}{t_1^3}\int_{\mathbb{R}^n}\frac{|x|^2}{(1+|x|^2)^n}+O(\tau^{\frac{5}{2}}|\log\tau|). \tag{B.5}$$

Lemma B.5 *Under the hypotheses of Lemma* B.4*, in addition that* Θ_5,Θ_6 *are positive constants independent of* τ*. Then, we have,*

$$\langle\delta_{P_1,t_1},\delta_{P_1,t_1}\rangle=2^{n-1}|\mathbb{S}^{n-1}|\mathrm{B}(\frac{n}{2},\frac{n}{2}), \tag{B.6}$$
$$\langle\delta_{P_1,t_1},\delta_{P_2,t_2}\rangle=O(\tau), \tag{B.7}$$
$$\Big\langle\frac{\partial\delta_{P_1,t_1}}{\partial t_1},\frac{\partial\delta_{P_1,t_1}}{\partial t_1}\Big\rangle=\Theta_5t_1^{-2}=O(\tau), \tag{B.8}$$
$$\Big\langle\frac{\partial\delta_{P_1,t_1}}{\partial P_1^{(\ell)}},\frac{\partial\delta_{P_1,t_1}}{\partial P_1^{(\ell)}}\Big\rangle=\Theta_6t_1^2,\quad\Big\langle\frac{\partial\delta_{P_1,t_1}}{\partial P_1^{(\ell)}},\frac{\partial\delta_{P_1,t_1}}{\partial P_1^{(m)}}\Big\rangle=0,\ \forall\,\ell\neq m, \tag{B.9}$$
$$\begin{aligned}&\|\delta^{n-2-\tau}_{P_1,t_1}\delta_{P_2,t_2}\|_{L^{n/(n-1)}(\mathbb{S}^n)}=O(\tau),\\&\|\delta^{n-3-\tau}_{P_1,t_1}\delta^2_{P_2,t_2}\|_{L^{n/(n-1)}(\mathbb{S}^n)}=O(\tau),\qquad\qquad(\text{B.10})\\&\|\delta^{n-1-\tau}_{P_1,t_1}-\delta^{n-1}_{P_1,t_1}\|_{L^{n/(n-1)}(\mathbb{S}^n)}=O(\tau|\log\tau|),\end{aligned}$$
$$\|\delta^{n-2-\tau}_{P_1,t_1}-\delta^{n-2}_{P_1,t_1}\|_{L^{n/(n-2)}(\mathbb{S}^n)}=O(\tau|\log\tau|), \tag{B.11}$$
$$\|\delta^{n-\tau}_{P_1,t_1}-\delta^{n}_{P_1,t_1}\|_{L^{1}(\mathbb{S}^n)}=O(\tau|\log\tau|), \tag{B.12}$$
$$\begin{aligned}&\big\||\cdot-P_1|\delta^{n-1}_{P_1,t_1}\big\|_{L^{n/(n-1)}(\mathbb{S}^n)}=O(\tau^{1/2}),\\&\big\||\cdot-P_1|^2\delta^{n-1}_{P_1,t_1}\big\|_{L^{n/(n-1)}(\mathbb{S}^n)}=O(\tau),\qquad\qquad(\text{B.13})\\&\big\||\cdot-P_1|\delta^{n-2}_{P_1,t_1}|\big\|_{L^{n/(n-2)}(\mathbb{S}^n)}=O(\tau^{1/2}),\end{aligned}$$

$$\left\| | \cdot - P_1 | \delta_{P_1,t_1}^{n-2-\tau} | \right\|_{L^{n/(n-2)}(\mathbb{S}^n)} = O(\tau^{1/2}), \tag{B.14}$$

$$\left\| \delta_{P_1,t_1}^{n-3-\tau} \delta_{P_2,t_2}^2 \frac{\partial \delta_{P_1,t_1}}{\partial t_1} \right\|_{L^1(\mathbb{S}^n)} = o(\tau^{3/2}), \tag{B.15}$$

$$\left\| \delta_{P_2,t_2}^{n-2-\tau} \frac{\partial \delta_{P_1,t_1}}{\partial t_1} \right\|_{L^{n/(n-1)}(\mathbb{S}^n)} = O(\tau^{3/2}), \tag{B.16}$$

$$\left\| \delta_{P_1,t_1}^{n-3-\tau} \delta_{P_2,t_2} \frac{\partial \delta_{P_1,t_1}}{\partial t_1} \right\|_{L^{n/(n-1)}(\mathbb{S}^n)} = O(\tau^{3/2}). \tag{B.17}$$

Lemma B.6 *In addition to the hypotheses of Lemma B.4, we assume that $K \in C^1(\mathbb{S}^n)$. Then*

$$\frac{\partial}{\partial t_1} \int_{\mathbb{S}^n} (K - K(P_1)) \delta_{P_2,t_2} \delta_{P_1,t_1}^{n-1-\tau} = O(\tau^2), \tag{B.18}$$

$$\frac{\partial}{\partial t_1} \int_{\mathbb{S}^n} (K - K(P_2)) \delta_{P_1,t_1} \delta_{P_2,t_2}^{n-1-\tau} = O(\tau^2). \tag{B.19}$$

Lemma B.7 *Let ε_0, τ, A be as in Lemma B.4, $P_1, P_2, P_3 \in \mathbb{S}^n$ satisfy $|P_i - P_j| \geq \varepsilon_0$, $i \neq j$, and $A^{-1}\tau^{-1/2} < t_1, t_2, t_3 \leq A\tau^{-1/2}$. Then, we have,*

$$\left\| \delta_{P_2,t_2}^{n-2-\tau} \delta_{P_3,t_3} \frac{\partial \delta_{P_1,t_1}}{\partial t_1} \right\|_{L^1(\mathbb{S}^n)} = o(\tau^{3/2}), \tag{B.20}$$

$$\int_{\mathbb{S}^n} \delta_{P_1,t_1}^{n-2-\tau} \delta_{P_2,t_2} \left| \frac{\partial \delta_{P_1,t_1}}{\partial P_1} \right| = O(\tau^{1/2}), \tag{B.21}$$

$$\left\| \delta_{P_1,t_1}^{n-3-\tau} \delta_{P_2,t_2} \left| \frac{\partial \delta_{P_1,t_1}}{\partial P_1} \right| \right\|_{L^{n/(n-1)}(\mathbb{S}^n)} = O(\tau^{1/2}), \tag{B.22}$$

$$\int_{\mathbb{S}^n} \delta_{P_1,t_1}^{n-3-\tau} \delta_{P_2,t_2}^2 \left| \frac{\partial \delta_{P_1,t_1}}{\partial P_1} \right| = O(\tau^{3/2}), \tag{B.23}$$

$$\left| \frac{\partial}{\partial P_1} \int_{\mathbb{S}^n} \delta_{P_2,t_2}^{n-1-\tau} \delta_{P_1,t_1} \right| = O(\tau). \tag{B.24}$$

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Calculus of Variations

An extended Flaherty-Keller formula for an elastic composite with densely packed convex inclusions

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Abstract
In this paper, we are concerned with the effective elastic property of a two-phase high-contrast periodic composite with densely packed inclusions. The equations of linear elasticity are assumed. We first give a novel proof of the Flaherty-Keller formula for elliptic inclusions, which improves a recent result of Kang and Yu (Calc Var Partial Differ Equ 59(1):3, 2020). We construct a family of auxiliary functions consisting of the Keller-type functions and additional correctors which depend on the coefficients of Lamé system and the geometry of inclusions, to capture the full singular term of the gradient. On the other hand, this method allows us to deal with the inclusions of arbitrary shape, even with zero curvature. An extended Flaherty-Keller formula is proved for m-convex inclusions, $m > 2$, curvilinear squares with round off angles, which minimize the elastic energy under the same volume fraction of hard inclusions.

Mathematics Subject Classification 74Q20 · 74B05 · 35J57

1 Introduction and main results

1.1 Background and Motivation

In a two-phase composite where inclusions are close to each other and the contrast is high between the material properties of the inclusions and the matrix, such as conductivity, elastic moduli. The study of various effective properties of such composite is an interesting and important topic, because they are always singular. As the distance between inclusions, ε,

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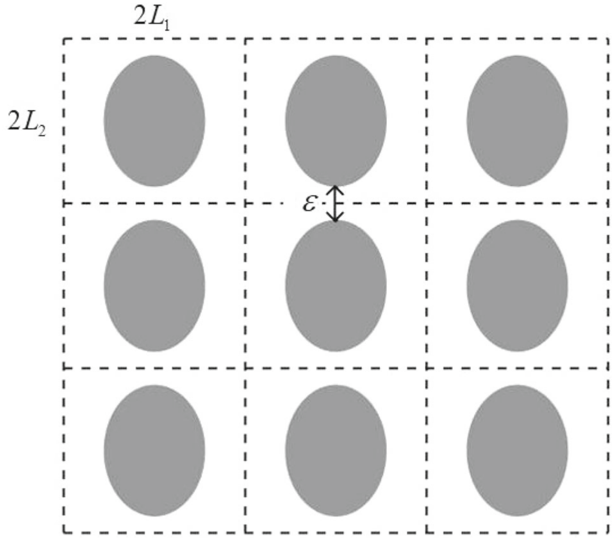

Fig. 1 Ellipse inclusions

tends to zero, several asymptotic formulae for the effective properties have been studied, for example, for effective electric or thermal conductivity problem in [7, 10, 16, 17] and Section 10.10 of [22], for effective shear and extensional modulus in [8, 9, 15].

We suppose that the fibers are rigid and rather closely packed, so that each fiber nearly touches the ones directly above and below it. For a rectangular array of cylinders in the nearly touching limit (see Fig. 1), Flaherty and Keller [9] obtained two asymptotic formulae for the effective shear modulus and extensional modulus. They also showed the numerical validity when the inclusions are rigid cylinders. Recently, Kang and Yu [15] gave a mathematically rigorous proof of the Flaherty-Keller formula, with a lower order term $O(\varepsilon^{-1/4})$, based on the primal-dual variational principle. The key of their proof lies in the contribution to which the dual energy principle is applied. However, these singular functions are only valid for two adjacent hard circular inclusions or elliptic inclusions. When the inclusions are of general convex shape, there will be trouble to apply the primal-dual principle, especially, to drive a suitable lower bound.

In this paper, we develop another method to overcome this limitation. We construct a family auxiliary functions, containing all geometry information of the inclusions of arbitrary shape. These functions consist of the Keller-type functions and additional correctors, stimulated by the idea that we construct the Green function of Laplace's equation. The introduction of such correctors is from an important observation. One can regard them as some variants of the basis of the linear space of rigid displacement coupled with the coefficients of Lamé system and the local geometric information of the inclusions. On the other hand, this construction can also improve the error term $O(\varepsilon^{-1/4})$ obtained in [15] to $O(1)$, so that the error becomes uniformly bounded, which may be helpful in view of numerical computations.

We would like to point out that our improvements result from precise gradient estimates see Proposition 1.3 below. The effective elastic properties (global properties) of a composite are closely related to the stress concentration phenomenon (local properties). When two inclusions with extreme material property are close to touching, the stress may blow up in between them. In fact, the dominant contribution to the effective elastic modulus comes from narrow gaps between closely spaced inclusions, while the stress field outside these gaps does not contribute to the leading term of the asymptotics of the effective elastic modulus. So the analysis of the local blow up of the gradient for the Lamé system with partially infinite coefficients has been an important topic in the field of partial differential equations particularly in the last two decades. The analogue in the scalar case, where u express the antiplane displacement, is also called the conductivity problem, because these two models are consistent in dimension two. We refer to [4–6, 8, 13, 14, 19] and references therein for developments in this topic.

To formulate our main results precisely, we first describe our domain and notations.

1.2 Formulation of the problem

We assume that the composite is spatially periodic, consisting of convex elastic inclusions embedded in an elastic matrix. Let $Y \subset \mathbb{R}^2$ be a rectangular unit period cell of size $2L_1$ along the x_1-axis and $2L_2$ along the x_2-axis, where $L_1,\ L_2 \in (0, +\infty)$. Let $D \subset Y$ be a convex domain, with C^2 boundary centered at the origin and symmetric with respect to the x_1- and x_2-axes. As in [9] we assume that D is close to the horizontal boundary of Y, but away from the vertical boundary. Let $\varepsilon/2$ be the distance between D and the horizontal boundary of Y, so that the distance between two adjacent inclusions is ε.

Assume that $Y\backslash\overline{D}$ is occupied by a homogeneous and isotropic materials with Lamé constants (λ, μ) satisfying the strong ellipticity conditions

$$\mu > 0, \quad and \quad \lambda + \mu > 0.$$

The elasticity tensors $\mathbb{C}$ is given by

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

where $i, j, k, l \in \{1, 2\}$ and δ_{ij} is the Kronecker symbol: $\delta_{ij} = 0$ for $i \neq j$, $\delta_{ij} = 1$ for $i = j$. The linear space of rigid displacements in $\mathbb{R}^2$ is

$$\Psi := \left\{\psi \in C^1(\mathbb{R}^2; \mathbb{R}^2) \mid \nabla\psi + (\nabla\psi)^T = 0\right\},$$

or equivalently

$$\Psi = \mathrm{span}\left\{\psi_1 = \begin{pmatrix}1\\0\end{pmatrix}, \psi_2 = \begin{pmatrix}0\\1\end{pmatrix}, \psi_3 = \begin{pmatrix}x_2\\-x_1\end{pmatrix}\right\}.$$

ψ_3 will be useful to construct auxiliary functions in the sequel.

In cell Y, for a composite with rigid inclusion D, we consider the following problem for the Lame$'$ system: for $i = 1, 2$,

$$\begin{cases} \mathcal{L}_{\lambda,\mu} v_i = \nabla \cdot \mathbb{C}e(v_i) = 0, & \text{in } Y\backslash\overline{D}, \\ v_i = 0, & \text{on } \partial D, \\ \partial_\nu v_i = 0, & \text{on } x_1 = \pm L_1, \\ v_i = \pm\frac{1}{2}\psi_i, & \text{on } x_2 = \pm L_2, \end{cases} \tag{1.1}$$

where $v_i = \left(v_i^{(1)}, v_i^{(2)}\right) \in H^1(Y\backslash\overline{D}; \mathbb{R}^2)$, represents the displacement field,

$$e(v_i) = \frac{1}{2}\Big(\nabla v_i + (\nabla v_i)^T\Big) \quad (T \text{ for transpose})$$

is the strain tensor, and the corresponding co-normal derivative on ∂Y is defined by

$$\partial_\nu v_i := \Big(\mathbb{C}e(v_i)\Big)\mathbf{n} = \lambda(\nabla \cdot v_i)\mathbf{n} + \mu\Big(\nabla v_i + (\nabla v_i)^T\Big)\mathbf{n},$$

and $\mathbf{n}$ is the unit outer normal vector of Y. As in [15], we extend v_i to the whole space $\mathbb{R}^2$ so that the extended function, denoted still by v_i, satisfies the following periodic conditions

$$v_i(x_1, x_2 \pm 2nL_2) = v_i(x_1, x_2) \mp n\psi_i, \quad v_i(x_1 \pm 2nL_1, x_2) = v_i(x_1, x_2).$$

Thus, $e(v_i)$ is periodic as well. The extended function v_j represents the displacement of the elastic periodic composite.

1.3 The Flaherty-Keller formula

For a composite with closely spaced rigid fibers the effective shear modulus μ^* and the effective extensional modulus E^* are defined as follows (see (2.1) and (2.2) in [9])

$$\mu^* = \frac{L_2}{L_1}\int_{-L_1}^{L_1} \partial_\nu v_1(x_1, L_2) \cdot \psi_1 \, dx_1, \tag{1.2}$$

and

$$\begin{aligned} E^* &= \frac{(1+\rho)(1-2\rho)}{1-\rho}\frac{L_2}{L_1}\int_{-L_1}^{L_1} \partial_\nu v_2(x_1, L_2) \cdot \psi_2 \, dx_1 \\ &= \frac{E}{\lambda + 2\mu}\frac{L_2}{L_1}\int_{-L_1}^{L_1} \partial_\nu v_2(x_1, L_2) \cdot \psi_2 \, dx_1, \end{aligned} \tag{1.3}$$

where

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \text{and} \quad \rho = \frac{\lambda}{2(\lambda + \mu)}$$

is, respectively, Young's modulus and Poisson's ratio of the matrix.

Assume that D is an ellipse,

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \le 1, \tag{1.4}$$

where a and b are the length of the short and long semi-axis, respectively, see Fig. 1. The boundary of D near points $(0, \pm b)$ can be written as, respectively,

$$x_2 = \pm b \mp \frac{\kappa_0}{2}x_1^2 + O(x_1^4), \tag{1.5}$$

where $\kappa_0 = b/a^2$ is the curvature of ∂D at the points $(0, \pm b)$. Because the first term in the right hand side of (1.5) is of order two, we call such elliptic inclusion 2-convex inclusion. The Flaherty-Keller formula is as follows.

Theorem 1.1 (The Flaherty-Keller formula) *Let D be as in* (1.4). *Then when* $2(L_2 - b) =: \varepsilon \to 0$, *the asymptotic formulae for the effective shear modulus as in* (1.2) *and the extensional modulus as in* (1.3) *satisfy, respectively,*

$$\mu_2^* = \mu\frac{L_2}{L_1}\frac{\pi}{\sqrt{\kappa_0}}\frac{1}{\sqrt{\varepsilon}} + O(1), \tag{1.6}$$

and

$$E_2^* = E\frac{L_2}{L_1}\frac{\pi}{\sqrt{\kappa_0}}\frac{1}{\sqrt{\varepsilon}} + O(1), \tag{1.7}$$

where κ_0 is the curvature of ∂D at the points $(0, \pm b)$.

Remark 1.1 We would like to remark that the error terms obtained by Kang and Yu in [15] are of $O(\varepsilon^{-1/4})$, which are now improved to $O(1)$ in Theorem 1.1. This improvement is due to a suitable construction of a family of auxiliary functions, see (1.15) and (1.16) below. Besides, we point out that here the constants a, b are fixed, while we can change the size of Y such that $L_2 \to b$ when $\varepsilon \to 0$.

Remark 1.2 As we know, the Keller-type functions (say, $\bar{u}_1$ in (1.15) and $\bar{u}_2$ in (1.16)) are not solutions of the Lamé system. Although it can be used to capture the main term of the gradient of v_i, as in [3–5], there will be a large error term when we calculate the effective modulus. So in order to prove Theorem 1.1, a novel auxiliary function is needed. Fortunately, a class of correctors depending on the Lamé parameters λ and μ can be constructed, together with the Keller-type function, to overcome this difficulty. Meanwhile, we as well improve the results on gradient estimates previously established in [4, 5, 13], for more detail see Proposition 1.3. More importantly, this kind of auxiliary functions allow us to deal with more general inclusions of arbitrary shape, see Theorem 1.2.

As an immediate consequence of Theorem 1.1, we obtain the asymptotic expansion for μ_2^* and E_2^* with respect to the volume fraction, when it is close to its maximum. For instance, $L_1 = L_2 = L$ and $a = b = r < L$, the volume fraction f_2, occupied by circular inclusions, given by

$$f_2 = \frac{\pi r^2}{4L^2},$$

and its maximum value is $\frac{\pi}{4}$ when the inclusions touch each other.

Corollary 1.1 *As $\frac{\pi}{4} - f_2$ tends to zero, we have the following asymptotic formulae for shear and extensional modulus, respectively,*

$$\mu_2^* = \mu\frac{\pi^{3/2}}{\sqrt{2}}\frac{1}{\sqrt{\frac{\pi}{4} - f_2}} + O(1),$$

and

$$E_2^* = E\frac{\pi^{3/2}}{\sqrt{2}}\frac{1}{\sqrt{\frac{\pi}{4} - f_2}} + O(1).$$

1.4 An extended Flaherty-Keller formula

The second contribution of this paper is that the method we developed in the proof of Theorem 1.1 allows us to deal with more general inclusions. Assume that the inclusion is nearly square (see Fig. 2), with a boundary defined by

$$|x_1|^m + |x_2|^m = r^m, \tag{1.8}$$

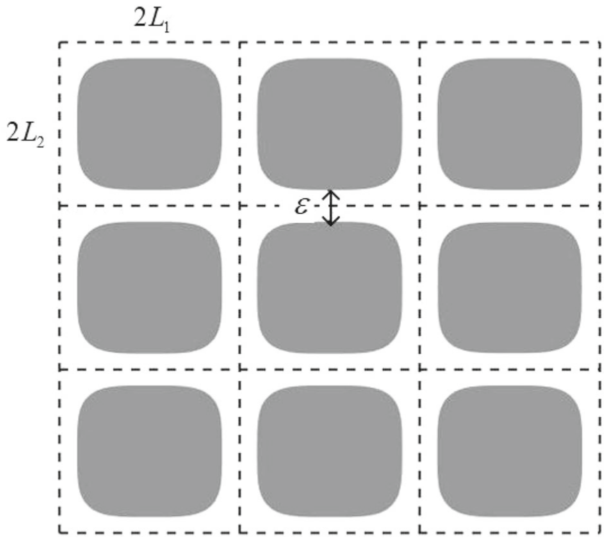

Fig. 2 m-covex inclusions, $m = 4$

where $m > 2$ and $r \in \mathbb{R}$ is a half-width of the inclusion. We call such inclusion a m-convex inclusion.

In this subsection we consider the case of m-convex inclusions. The reason why we study such kind of inclusions will be explained later after Corollary 1.2. We take D as the following a curvilinear square with round off angles, with $m > 2$,

$$|x_1|^m + |x_2|^m \leq r^m,$$

where r is a half-width of the inclusion (See Fig. 2). An extended Flaherty-Keller formula for the effective elastic moduli is as follows:

Theorem 1.2 (An extended Flaherty-Keller formula) *Given $m > 2$, then, as the distance between two inclusions $\varepsilon = 2(L_2 - r) \to 0$, the asymptotic formulae for the effective shear modulus defined in* (1.2) *and the extensional modulus in* (1.3) *are, respectively,*

$$\mu_m^* = 2\mu \frac{L_2}{L_1} \frac{\pi}{m \sin \frac{\pi}{m}} \frac{1}{\kappa_0^{\frac{1}{m}}} \frac{1}{\varepsilon^{1-\frac{1}{m}}} + O(1),$$

and

$$E_m^* = 2E \frac{L_2}{L_1} \frac{\pi}{m \sin \frac{\pi}{m}} \frac{1}{\kappa_0^{\frac{1}{m}}} \frac{1}{\varepsilon^{1-\frac{1}{m}}} + O(1),$$

where $\kappa_0 := \frac{2}{m} r^{1-m}$.

Similarly as in Corollary 1.1, let f_m be the volume fraction occupied by curvilinear squares. Then under the assumption that $L_1 = L_2 = L$,

$$f_m = \frac{r^2}{2mL^2} \frac{\Gamma(\frac{1}{m})^2}{\Gamma(\frac{2}{m})},$$

and its maximum value is $\frac{\Gamma(\frac{1}{m})^2}{2m\Gamma(\frac{2}{m})}$ when the inclusions touch each other.

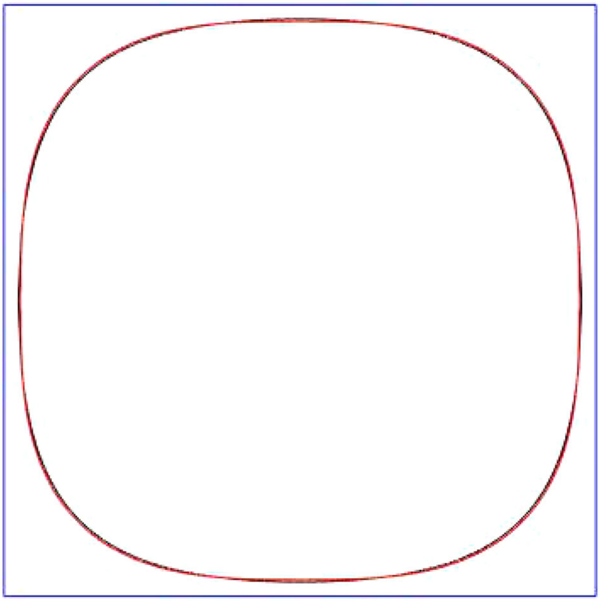

Fig. 3 A Vigdergauz inclusion (black) and an m-convex inclusion (red) with the same volume fraction

Corollary 1.2 *As* $\delta_m = \frac{\Gamma(\frac{1}{m})^2}{2m\Gamma(\frac{2}{m})} - f_m$ *tends to zero, we have*

$$\mu_m^* = \mu \frac{\pi}{\sin\frac{\pi}{m}} \left(\frac{2}{m^2} \frac{\Gamma(\frac{1}{m})^2}{\Gamma(\frac{2}{m})} \right)^{1-\frac{1}{m}} \frac{1}{\delta_m^{1-\frac{1}{m}}} + O(1)$$

and

$$E_m^* = E \frac{\pi}{\sin\frac{\pi}{m}} \left(\frac{2}{m^2} \frac{\Gamma(\frac{1}{m})^2}{\Gamma(\frac{2}{m})} \right)^{1-\frac{1}{m}} \frac{1}{\delta_m^{1-\frac{1}{m}}} + O(1).$$

Here we explain the relationship between m-convex inclusions and "Vigdergauz inclusions", since the latter minimize the maximum stress concentration in the theory of structural optimization. This kind of inclusions was first discovered by Vigdergauz in a series of papers, [23, 24]. Indeed, composites that achieve extremal effective properties have received a lot of attention in structural optimization problems, see [2, 12, 16, 18, 20, 21]. In [11], a shape of an optimal inclusion is given in terms of the elliptic integrals of the first kind, which is called "Vigdergauz inclusion".

For the square periodicity cell, given the volume fraction f of the inclusions, the parameter $h(0 < h < 1)$ depending only on f is the solution of the equation $h = (1 - f)/(1 + f)$. The incomplete and complete elliptic integrals of the first kind are, respectively,

$$F(x \mid \mu) = \int_0^x \frac{ds}{\sqrt{(1 - s^2)(1 - \mu s^2)}}, \quad K(\mu) = F(1 \mid \mu),$$

where the parameter $\mu(\frac{1}{2} < \mu < 1)$ is a solution of the equation $h = K(1 - \mu)/K(\mu)$. Then the quarter of the boundary of the Vigdergauz inclusion can be given by the following parametrization [11]:

$$\begin{cases} x(t) = -\frac{1}{2(1+h)K(\mu)} F(\sqrt{1-t} \mid \mu), \\ y(t) = \frac{1}{2(1+h)K(\mu)} F\left(\sqrt{1-\frac{M}{t}} \,\middle|\, \mu\right), \end{cases} \tag{1.9}$$

where the parameter $t \in [M, 1]$ and $M = (1-\mu)^2/\mu^2$. It was found in [10] that such inclusions have a nearly square shape. It is very close to an m-convex inclusion, under the same volume fraction (see Fig. 3). So it is also very interesting to consider m-convex inclusion, with simple curve boundary (1.8), to describe the nearly square shape. We would like to point out that an asymptotic formula for the effective conductivity of a composite with m-convex inclusion was derived in [10].

As shown above, we have obtained the asymptotic formula for the elastic moduli near the maximum volume fractions at $m = 2$ (Corollary 1.1) and $m > 2$ (Corollary 1.2). It is shown in [11] that composites with"Vigdergauz inclusions" minimize the overall energy at a given strain, among all composites made from the same components in the same volume fraction. For this purpose, we compare the elastic moduli of the two cases under the same volume fraction.

Remark 1.3 For circular inclusions and curvilinear square inclusions with $m = 4$, for example, when their volume fractions are the same, we can calculate the corresponding elastic moduli of the composite. Namely, for $\delta_2 := \frac{\pi}{4} - f_2 > 0$, it is easy to see, from

$$\frac{\pi}{4} - \delta_2 = f_2 = f_4 = \frac{1}{8}\frac{\Gamma(\frac{1}{4})^2}{\Gamma(\frac{1}{2})} - \delta_4, \tag{1.10}$$

that $\delta_4 > \delta_2$. So $\mu_4^* < \mu_2^*$ and $E_4^* < E_2^*$. For example, taking $\delta_2 = 0.01$, we have

$$\mu_2^* \approx 12.53\pi\mu, \quad E_2^* \approx 12.53\pi E,$$

while by (1.10), $\delta_4 \approx 0.15$, and

$$\mu_4^* \approx 5.56\pi\mu, \quad E_4^* \approx 5.56\pi E.$$

This shows that under the same volume fraction the elastic moduli at $m = 4$ is exactly smaller than at $m = 2$, which is consistent with the conclusion in [11] that "Vigdergauz inclusions" minimize the elastic energy. From (1.9), we can only approximately solve $x_2 = h(x_1)$ by Taylor expansions, then we also can compute the corresponding effective modulus.

1.5 Outline of the Proof of Theorem 1.1

We next outline our main idea to prove Theorem 1.1. As (2.5) in [15], we first extend v_i to the whole space $\mathbb{R}^2$ by periodicity so that the extended function, denoted still by v_i as defined in (1.1), satisfies the following periodic conditions

$$\begin{aligned} v_i(x_1, x_2 + 2L_2) &= v_i(x_1, x_2) + \psi_i, \\ v_i(x_1 + 2L_1, x_2) &= v_i(x_1, x_2). \end{aligned}$$

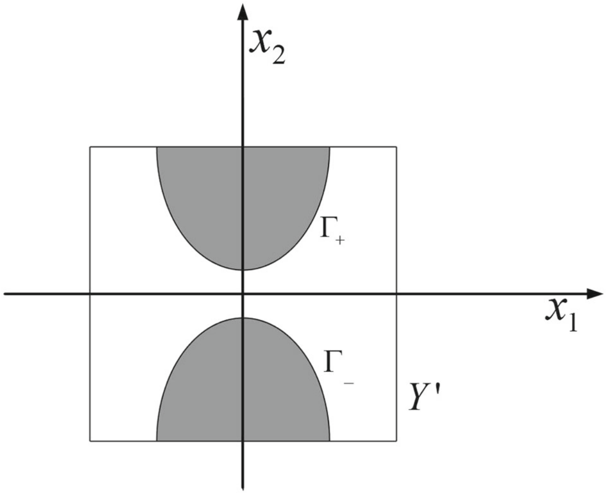

Fig. 4 Elliptic inclusions

Using $\mathbf{n}\big|_{x_2=L_2} = -\mathbf{n}\big|_{x_2=-L_2}$ and the boundary condition of v_i in (1.1), we have

$$\int_{-L_1}^{L_1} \partial_\nu v_i(x_1, L_2)\cdot\psi_i = \int_{-L_1}^{L_1} \partial_\nu v_i(x_1, -L_2)\cdot\Big(-\frac{1}{2}\psi_i\Big) + \int_{-L_1}^{L_1} \partial_\nu v_i(x_1, L_2)\cdot\Big(\frac{1}{2}\psi_i\Big)$$
$$= \int_{\partial(Y\setminus\overline{D})} \partial_\nu v_i\cdot v_i = \int_{Y\setminus\overline{D}} (\mathbb{C}e(v_i), e(v_i)) =: \mathcal{E}_i.$$

In view of (1.2) and (1.3), the effective moduli μ^* and E^* can be expressed in terms of the energy integral, namely,

$$\mu^* = \frac{L_2}{L_1}\mathcal{E}_1 \quad\text{and}\quad E^* = \frac{E}{\lambda+2\mu}\frac{L_2}{L_1}\mathcal{E}_2. \tag{1.11}$$

Now it is more convenient to consider the energy integral $\mathcal{E}_i$ in a translated cell $Y_{tr} := (-L_1, L_1)\times(0, 2L_2)$, by translating the x_1-axis by L_2 along the x_2-direction. Let the upper half-inclusion be D_1 and the lower one be D_2, and $Y' = Y_{tr}\setminus\overline{D_1\cup D_2}$. See Fig. 4. Set $\varGamma_+ = (\partial D_1\cup\{x_2 = L_2\})\cap\partial Y'$ and $\varGamma_- = (\partial D_2\cup\{x_2 = -L_2\})\cap\partial Y'$. We still denote v_i after translation. By the periodicity, note that, for any $i = 1, 2$, $v_i|_{Y'}\in H^1(Y')$ is the solution to the following problem:

$$\begin{cases} \mathcal{L}_{\lambda,\mu} v_i = \nabla\cdot\mathbb{C}e(v_i) = 0, & \text{in } Y',\\ v_i = \psi_i, & \text{on } \varGamma_+,\\ v_i = 0, & \text{on } \varGamma_-,\\ \partial_\nu v_i = 0, & \text{on } \{x_1 = \pm L_1\}. \end{cases} \tag{1.12}$$

Then,

$$\mathcal{E}_i = \int_{Y'} (\mathbb{C}e(v_i), e(v_i))\,dx. \tag{1.13}$$

Denote the two points on ∂D_1 and ∂D_2, achieving the distance between D_1 and D_2,

$$P_1 = \Big(0, \frac{\varepsilon}{2}\Big)\in\partial D_1 \quad\text{and}\quad P_2 = \Big(0, -\frac{\varepsilon}{2}\Big)\in\partial D_2.$$

Then the parts of ∂D_1 and ∂D_2 near P_1 and P_2, respectively, can be represented as follows

$$x_2 = \frac{\varepsilon}{2} + h_1(x_1) = \frac{\varepsilon}{2} - \frac{\kappa_0}{2}x_1^2 + O(x_1^4),$$
$$x_2 = -\frac{\varepsilon}{2} + h_2(x_1) = -\frac{\varepsilon}{2} + \frac{\kappa_0}{2}x_1^2 + O(x_1^4), .$$

for $|x_1| \le a$. We always use $\delta(x_1)$ to denote the vertical distance between the inclusions,

$$\delta(x_1) := \varepsilon + h_1(x_1) - h_2(x_1), \quad \text{for } |x_1| \le a$$

and for $0 \le s \le r$,

$$\Omega_s := \left\{ x = (x_1, x_2) \in \mathbb{R}^2 \;\Big|\; -\frac{\varepsilon}{2} + h_2(x_1) < x_2 < \frac{\varepsilon}{2} + h_1(x_1),\ |x_1| < s \right\}. \tag{1.14}$$

Because, as mentioned before, the greatest stress occurs in the narrow gaps between D_1 and D_2 while the outside stress does not contribute to the singularity, we now construct two auxiliary functions $u_i \in C^2(\mathbb{R}^2)$, such that $u_i = \Psi_i$ on Γ_+, $u_i = 0$ on Γ_-, $\partial_\nu u_i = 0$ on $x_1 = \pm L_1$, and for $x \in \Omega_{\frac{r}{2}}$,

$$u_1 := \bar{u}_1 + \tilde{u}_1 := \frac{2x_2 + \delta(x_1)}{2\delta(x_1)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \Big((\frac{x_2}{\delta(x_1)})^2 - \frac{1}{4} \Big) \begin{pmatrix} (2 - \frac{\mu}{\lambda+2\mu})\frac{\kappa_0}{3}x_2 \\ (1 - \frac{\mu}{\lambda+2\mu})\kappa_0 x_1 \end{pmatrix}, \tag{1.15}$$

$$u_2 := \bar{u}_2 + \tilde{u}_2 := \frac{2x_2 + \delta(x_1)}{2\delta(x_1)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \Big((\frac{x_2}{\delta(x_1)})^2 - \frac{1}{4} \Big) \begin{pmatrix} \frac{\lambda+\mu}{\mu}\kappa_0 x_1 \\ -\frac{\lambda}{3\mu}\kappa_0 x_2 \end{pmatrix}, \tag{1.16}$$

where $\kappa_0 = 1/r$, and

$$\|u_i\|_{C^2(\mathbb{R}^2 \setminus \Omega_{\frac{r}{2}})} \le C. \tag{1.17}$$

We note that the parts $\tilde{u}_i$ can be regarded as variants of $\psi_3 = (x_2, -x_1)^T$, but they also depend on the coefficients of Lamé system. Then we can use an adapted version the energy iteration technique developed in [4, 5], together with the rescaling argument, $W^{2,p}$ estimates, and Sobolev embedding theorem, to obtain the following improved estimates.

Proposition 1.3 *For $i = 1, 2$, we have*

$$|\nabla(v_i - u_i)| \le C, \tag{1.18}$$

where C is independent of ε.

Consequently,

$$\nabla v_i = \nabla \bar{u}_i + \nabla \tilde{u}_i + O(1), \quad i = 1, 2. \tag{1.19}$$

Remark 1.4 We remark that (1.19) is an improvement of the results in [3, 4], where the lower and upper bounds of $|\nabla v_i^\alpha|, i, \alpha = 1, 2$ (see (2.2) in [4] for the definition) are obtained. While (1.19) captures the full singular terms of ∇v_i. It is because our novel constructions of $\tilde{u}_i$ that we can use Proposition 1.3 to prove Theorem 1.1. In particular, it improves the error term to the order of $O(1)$. On the other hand, this construction of u_i allows us to study m-convex inclusions, even when they have zero curvature when $m > 2$. It can also be used to deal with more general convex inclusions, see [19].

For m-convex inclusions, we suppose that

$$h_1(x_1)=\frac{\kappa_0}{2}|x_1|^m+O(|x_1|^m)\quad\text{and}\quad h_2(x_1)=-\frac{\kappa_0}{2}|x_1|^m+O(|x_1|^m),\quad m>2,$$

where $\kappa_0=2r^{1-m}/m$. Instead of (1.15) and (1.16), by constructing the auxiliary functions $u_i\in C^2(\mathbb{R}^2)$ in $\Omega_{\frac{r}{2}}$,

$$u_1=\bar{u}_1+\tilde{u}_1:=\frac{2x_2+\delta(x_1)}{2\delta(x_1)}\begin{pmatrix}1\\0\end{pmatrix}+\Big(\big(\frac{x_2}{\delta(x_1)}\big)^2-\frac{1}{4}\Big)\begin{pmatrix}(2-\frac{\mu}{\lambda+2\mu})\frac{\kappa_0}{3}\frac{m(m-1)}{2}x_2x_1^{m-2}\\ (1-\frac{\mu}{\lambda+2\mu})\kappa_0\frac{m}{2}x_1^{m-1}\end{pmatrix}$$

and

$$u_2=\bar{u}_2+\tilde{u}_2:=\frac{2x_2+\delta(x_1)}{2\delta(x_1)}\begin{pmatrix}0\\1\end{pmatrix}+\Big(\big(\frac{x_2}{\delta(x_1)}\big)^2-\frac{1}{4}\Big)\begin{pmatrix}\frac{\lambda+\mu}{\mu}\kappa_0\frac{m}{2}x_1^{m-1}\\ -\frac{\lambda}{3\mu}\kappa_0\frac{m(m-1)}{2}x_2x_1^{m-2}\end{pmatrix},$$

where $\delta(x_1)=\varepsilon+h_1(x_1)-h_2(x_1)$, we can prove the extended Flaherty-Keller formula in Theorem 1.2.

The rest of this paper is organized as follows. In Sect. 2, we first present some elementary calculations of the auxiliary functions, constructed in (1.15) and (1.16), then use them to prove Proposition 1.3, finally give a new proof of the Flaherty-Keller formula in Theorem 1.1. By this method, the extended Flaherty-Keller formula is proved in Sect. 3, with the main differences provided.

2 Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. We first reduce its proof to the asymptotic formula of $\mathcal{E}_i$, Theorem 2.1 below, then we construct an auxiliary function, which depends on the Lamé system to capture the main terms up to $O(1)$. Finally, we use the asymptotics of the ∇v_i's to prove Theorem 1.1.

Throughout the paper, unless otherwise stated, we use C to denote some positive constant, whose values may vary from line to line, depending only on $a,\ b,\ r$, and an upper bound of the C^2, norms of ∂D_1, ∂D_2 and $\partial Y'$, but not on ε. We call a constant having such dependence a universal constant. First, by the standard theory for elliptic systems, we have

$$\|\nabla v_i\|_{L^\infty(Y'\setminus\Omega_{\frac{r}{2}})}\le C,\quad i=1,2.$$

It follows that

$$\int_{Y'\setminus\Omega_{\frac{r}{2}}}(\mathbb{C}e(v_i),e(v_i))\,dx\le C,\quad i=1,2. \tag{2.1}$$

Thus, in the following we only need to deal with the integrals in $\Omega_{\frac{r}{2}}$. For readers' convenience, in what follows we assume $a=b=r$, and

$$h_1(x_1)=\frac{\kappa_0}{2}x_1^2\quad\text{and}\quad h_2(x_1)=-\frac{\kappa_0}{2}x_1^2,$$

omitting the term $O(x_1^4)$, where $\kappa_0=1/r$ and

$$\delta(x_1)=\varepsilon+\kappa_0x_1^2.$$

We have the following conclusion.

Theorem 2.1 *The energy integral $\mathcal{E}_1$ and $\mathcal{E}_2$, defined in (1.13), have the following expansion, namely*

$$\mathcal{E}_1 = \frac{\pi\mu}{\sqrt{\kappa_0}}\frac{1}{\sqrt{\varepsilon}} + O(1) \quad and \quad \mathcal{E}_2 = \frac{\pi(\lambda+2\mu)}{\sqrt{\kappa_0}}\frac{1}{\sqrt{\varepsilon}} + O(1), \tag{2.2}$$

where $\kappa_0 = 1/r$, as $\varepsilon \to 0$.

It is clear that Theorem 1.1 is an immediate consequence of Theorem 2.1.

Proof (Proof of Theorem 1.1) Recalling (1.11), we have the effective elastic moduli

$$\mu^* = \mu\frac{L_2}{L_1}\frac{\pi}{\sqrt{\kappa_0}}\frac{1}{\sqrt{\varepsilon}} + O(1)$$

and

$$E^* = \frac{E}{\lambda+2\mu}\frac{L_2}{L_1}\frac{(\lambda+2\mu)\pi}{\sqrt{\kappa_0}}\frac{1}{\sqrt{\varepsilon}} + O(1) = E\frac{L_2}{L_1}\frac{\pi}{\sqrt{\kappa_0}}\frac{1}{\sqrt{\varepsilon}} + O(1),$$

as $\varepsilon \to 0$. This completes the proof of Theorem1.1. □

In what follows we will use Proposition 1.3 to prove Theorem 2.1. To this end, we first give some elementary estimates.

2.1 Some elementary estimates

A direct calculation gives the first order derivatives of u_1, defined by (1.15),

$$\partial_{x_1}\bar{u}_1^{(1)} = -2\kappa_0\frac{x_1x_2}{\delta^2(x_1)}, \quad \partial_{x_2}\bar{u}_1^{(1)} = \frac{1}{\delta(x_1)}, \quad \partial_{x_2}\tilde{u}_1^{(2)} = \frac{2(\lambda+\mu)\kappa_0}{\lambda+2\mu}\frac{x_1x_2}{\delta^2(x_1)}, \tag{2.3}$$

and the following estimates for the other terms

$$|\partial_{x_1}\bar{u}_1^{(1)}|,\ |\partial_{x_2}\tilde{u}_1^{(2)}| \le \frac{C|x_1|}{\delta(x_1)}, \qquad |\partial_{x_1}\tilde{u}_1^{(1)}|,\ |\partial_{x_2}\tilde{u}_1^{(1)}|,\ |\partial_{x_1}\tilde{u}_1^{(2)}| \le C. \tag{2.4}$$

Further, for second order derivatives, we have

$$\partial_{x_1x_1}\bar{u}_1^{(1)} = -2\kappa_0\frac{x_2}{\delta^2(x_1)} + \mathcal{R}_{11}^{11}, \quad \partial_{x_1x_2}\bar{u}_1^{(1)} = -2\kappa_0\frac{x_1}{\delta^2(x_1)}, \tag{2.5}$$

$$\partial_{x_2x_2}\tilde{u}_1^{(1)} = \frac{2(2\lambda+3\mu)\kappa_0}{\lambda+2\mu}\frac{x_2}{\delta^2(x_1)}, \quad \partial_{x_1x_2}\tilde{u}_1^{(2)} = \frac{2(\lambda+\mu)\kappa_0}{\lambda+2\mu}\frac{x_2}{\delta^2(x_1)} + \mathcal{R}_{12}^{12}, \tag{2.6}$$

$$\partial_{x_2x_2}\tilde{u}_1^{(2)} = \frac{2(\lambda+\mu)\kappa_0}{\lambda+2\mu}\frac{x_1}{\delta^2(x_1)}, \tag{2.7}$$

where

$$\mathcal{R}_{11}^{11} = 8\kappa_0^2\frac{x_1^2x_2}{\delta^3(x_1)}, \qquad \mathcal{R}_{12}^{12} = \frac{-8\kappa_0^2(\lambda+\mu)}{\lambda+2\mu}\frac{x_1^2x_2}{\delta^3(x_1)}. \tag{2.8}$$

It is clear that

$$\left|\mathcal{R}_{11}^{11}\right|,\ \left|\mathcal{R}_{12}^{12}\right| \le \frac{C}{\delta(x_1)}, \quad \left|\partial_{x_1x_1}\tilde{u}_1^{(1)}\right| \le C, \tag{2.9}$$

$$\left|\partial_{x_1x_2}\tilde{u}_1^{(1)}\right|,\ \left|\partial_{x_1x_1}\tilde{u}_1^{(2)}\right| \le \frac{C|x_1|}{\delta(x_1)}. \tag{2.10}$$

Recalling $u_1 = \bar{u}_1 + \tilde{u}_1$ and (2.5), (2.6) and (2.7), we have

$$(\lambda + 2\mu)(\partial_{x_1x_1}\bar{u}_1^{(1)} - \mathcal{R}_{11}^{11}) + \mu\partial_{x_2x_2}\tilde{u}_1^{(1)} + (\lambda + \mu)(\partial_{x_1x_2}\tilde{u}_1^{(2)} - \mathcal{R}_{12}^{12}) = 0.$$

So,

$$\begin{aligned}(\mathcal{L}_{\lambda,\mu}u_1)^{(1)} &= \mu\triangle u_1^{(1)} + (\lambda + \mu)\Big(\partial_{x_1x_1}u_1^{(1)} + \partial_{x_1x_2}u_1^{(2)}\Big)\\ &= (\lambda + 2\mu)\partial_{x_1x_1}\tilde{u}_1^{(1)} + (\lambda + 2\mu)\mathcal{R}_{11}^{11} + (\lambda + \mu)\mathcal{R}_{12}^{12}.\end{aligned} \tag{2.11}$$

By (2.9), we obtain

$$\Big|(\mathcal{L}_{\lambda,\mu}u_1)^{(1)}\Big| \leq C\Big(\frac{1}{\delta(x_1)} + 1\Big). \tag{2.12}$$

On the other hand, using (2.5), (2.6) and (2.7) again, we have

$$(\lambda + 2\mu)\partial_{x_2x_2}\tilde{u}_1^{(2)} + (\lambda + \mu)\partial_{x_2x_1}\bar{u}_1^{(1)} = 0.$$

Thus,

$$\begin{aligned}(\mathcal{L}_{\lambda,\mu}u_1)^{(2)} &= \mu\triangle u_1^{(2)} + (\lambda + \mu)\Big(\partial_{x_2x_1}u_1^{(1)} + \partial_{x_2x_2}u_1^{(2)}\Big)\\ &= \mu\partial_{x_1x_1}\tilde{u}_1^{(2)} + (\lambda + \mu)\partial_{x_2x_1}\tilde{u}_1^{(1)}.\end{aligned} \tag{2.13}$$

Using (2.10),

$$\Big|(\mathcal{L}_{\lambda,\mu}u_1)^{(2)}\Big| \leq \frac{C|x_1|}{\delta(x_1)}. \tag{2.14}$$

This, together with (2.12), yields

$$\Big|\mathcal{L}_{\lambda,\mu}u_1\Big| \leq \Big|(\mathcal{L}_{\lambda,\mu}u_1)^{(1)}\Big| + \Big|(\mathcal{L}_{\lambda,\mu}u_1)^{(2)}\Big| \leq \frac{C}{\delta(x_1)}. \tag{2.15}$$

Similarly, for $i = 2$, a direct calculation gives the first order derivatives of u_2, defined by (1.16),

$$\partial_{x_1}\bar{u}_2^{(2)} = -2\kappa_0\frac{x_1x_2}{\delta^2(x_1)}, \quad \partial_{x_2}\bar{u}_2^{(2)} = \frac{1}{\delta(x_1)}, \quad \partial_{x_2}\tilde{u}_2^{(1)} = \frac{2(\lambda + \mu)\kappa_0}{\mu}\frac{x_1x_2}{\delta^2(x_1)}. \tag{2.16}$$

It is easy to see that

$$\Big|\partial_{x_1}\bar{u}_2^{(2)}\Big|, \Big|\partial_{x_2}\tilde{u}_2^{(1)}\Big| \leq \frac{C|x_1|}{\delta(x_1)}, \qquad \Big|\partial_{x_1}\tilde{u}_2^{(1)}\Big|, \Big|\partial_{x_1}\tilde{u}_2^{(2)}\Big|, \Big|\partial_{x_2}\tilde{u}_2^{(2)}\Big| \leq C. \tag{2.17}$$

Further,

$$\partial_{x_1x_1}\bar{u}_2^{(2)} = -2\kappa_0\frac{x_2}{\delta^2(x_1)} + \mathcal{R}_{22}^{11}, \quad \partial_{x_1x_2}\bar{u}_2^{(2)} = -2\kappa_0\frac{x_1}{\delta^2(x_1)}, \tag{2.18}$$

$$\partial_{x_1x_2}\tilde{u}_2^{(1)} = \frac{2\kappa_0(\lambda + \mu)}{\mu}\frac{x_2}{\delta^2(x_1)} + \mathcal{R}_{21}^{12}, \quad \partial_{x_2x_2}\tilde{u}_2^{(1)} = \frac{2\kappa_0(\lambda + \mu)}{\mu}\frac{x_1}{\delta^2(x_1)}, \tag{2.19}$$

$$\partial_{x_2x_2}\tilde{u}_2^{(2)} = \frac{-2\lambda\kappa_0}{\mu}\frac{x_2}{\delta^2(x_1)}, \tag{2.20}$$

where

$$\mathcal{R}_{22}^{11} = 8\kappa_0^2\frac{x_1^2x_2}{\delta^3(x_1)}, \quad \mathcal{R}_{21}^{12} = \frac{-8\kappa_0^2(\lambda + \mu)}{\mu}\frac{x_1^2x_2}{\delta^3(x_1)}. \tag{2.21}$$

It is clear that

$$\left|\mathcal{R}_{22}^{11}\right|, \left|\mathcal{R}_{21}^{12}\right| \leq \frac{C}{\delta(x_1)}, \qquad \left|\partial_{x_1x_1}\tilde{u}_2^{(2)}\right| \leq C, \tag{2.22}$$

$$\left|\partial_{x_1x_1}\tilde{u}_2^{(1)}\right|, \left|\partial_{x_1x_2}\tilde{u}_2^{(2)}\right| \leq \frac{C|x_1|}{\delta(x_1)}. \tag{2.23}$$

Recalling $u_2 = \bar{u}_2 + \tilde{u}_2$ and (2.18), (2.19) and (2.20), we have

$$(\lambda+\mu)\partial_{x_1x_2}\bar{u}_2^{(2)} + \mu\partial_{x_2x_2}\tilde{u}_2^{(1)} = 0.$$

Thus

$$\begin{aligned}(\mathcal{L}_{\lambda,\mu}u_2)^{(1)} &= \mu\triangle u_2^{(1)} + (\lambda+\mu)\Big(\partial_{x_1x_1}u_2^{(1)} + \partial_{x_1x_2}u_2^{(2)}\Big)\\ &= (\lambda+2\mu)\partial_{x_1x_1}\tilde{u}_2^{(1)} + (\lambda+\mu)\partial_{x_1x_2}\tilde{u}_2^{(2)}.\end{aligned} \tag{2.24}$$

This, combining with (2.23), yields

$$\left|(\mathcal{L}_{\lambda,\mu}u_2)^{(1)}\right| \leq \frac{C|x_1|}{\delta(x_1)}. \tag{2.25}$$

By the same way, using (2.18), (2.19) and (2.20), we have

$$(\lambda+2\mu)\partial_{x_2x_2}\tilde{u}_2^{(2)} + \mu(\partial_{x_1x_1}\bar{u}_2^{(2)} - \mathcal{R}_{22}^{11}) + (\lambda+\mu)(\partial_{x_1x_2}\tilde{u}_2^{(1)} - \mathcal{R}_{21}^{12}) = 0.$$

So

$$\begin{aligned}(\mathcal{L}_{\lambda,\mu}u_2)^{(2)} &= \mu\triangle u_2^{(2)} + (\lambda+\mu)\Big(\partial_{x_2x_2}u_2^{(1)} + \partial_{x_2x_1}u_2^{(2)}\Big)\\ &= \mu\partial_{x_1x_1}\tilde{u}_2^{(2)} + \mu\mathcal{R}_{22}^{11} + (\lambda+\mu)\mathcal{R}_{21}^{12},\end{aligned} \tag{2.26}$$

combining with (2.22), yields

$$\left|(\mathcal{L}_{\lambda,\mu}u_2)^{(2)}\right| \leq C(\frac{1}{\delta(x_1)} + 1). \tag{2.27}$$

Therefore, we have

$$\left|\mathcal{L}_{\lambda,\mu}u_2\right| \leq \left|(\mathcal{L}_{\lambda,\mu}u_2)^{(1)}\right| + \left|(\mathcal{L}_{\lambda,\mu}u_2)^{(2)}\right| \leq \frac{C}{\delta(x_1)}. \tag{2.28}$$

We remark that estimates (2.15) and (2.28) will improve the gradient estimates obtained in [4].

2.2 Proof of Proposition 1.3

For $i = 1, 2$, let $w_i := v_i - u_i$. Thus w_i is the solution to the following problem

$$\begin{cases}\mathcal{L}_{\lambda,\mu}w_i = -\mathcal{L}_{\lambda,\mu}u_i, & \text{in } Y',\\ w_i = 0, & \text{on } \Gamma_+,\\ w_i = 0, & \text{on } \Gamma_-,\\ \partial_\nu w_i = 0, & \text{on } x_1 = \pm L_1.\end{cases} \tag{2.29}$$

In order to prove Proposition 1.3, we only need to prove the order of $|\nabla w_i|$ is $O(1)$. The following two Lemmas are needed. The first one is to show that the global energy of w_i is bounded.

Lemma 2.1 *For $i=1,2$, the energy of w_i on Y' is bounded by C, that is,*

$$\int_{Y'} |\nabla w_i|^2 dx \le C. \tag{2.30}$$

Proof For the case $i=1$.

From (3.25) of [4], there exists $r_0 \in (r/4, r/3)$ such that

$$\int_{\substack{|x_1|=r_0,\\ -\varepsilon/2+h_2(x_1)<x_2<\varepsilon/2+h_1(x_1)}} |w_1|\, dx_2 \le C\left(\int_{Y'} |\nabla w_1|^2\, dx\right)^{1/2}, \tag{2.31}$$

and by (3.26) in [4], we have

$$\begin{aligned}\int_{Y'} |\nabla w_1|^2\, dx \le C\left(\left|\int_{\Omega_{r_0}} w_1^{(1)} \left(\mathcal{L}_{\lambda,\mu} u_1\right)^{(1)}\, dx\right| + \left|\int_{\Omega_{r_0}} w_1^{(2)} \left(\mathcal{L}_{\lambda,\mu} u_1\right)^{(2)}\, dx\right|\right)\\ + C\left(\int_{Y'\setminus\Omega_{r_0}} |\nabla w_1|^2\, dx\right)^{1/2}.\end{aligned} \tag{2.32}$$

For the first term in the right hand side of (2.32), to use integration by parts for (2.11), recalling (2.8) we introduce two functions

$$\mathcal{T}_{11}^{11} = 4\kappa_0^2 \frac{x_1^2 x_2^2}{\delta^3(x_1)}, \quad \mathcal{T}_{12}^{12} = \frac{-4\kappa_0^2(\lambda+\mu)}{\lambda+2\mu}\frac{x_1^2 x_2^2}{\delta^3(x_1)}, \tag{2.33}$$

such that

$$\partial_{x_2}\mathcal{T}_{11}^{11} = \mathcal{R}_{11}^{11}, \qquad \partial_{x_2}\mathcal{T}_{12}^{12} = \mathcal{R}_{12}^{12}.$$

Notice that

$$\left|\mathcal{T}_{11}^{11}\right| \le C, \quad \left|\mathcal{T}_{12}^{12}\right| \le C. \tag{2.34}$$

Thus,

$$\begin{aligned}&\left|\int_{\Omega_{r_0}} w_1^{(1)}\Big((\lambda+2\mu)\mathcal{R}_{11}^{11} + (\lambda+\mu)\mathcal{R}_{12}^{12}\Big)\, dx\right|\\ &= \left|\int_{\Omega_{r_0}} w_1^{(1)}\partial_{x_2}\Big((\lambda+2\mu)\mathcal{T}_{11}^{11} + (\lambda+\mu)\mathcal{T}_{12}^{12}\Big)\, dx\right|\\ &= \left|-\int_{\Omega_{r_0}} \partial_{x_2} w_1^{(1)}\Big((\lambda+2\mu)\mathcal{T}_{11}^{11} + (\lambda+\mu)\mathcal{T}_{12}^{12}\Big)\, dx\right|\\ &\le C\left(\int_{Y'} |\nabla w_1|^2\, dx\right)^{1/2}.\end{aligned} \tag{2.35}$$

By (2.4), we have

$$\int_{\Omega_{r_0}} \left|\partial_{x_1}\tilde{u}_1^{(1)}\right|^2\, dx \le C. \tag{2.36}$$

Combining with (2.31), we obtain

$$\begin{aligned}
&\left|\int_{\Omega_{r_0}} w_1^{(1)}\partial_{x_1x_1}\tilde{u}_1^{(1)}\,dx\right|\\
&\quad\le\left|-\int_{\Omega_{r_0}}\partial_{x_1}w_1^{(1)}\partial_{x_1}\tilde{u}_1^{(1)}\,dx\right|+\left|\int_{\substack{|x_1|=r_0,\\ -\varepsilon/2+h_2(x_1)<x_2<\varepsilon/2+h_1(x_1)}} w_1^{(1)}\partial_{x_1}\tilde{u}_1^{(1)}\,dx_2\right|\\
&\quad\le C\left(\int_{Y'}|\nabla w_1|^2\,dx\right)^{1/2}.
\end{aligned}\tag{2.37}$$

Thus, recalling (2.11), we have

$$\left|\int_{\Omega_{r_0}} w_1^{(1)}\left(\mathcal{L}_{\lambda,\mu}u_1\right)^{(1)}\,dx\right|\le C\left(\int_{Y'}|\nabla w_1|^2\,dx\right)^{1/2}.\tag{2.38}$$

By using (2.4),

$$\int_{\Omega_{r_0}}\left|\partial_{x_1}\tilde{u}_1^{(2)}\right|^2\,dx\le C.\tag{2.39}$$

Similar to (2.37), combining with (2.31), we obtain

$$\left|\int_{\Omega_{r_0}} w_1^{(2)}\left(\partial_{x_1x_1}\tilde{u}_1^{(2)}\right)dx\right|\le C\left(\int_{Y'}|\nabla w_1|^2\,dx\right)^{1/2}.\tag{2.40}$$

In view of (2.36), we have

$$\begin{aligned}
\left|\int_{\Omega_{r_0}} w_1^{(1)}\left(\partial_{x_2x_1}\tilde{u}_1^{(1)}\right)dx\right|&=\left|-\int_{\Omega_{r_0}}\partial_{x_2}w_1^{(2)}\partial_{x_1}\tilde{u}_1^{(1)}\,dx\right|\\
&\le\left(\int_{\Omega_{r_0}}\left|\partial_{x_1}\tilde{u}_1^{(1)}\right|^2\,dx\right)^{1/2}\left(\int_{Y'}|\nabla w_1|^2\,dx\right)^{1/2}\\
&\le C\left(\int_{Y'}|\nabla w_1|^2\,dx\right)^{1/2}.
\end{aligned}\tag{2.41}$$

Thus, recalling (2.13),

$$\left|\int_{\Omega_{r_0}} w_1^{(2)}\left(\mathcal{L}_{\lambda,\mu}u_1\right)^{(2)}\,dx\right|\le C\left(\int_{Y'}|\nabla w_1|^2\,dx\right)^{1/2}.\tag{2.42}$$

By (2.32), (2.38) and (2.42), we have

$$\int_{Y'}|\nabla w_1|^2\,dx\le C\left(\int_{Y'}|\nabla w_1|^2\right)^{1/2}.\tag{2.43}$$

This implies (2.30) hold.

For the case $i=2$. Instead of (2.33), we can use

$$\mathcal{T}_{22}^{11}=4\kappa_0^2\frac{x_1^2x_2^2}{\delta^3(x_1)},\quad \mathcal{T}_{21}^{12}=\frac{-4\kappa_0^2(\lambda+\mu)}{\mu}\frac{x_1^2x_2^2}{\delta^3(x_1)},\tag{2.44}$$

such that

$$\partial_{x_2}\mathcal{T}_{22}^{11}=\mathcal{R}_{22}^{11},\qquad \partial_{x_2}\mathcal{T}_{21}^{12}=\mathcal{R}_{21}^{12},$$

to obtain (2.30), by the same way as in case $i=1$. □

For $|z_1|\le r/4,\ s<r/4$, set

$$\widehat{\Omega}_s(z_1):=\{(x_1,x_2)\,|-\frac{\varepsilon}{2}+h_2(x_1)<x_2<\frac{\varepsilon}{2}+h_1(x_1),\ |x_1-z_1|<s\}.$$

We now use the iteration technique developed in [4] to estimate the scaling of the local energy of w_i in a small region.

Lemma 2.2 *For a given $|z_1|\le r/4$, the integral of $|\nabla w_i|^2$ over a small region $\widehat{\Omega}_{\delta(z_1)}(z_1)$ satisfies the following estimate*

$$\int_{\widehat{\Omega}_{\delta(z_1)}(z_1)}|\nabla w_i|^2\,dx\le\begin{cases}C|z_1|^4, & \sqrt{\varepsilon}<|z_1|\le r/4,\\ C\varepsilon^2, & |z_1|\le\sqrt{\varepsilon}.\end{cases}\tag{2.45}$$

Proof The iteration scheme we use is similar in spirit to that in [4]. For $0<t<s<r/4$, let η be a smooth function satisfying $\eta(x_1)=1$ if $|x_1-z_1|<t,\ \eta(x_1)=0$ if $|x_1-z_1|>s,\ 0\le\eta(x_1)\le1$ if $t\le|x_1-z_1|\le s$, and $|\eta'(x_1)|\le\frac{2}{s-t}$. Multiplying the equation in (2.29) by $w\eta^2$ and integrating by parts leads to the following inequality, the same as in (3.30) in [4],

$$\int_{\widehat{\Omega}_t(z_1)}|\nabla w_i|^2\,dx\le\frac{C}{(s-t)^2}\int_{\widehat{\Omega}_s(z_1)}|w_i|^2\,dx+(s-t)^2\int_{\widehat{\Omega}_s(z_1)}|\mathcal{L}_{\lambda,\mu}u_i|^2\,dx.\tag{2.46}$$

For the case: $\sqrt{\varepsilon}<|z_1|\le r/4$.

Note that for $0<s<\frac{2|z_1|}{3}$, by (3.31) in [4] we have,

$$\int_{\widehat{\Omega}_s(z_1)}|w_i|^2\,dx\le C|z_1|^4\int_{\widehat{\Omega}_s(z_1)}|\nabla w_i|^2\,dx.$$

By (2.15) and (2.28), we have

$$\int_{\widehat{\Omega}_s(z_1)}|\mathcal{L}_{\lambda,\mu}u_i|^2\,dx\le\frac{Cs}{|z_1|^2},\quad 0<s<\frac{2|z_1|}{3},\tag{2.47}$$

which exactly is an improvement of (3.32) in [4]. Denote

$$F(t):=\int_{\widehat{\Omega}_t(z_1)}|\nabla w_i|^2\,dx.$$

It follows from (2.46) that

$$F(t)\le\left(\frac{C_0|z_1|^2}{s-t}\right)^2F(s)+C(s-t)^2\frac{s}{|z_1|^2},\quad\forall\,0<t<s<\frac{2|z_1|}{3},\tag{2.48}$$

where C_0 is also a universal constant.

Let $t_j=2C_0j|z_1|^2,\ j=1,2,\ldots$, then

$$\frac{C_0|z_1|^2}{t_{j+1}-t_j}=\frac{1}{2},$$

taking $s = t_{j+1}$ and $t = t_j$ in (2.48), we have

$$F(t_j) \le \frac{1}{4}F(t_{j+1}) + \frac{C(t_{j+1}-t_j)^2 t_{j+1}}{|z_1|^2} \le \frac{1}{4}F(t_{j+1}) + C(j+1)|z_1|^4,$$

After $k = \left[\frac{1}{4C_0|z_1|}\right]$ iterations, and using (2.30), we have

$$\begin{aligned} F(t_1) &\le \left(\frac{1}{4}\right)^k F(t_{k+1}) + C|z_1|^4 \sum_{l=1}^{k} \left(\frac{1}{4}\right)^{l-1} (l+1) \\ &\le C\left(\frac{1}{4}\right)^k + C|z_1|^4 \sum_{l=1}^{k} \left(\frac{1}{4}\right)^{l-1} (l+1) \\ &\le C|z_1|^4, \end{aligned}$$

for sufficiently small $|z_1|$.

For the case: $|z_1| \le \sqrt{\varepsilon}$.

Note that for $0 < t < s < \sqrt{\varepsilon}$, we still have (2.76). By (3.34) in [4], we have

$$\int_{\widehat{\Omega}_s(z_1)} |w_i|^2\, dx \le C\varepsilon^2 \int_{\widehat{\Omega}_s(z_1)} |\nabla w_i|^2\, dx.$$

Estimate (2.47) becomes

$$\int_{\widehat{\Omega}_s(z_1)} |\mathcal{L}_{\lambda,\mu} u_i|^2\, dx \le \frac{Cs}{\varepsilon}, \quad 0 < s < \sqrt{\varepsilon}.$$

Estimate (2.48) becomes, in view of (2.46),

$$F(t) \le \left(\frac{C_0\varepsilon}{s-t}\right)^2 F(s) + C(s-t)^2\frac{s}{\varepsilon}, \quad \forall\, 0 < t < s < \sqrt{\varepsilon}, \tag{2.49}$$

where C_0 is also a universal constant. Similarly, let $t_j = 2C_0 j\varepsilon,\ j = 1, 2, \ldots,$ then

$$\frac{C_0\varepsilon}{t_{j+1}-t_j} = \frac{1}{2}.$$

By (2.49) with $s = t_{j+1}$ and $t = t_j$, we have

$$F(t_j) \le \frac{1}{4}F(t_{j+1}) + C(j+1)\varepsilon^2,$$

then after $k = [\frac{1}{4C_0\sqrt{\varepsilon}}]$ iterations, we have

$$\begin{aligned} F(t_1) &\le \left(\frac{1}{4}\right)^k F(t_{k+1}) + C\varepsilon^2 \sum_{l=1}^{k} \left(\frac{1}{4}\right)^{l-1} (l+1) \\ &\le C\left(\frac{1}{4}\right)^k + C\varepsilon^2 \sum_{l=1}^{k} \left(\frac{1}{4}\right)^{l-1} (l+1) \\ &\le C\varepsilon^2, \end{aligned}$$

for sufficiently small ε. Then (2.45) is proved. □

Proof (Proof of Proposition 1.3) By the scaling argument, $W^{2,p}$ estimate bootstrap argument (see [1]), and Sobolev embedding theorems, it follows from (3.40) in [4] that

$$\|\nabla w_i\|_{L^\infty(\widehat{\Omega}_{\frac{\delta}{2}}(z_1))} \leq \frac{C}{\delta}\left(\|\nabla w_i\|_{L^2(\widehat{\Omega}_\delta(z_1))} + \delta^2\|\mathcal{L}_{\lambda,\mu}u_i\|_{L^\infty(\widehat{\Omega}_\delta(z_1))}\right), \tag{2.50}$$

where $\delta = \delta(z_1)$.

For the case: $\sqrt{\varepsilon} < |z_1| \leq r/4$.

By (2.45),

$$\int_{\widehat{\Omega}_\delta(z_1)} |\nabla w_i|^2\, dx \leq C|z_1|^4.$$

By (2.15) and (2.28), we have

$$\delta^2\left|\mathcal{L}_{\lambda,\mu}u_i\right| \leq |z_1|^4\frac{C}{|z_1|^2} \leq C|z_1|^2, \quad \text{in } \Omega_\delta(z_1).$$

We deduce from (2.50) that

$$|\nabla w_i(z_1,x_2)| \leq \frac{C|z_1|^2}{\delta} \leq C, \quad \forall -\frac{\varepsilon}{2} + h_2(z_1) < x_2 < \frac{\varepsilon}{2} + h_1(z_1).$$

For the case: $|z_1| \leq \sqrt{\varepsilon}$.

Using (2.45),

$$\int_{\widehat{\Omega}_\delta(z_1)} |\nabla w_i|^2\, dx \leq C\varepsilon^2.$$

By (2.15) and (2.28), we have

$$\delta^2\left|\mathcal{L}_{\lambda,\mu}u_i\right| \leq C\varepsilon, \quad \text{in } \widehat{\Omega}_\delta(z_1).$$

It follows that

$$|\nabla w_i(z_1,x_2)| \leq \frac{C\varepsilon}{\delta} \leq C, \quad \forall -\frac{\varepsilon}{2} + h_2(z_1) < x_2 < \frac{\varepsilon}{2} + h_1(z_1).$$

The proof of (1.18) is completed. □

2.3 Proof of Theorem 2.1

Notice that the components C_{ijkl} possess symmetry property:

$$C_{ijkl} = C_{klij} = C_{klji}, \quad i,j,k,l = 1,2.$$

For 2×2 matrices $A = (A_{ij}),\ B = (B_{ij})$, denote

$$(\mathbb{C}A)_{ij} = \sum_{k,l=1}^{2} C_{ijkl}A_{kl}, \quad \text{and} \quad (A,B) \equiv A : B = \sum_{i,j=1}^{2} A_{ij}B_{ij}.$$

Clearly,

$$(\mathbb{C}A, B) = (A, \mathbb{C}B). \tag{2.51}$$

Therefore, for $i = 1,2$ we have

$$(\mathbb{C}e(v_i), e(v_i)) = (\mathbb{C}\nabla v_i, \nabla v_i). \tag{2.52}$$

Proof (Proof of Theorem 2.1) Recalling $w_i := v_i - u_i$, and in view of (2.51) and (2.52), we divide $(\mathbb{C}e(v_i), e(v_i))$ into three parts,

$$\begin{aligned}(\mathbb{C}e(v_i), e(v_i)) &= (\mathbb{C}\nabla u_i, \nabla u_i) + 2(\mathbb{C}\nabla u_i, \nabla w_i) + (\mathbb{C}\nabla w_i, \nabla w_i)\\ &:= \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned}\tag{2.53}$$

For case $i = 1$. First, recalling (2.4) and Proposition 1.3, we have

$$\begin{aligned}\left|\int_{\Omega_{\frac{r}{2}}} \mathrm{II}\, dx\right| &\le C \int_{\Omega_{\frac{r}{2}}} |\nabla u_1||\nabla w_1|\, dx\\ &\le C \int_{\Omega_{\frac{r}{2}}} \left(\frac{1}{\delta(x_1)} + \frac{|x_1|}{\delta(x_1)} + 1\right) dx \le C,\end{aligned}\tag{2.54}$$

and

$$\left|\int_{\Omega_{\frac{r}{2}}} \mathrm{III}\, dx\right| \le C \int_{\Omega_{\frac{r}{2}}} |\nabla w_1|^2 dx \le C.\tag{2.55}$$

We further denote

$$\mathrm{I} = (\mathbb{C}\nabla u_1, \nabla u_1) = \mathrm{I}_{11} + \mathrm{I}_{12} + \mathrm{I}_{21} + \mathrm{I}_{22},\tag{2.56}$$

where

$$\begin{aligned}\mathrm{I}_{11} :&= \left((\lambda + 2\mu)\sum_{i=1}^{2} \partial_{x_i} u_1^{(1)} + \lambda \sum_{i=1}^{2} \partial_{x_i} u_1^{(2)}\right) \partial_{x_1} u_1^{(1)},\\ \mathrm{I}_{12} :&= \left(\mu \sum_{i,j=1}^{2} \partial_{x_i} u_1^{(j)}\right) \partial_{x_2} u_1^{(1)}, \qquad \mathrm{I}_{21} := \left(\mu \sum_{i,j=1}^{2} \partial_{x_i} u_1^{(j)}\right) \partial_{x_1} u_1^{(2)},\\ \mathrm{I}_{22} :&= \left(\lambda \sum_{i=1}^{2} \partial_{x_i} u_1^{(1)} + (\lambda + 2\mu)\sum_{i=1}^{2} \partial_{x_i} u_1^{(2)}\right) \partial_{x_2} u_1^{(2)}.\end{aligned}\tag{2.57}$$

By observation, we find that among all the terms of I, except three of them,

$$\begin{aligned}&\partial_{x_1}\bar{u}_1^{(1)}\partial_{x_2}\bar{u}_1^{(1)} = -2\kappa_0 \frac{x_1 x_2}{\delta^3(x_1)}, \quad \left|\partial_{x_2}\bar{u}_1^{(1)}\right|^2 = \frac{1}{\delta^2(x_1)},\\ &\partial_{x_2}\bar{u}_1^{(1)}\partial_{x_2}\tilde{u}_1^{(2)} = \frac{2\kappa_0(\lambda + \mu)}{\lambda + 2\mu}\frac{x_1 x_2}{\delta^3(x_1)},\end{aligned}\tag{2.58}$$

all the other terms can be controlled by $\frac{C}{\delta(x_1)}$, by using (2.4). Because

$$\int_{\Omega_{\frac{r}{2}}} \frac{1}{\delta(x_1)}\, dx \le C,\tag{2.59}$$

they all are good terms.

Thus, it is easy to see from (2.57) that

$$\begin{aligned}\left|\int_{\Omega_{\frac{r}{2}}} \mathrm{I}_{21}\, dx\right| &\le C \int_{\Omega_{\frac{r}{2}}} \left|\sum_{i=1}^{2}\left(\partial_{x_i}\bar{u}_1^{(1)} + \partial_{x_i}\tilde{u}_1^{(1)} + \partial_{x_i}\tilde{u}_1^{(2)}\right)\partial_{x_1}\tilde{u}_1^{(2)}\right| dx\\ &\le C \int_{\Omega_{\frac{r}{2}}} \frac{1}{\delta(x_1)} dx \le C.\end{aligned}\tag{2.60}$$

On the other hand, since $\frac{x_1x_2}{\delta^3(x_1)}$ is an odd function of x_2, it follows that

$$\int_{\Omega_{\frac{r}{2}}} \partial_{x_1}\bar{u}_1^{(1)}\partial_{x_2}\bar{u}_1^{(1)}\,dx = \int_{\Omega_{\frac{r}{2}}} \partial_{x_2}\bar{u}_1^{(1)}\partial_{x_2}\tilde{u}_1^{(2)}\,dx = \int_{\Omega_{\frac{r}{2}}} \frac{x_1x_2}{\delta^3(x_1)}\,dx = 0. \tag{2.61}$$

So we have

$$\int_{\Omega_{\frac{r}{2}}} \mathrm{I}_{11}\,dx = (\lambda+2\mu)\int_{\Omega_{\frac{r}{2}}} \partial_{x_2}\bar{u}_1^{(1)}\partial_{x_1}\bar{u}_1^{(1)}\,dx + O(1) = 0 + O(1), \tag{2.62}$$

and

$$\int_{\Omega_{\frac{r}{2}}} \mathrm{I}_{22}\,dx = \lambda\int_{\Omega_{\frac{r}{2}}} \partial_{x_2}\bar{u}_1^{(1)}\partial_{x_2}\tilde{u}_1^{(2)}\,dx + O(1) = 0 + O(1). \tag{2.63}$$

Now for I_{12}, we write it as

$$\begin{aligned}\mathrm{I}_{12} &= \mu\partial_{x_1}u_1^{(1)}\partial_{x_2}u_1^{(1)} + \mu\left|\partial_{x_2}u_1^{(1)}\right|^2 + \mu\partial_{x_1}u_1^{(2)}\partial_{x_2}u_1^{(1)} + \mu\partial_{x_2}u_1^{(2)}\partial_{x_2}u_1^{(1)}\\ &:= \mathrm{I}_{12}^1 + \mathrm{I}_{12}^2 + \mathrm{I}_{12}^3 + \mathrm{I}_{12}^4.\end{aligned} \tag{2.64}$$

By using (2.58) and (2.61), we have

$$\int_{\Omega_{\frac{r}{2}}} \mathrm{I}_{12}^1\,dx = \mu\int_{\Omega_{\frac{r}{2}}} \partial_{x_1}\bar{u}_1^{(1)}\partial_{x_2}\bar{u}_1^{(1)}\,dx + O(1) = 0 + O(1), \tag{2.65}$$

and

$$\int_{\Omega_{\frac{r}{2}}} \mathrm{I}_{12}^4\,dx = \mu\int_{\Omega_{\frac{r}{2}}} \partial_{x_2}\tilde{u}_1^{(2)}\partial_{x_2}\bar{u}_1^{(1)}\,dx + O(1) = 0 + O(1). \tag{2.66}$$

By (2.59),

$$\left|\int_{\Omega_{\frac{r}{2}}} \mathrm{I}_{12}^3 dx\right| = \left|\int_{\Omega_{\frac{r}{2}}} \mu\partial_{x_1}\tilde{u}_1^{(2)}(\partial_{x_2}\bar{u}_1^{(1)} + \partial_{x_2}\tilde{u}_1^{(1)})dx\right| \le C\int_{\Omega_{\frac{r}{2}}} \frac{1}{\delta(x_1)}dx \le C. \tag{2.67}$$

Recalling that $u_1 = \bar{u}_1 + \tilde{u}_1$, we have

$$\begin{aligned}\int_{\Omega_{\frac{r}{2}}} \mathrm{I}_{12}^2\,dx &= \mu\int_{\Omega_{\frac{r}{2}}} \left|\partial_{x_2}\bar{u}_1^{(1)}\right|^2 dx + O(1)\\ &= \mu\int_{|x_1|<\frac{r}{2}} \frac{1}{\delta(x_1)}\,dx_1 + O(1) = \frac{\mu\pi}{\sqrt{\kappa_0}}\frac{1}{\sqrt{\varepsilon}} + O(1).\end{aligned} \tag{2.68}$$

This, together with (2.64)-(2.67), yields

$$\int_{\Omega_{\frac{r}{2}}} \mathrm{I}_{12}\,dx = \int_{\Omega_{\frac{r}{2}}} (\mathrm{I}_{12}^1 + \mathrm{I}_{12}^2 + \mathrm{I}_{12}^3 + \mathrm{I}_{12}^4)\,dx = \frac{\mu\pi}{\sqrt{\kappa_0}}\frac{1}{\sqrt{\varepsilon}} + O(1). \tag{2.69}$$

Thus, combining (2.60), (2.62) and (2.63), we obtain

$$\int_{\Omega_{\frac{r}{2}}} (\mathbb{C}\nabla u_1, \nabla u_1)dx = \int_{\Omega_{\frac{r}{2}}} \mathrm{I}_{12}\,dx + \int_{\Omega_{\frac{r}{2}}} (\mathrm{I}_{11} + \mathrm{I}_{21} + \mathrm{I}_{22})\,dx = \frac{\mu\pi}{\sqrt{\kappa_0}}\frac{1}{\sqrt{\varepsilon}} + O(1). \tag{2.70}$$

Therefore, substituting (2.54), (2.55) and (2.70) into (2.53), combining with (2.1), we have

$$\begin{aligned}\mathcal{E}_1 &= \int_{\Omega_{\frac{r}{2}}} (\mathbb{C}\nabla v_1, \nabla v_1)dx + \int_{Y'\setminus\Omega_{\frac{r}{2}}} (\mathbb{C}\nabla v_1, \nabla v_1)dx \\ &= \int_{\Omega_{\frac{r}{2}}} (\mathrm{I}+\mathrm{II}+\mathrm{III})\,dx + O(1) \\ &= \frac{\mu\pi}{\sqrt{\kappa_0}}\frac{1}{\sqrt{\varepsilon}} + O(1). \end{aligned} \tag{2.71}$$

For case $i = 2$. The process is the same. We only point out the differences. By (2.17) and Proposition 1.3, we have

$$\left|\int_{\Omega_{\frac{r}{2}}} (\mathrm{II}+\mathrm{III})\,dx\right| \leq \int_{\Omega_{\frac{r}{2}}} \left|2(\mathbb{C}\nabla u_2, \nabla w_2) + (\mathbb{C}\nabla w_2, \nabla w_2)\right| dx \leq C. \tag{2.72}$$

Let

$$\mathrm{I} := (\mathbb{C}\nabla u_2, \nabla u_2) = \mathrm{I}_{11} + \mathrm{I}_{12} + \mathrm{I}_{21} + \mathrm{I}_{22}, \tag{2.73}$$

where

$$\begin{aligned} \mathrm{I}_{11} &:= \left((\lambda+2\mu)\sum_{i=1}^{2}\partial_{x_i}u_2^{(1)} + \lambda\sum_{i=1}^{2}\partial_{x_i}u_2^{(2)}\right)\partial_{x_1}u_2^{(1)}, \\ \mathrm{I}_{12} &:= \left(\mu\sum_{i,j=1}^{2}\partial_{x_i}u_2^{(j)}\right)\partial_{x_2}u_2^{(1)}, \qquad \mathrm{I}_{21} := \left(\mu\sum_{i,j=1}^{2}\partial_{x_i}u_2^{(j)}\right)\partial_{x_1}u_2^{(2)}, \\ \mathrm{I}_{22} &:= \left(\lambda\sum_{i=1}^{2}\partial_{x_i}u_2^{(1)} + (\lambda+2\mu)\sum_{i=1}^{2}\partial_{x_i}u_2^{(2)}\right)\partial_{x_2}u_2^{(2)}. \end{aligned} \tag{2.74}$$

Similarly as before, among all the terms of I, except three of them

$$\begin{aligned} &\partial_{x_1}\bar{u}_2^{(2)}\partial_{x_2}\bar{u}_2^{(2)} = -2\kappa_0\frac{x_1x_2}{\delta^3(x_1)}, \quad \left|\partial_{x_2}\bar{u}_2^{(2)}\right|^2 = \frac{1}{\delta^2(x_1)}, \\ &\partial_{x_2}\bar{u}_2^{(2)}\partial_{x_2}\tilde{u}_2^{(1)} = \frac{2\kappa_0(\lambda+\mu)}{\mu}\frac{x_1x_2}{\delta^3(x_1)}, \end{aligned} \tag{2.75}$$

all the others can be controlled by $\frac{C}{\delta(x_1)}$, by using (2.17). In view of (2.61), it follows that

$$\int_{\Omega_{\frac{r}{2}}} \partial_{x_1}\bar{u}_2^{(2)}\partial_{x_2}\bar{u}_2^{(2)}\,dx = \int_{\Omega_{\frac{r}{2}}} \partial_{x_2}\bar{u}_2^{(2)}\partial_{x_2}\tilde{u}_2^{(1)}\,dx = \int_{\Omega_{\frac{r}{2}}} \frac{x_1x_2}{\delta^3(x_1)}\,dx = 0.$$

Then,

$$\begin{aligned} \int_{\Omega_{\frac{r}{2}}} \mathrm{I}\,dx &= \int_{\Omega_{\frac{r}{2}}} \mathrm{I}_{22}\,dx + O(1) = (\lambda+2\mu)\int_{\Omega_{\frac{r}{2}}} \left|\partial_{x_2}\bar{u}_2^{(2)}\right|^2 dx + O(1) \\ &= \mu\int_{\Omega_{\frac{r}{2}}} \frac{1}{\delta^2(x_1)}\,dx + O(1) = \frac{(\lambda+2\mu)\pi}{\sqrt{\kappa_0}}\frac{1}{\sqrt{\varepsilon}} + O(1). \end{aligned} \tag{2.76}$$

By (2.1), (2.53), (2.72) and (2.76) we have

$$\begin{aligned}\mathcal{E}_2 &= \int_{\Omega_{\frac{r}{2}}} (\mathbb{C}\nabla v_2, \nabla v_2)dx + \int_{Y'\setminus\Omega_{\frac{r}{2}}} (\mathbb{C}\nabla v_2, \nabla v_2)dx \\ &= \int_{\Omega_{\frac{r}{2}}} \mathrm{I}\, dx + O(1) = \frac{(\lambda+2\mu)\pi}{\sqrt{\kappa_0}} \frac{1}{\sqrt{\varepsilon}} + O(1). \end{aligned} \tag{2.77}$$

The proof of Theorem 2.1 is completed. □

3 Proof of Theorem 1.2

In this section we consider the m-convex inclusion, which is the curvilinear square with rounded-off angles, namely $|x_1|^m + |x_2|^m \le r^m$. Instead of Theorem 2.1, we have

Theorem 3.1 *Let $m > 2$, then energy integral $\mathcal{E}_i$ as in (1.13) has the following expansions,*

$$\mathcal{E}_1 = 2\mu \frac{\pi}{m \sin\frac{\pi}{m}} \frac{1}{\kappa_0^{\frac{1}{m}}} \frac{1}{\varepsilon^{1-\frac{1}{m}}} + O(1),$$

and

$$\mathcal{E}_2 = 2(\lambda+2\mu) \frac{\pi}{m \sin\frac{\pi}{m}} \frac{1}{\kappa_0^{\frac{1}{m}}} \frac{1}{\varepsilon^{1-\frac{1}{m}}} + O(1),$$

where $\kappa_0 = 2r^{1-m}/m$, as $\varepsilon \to 0$.

Thus, Theorem 1.2 is an immediate consequence. In the following we give some elementary estimates of u_i, for $m > 2$.

In this case, for $i = 1$, the auxiliary function can be constructed in $\Omega_{\frac{r}{2}}$ as defined in (1.14), by a modification of (1.15)

$$u_1 = \bar{u}_1 + \tilde{u}_1 := \frac{2x_2 + \delta(x_1)}{2\delta(x_1)} \begin{pmatrix}1\\0\end{pmatrix} + \left(\left(\frac{x_2}{\delta(x_1)}\right)^2 - \frac{1}{4}\right) \begin{pmatrix} (2 - \frac{\mu}{\lambda+2\mu})\frac{\kappa_0}{3}\frac{m(m-1)}{2} x_2 x_1^{m-2} \\ (1 - \frac{\mu}{\lambda+2\mu})\kappa_0 \frac{m}{2} x_1^{m-1} \end{pmatrix}$$

and still satisfies $u_1 = \Psi_1$ on Γ_+, $u_1 = 0$ on Γ_- and $\partial_\nu u_1 = 0$ on $x_1 = \pm L_1$. To simplify of calculation (Fig. 5), we still assume that

$$h_1(x_1) = \frac{\kappa_0}{2}|x_1|^m \quad \text{and} \quad h_2(x_1) = -\frac{\kappa_0}{2}|x_1|^m,$$

where $\kappa_0 = 2r^{1-m}/m$. Then

$$\delta(x_1) = \varepsilon + \kappa_0 |x_1|^m.$$

A direct calculation gives

$$\partial_{x_1}\bar{u}_1^{(1)} = -m\kappa_0 \frac{x_1^{m-1}x_2}{\delta^2(x_1)}, \quad \partial_{x_2}\bar{u}_1^{(1)} = \frac{1}{\delta(x_1)}, \qquad \partial_{x_2}\tilde{u}_1^{(2)} = \frac{m\kappa_0(\lambda+\mu)}{\lambda+2\mu}\frac{x_1^{m-1}x_2}{\delta^2(x_1)}, \tag{3.1}$$

and

$$|\partial_{x_1}\bar{u}_1^{(1)}|,\ |\partial_{x_2}\tilde{u}_1^{(2)}| \le \frac{C|x_1|^{m-1}}{\delta(x_1)}, \qquad |\partial_{x_1}\tilde{u}_1^{(1)}|,\ |\partial_{x_2}\tilde{u}_1^{(1)}|,\ |\partial_{x_1}\tilde{u}_1^{(2)}| \le C. \tag{3.2}$$

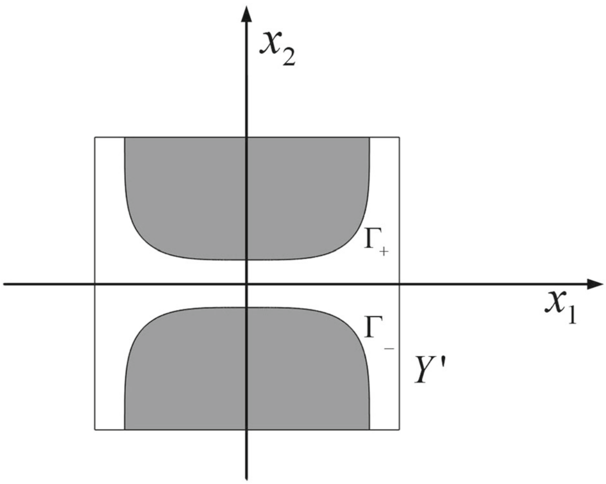

Fig. 5 m-covex inclusions, $m=4$

Further,

$$\partial_{x_1x_1}\bar{u}_1^{(1)} = -m\kappa_0(m-1)\frac{x_1^{m-2}x_2}{\delta^2(x_1)} + \mathcal{R}_{11}^{11}, \quad \partial_{x_1x_2}\bar{u}_1^{(1)} = -m\kappa_0\frac{x_1^{m-1}}{\delta^2(x_1)}, \tag{3.3}$$

$$\partial_{x_2x_2}\tilde{u}_1^{(1)} = \frac{m\kappa_0(m-1)(2\lambda+3\mu)}{\lambda+2\mu}\frac{x_1^{m-2}x_2}{\delta^2(x_1)}, \quad \partial_{x_2x_2}\tilde{u}_1^{(2)} = \frac{m\kappa_0(\lambda+\mu)}{\lambda+2\mu}\frac{x_1^{m-1}}{\delta^2(x_1)}, \tag{3.4}$$

$$\partial_{x_1x_2}\tilde{u}_1^{(2)} = \frac{m\kappa_0(m-1)(\lambda+\mu)}{\lambda+2\mu}\frac{x_1^{m-2}x_2}{\delta^2(x_1)} + \mathcal{R}_{12}^{12}, \tag{3.5}$$

where

$$\mathcal{R}_{11}^{11} = 2m^2\kappa_0^2\frac{x_1^{2m-2}x_2}{\delta^3(x_1)}, \quad \mathcal{R}_{12}^{12} = -\frac{2m^2\kappa_0^2(\lambda+\mu)}{\lambda+2\mu}\frac{x_1^{2m-2}x_2}{\delta^3(x_1)}. \tag{3.6}$$

From the above, we can see that (2.11) and (2.13) still hold. Because

$$|\mathcal{R}_{11}^{11}|,\ |\mathcal{R}_{12}^{12}| \le \frac{C|x_1|^{m-2}}{\delta(x_1)}, \quad |\partial_{x_1x_1}\tilde{u}_1^{(1)}| \le C, \tag{3.7}$$

$$|\partial_{x_1x_2}\tilde{u}_1^{(1)}|,\ |\partial_{x_1x_1}\tilde{u}_1^{(2)}| \le C|x_1|^{m-3}, \tag{3.8}$$

estimate (2.15) becomes

$$\left|\mathcal{L}_{\lambda,\mu}u_1\right| \le C\Big(\frac{|x_1|^{m-2}}{\delta(x_1)} + 1\Big). \tag{3.9}$$

For $i=2$, instead of (1.16), in $\Omega_{r/2}$ as defined in (1.14), u_2 takes the form

$$u_2 = \bar{u}_2 + \tilde{u}_2 := \frac{2x_2+\delta(x_1)}{2\delta(x_1)}\begin{pmatrix}0\\1\end{pmatrix} + \Big(\big(\frac{x_2}{\delta(x_1)}\big)^2 - \frac{1}{4}\Big)\begin{pmatrix}\frac{\lambda+\mu}{\mu}\kappa_0\frac{m}{2}x_1^{m-1}\\ -\frac{\lambda}{3\mu}\kappa_0\frac{m(m-1)}{2}x_2x_1^{m-2}\end{pmatrix}$$

and still satisfies the same boundary conditions as (1.16). A direct calculation gives

$$\partial_{x_1}\bar{u}_2^{(2)} = -m\kappa_0\frac{x_1^{m-1}x_2}{\delta^2(x_1)}, \quad \partial_{x_2}\bar{u}_2^{(2)} = \frac{1}{\delta(x_1)}, \quad \partial_{x_2}\tilde{u}_2^{(1)} = \frac{m\kappa_0(\lambda+\mu)}{\mu}\frac{x_1^{m-1}x_2}{\delta^2(x_1)}, \tag{3.10}$$

and

$$|\partial_{x_1}\bar{u}_2^{(2)}|,\ |\partial_{x_2}\tilde{u}_2^{(1)}| \le \frac{C|x_1|^{m-1}}{\delta(x_1)}, \qquad |\partial_{x_1}\tilde{u}_2^{(1)}|,\ |\partial_{x_1}\tilde{u}_2^{(2)}|,\ |\partial_{x_2}\tilde{u}_2^{(2)}| \le C. \tag{3.11}$$

Further,

$$\partial_{x_1x_1}\bar{u}_2^{(2)} = -m\kappa_0(m-1)\frac{x_1^{m-2}x_2}{\delta^2(x_1)} + \mathcal{R}_{22}^{11}, \quad \partial_{x_1x_2}\bar{u}_2^{(2)} = -m\kappa_0\frac{x_1^{m-1}}{\delta^2(x_1)}, \tag{3.12}$$

$$\partial_{x_2x_2}\tilde{u}_2^{(1)} = \frac{m\kappa_0(\lambda+\mu)}{\mu}\frac{x_1^{m-1}}{\delta^2(x_1)}, \qquad \partial_{x_2x_2}\tilde{u}_2^{(2)} = \frac{-\lambda m\kappa_0(m-1)}{\mu}\frac{x_1^{m-2}x_2}{\delta^2(x_1)}, \tag{3.13}$$

$$\partial_{x_1x_2}\tilde{u}_2^{(1)} = \frac{m\kappa_0(m-1)(\lambda+\mu)}{\mu}\frac{x_1^{m-2}x_2}{\delta^2(x_1)} + \mathcal{R}_{21}^{12}, \tag{3.14}$$

where

$$\mathcal{R}_{22}^{11} = 2m^2\kappa_0^2\frac{x_1^{2m-2}x_2}{\delta^3(x_1)}, \qquad \mathcal{R}_{21}^{12} = -\frac{2m^2\kappa_0^2(\lambda+\mu)}{\mu}\frac{x_1^{2m-2}x_2}{\delta^3(x_1)}.$$

From the above, we can see that (2.24) and (2.26) still hold. Since

$$|\mathcal{R}_{22}^{11}|,|\mathcal{R}_{21}^{12}| \le \frac{C|x_1|^{m-2}}{\delta(x_1)}, \quad |\partial_{x_1x_1}\tilde{u}_2^{(2)}| \le C, \tag{3.15}$$

$$|\partial_{x_1x_1}\tilde{u}_2^{(1)}|,\ |\partial_{x_1x_2}\tilde{u}_2^{(2)}| \le C|x_1|^{m-3}. \tag{3.16}$$

Instead (2.28), we have

$$|\mathcal{L}_{\lambda,\mu}u_2| \le C\Big(\frac{|x_1|^{m-2}}{\delta(x_1)} + 1\Big). \tag{3.17}$$

Recalling $w_i := v_i - u_i$, we still have Proposition 1.3 holds. Let us first show Lemma 2.1 in this case.

Proof For case $i = 1$. Instead of (2.33), we have

$$\mathcal{T}_{11}^{11} := m^2\kappa_0^2\frac{x_1^{2m-2}x_2^2}{\delta^3(x_1)}, \quad \mathcal{T}_{12}^{12} := -\frac{m^2\kappa_0^2(\lambda+\mu)}{\lambda+2\mu}\frac{x_1^{2m-2}x_2^2}{\delta^3(x_1)}.$$

By (3.2) we still have

$$\int_{\Omega_{r_0}} |\partial_{x_1}\tilde{u}_1^{(1)}|\,dx \le C \quad \text{and} \quad \int_{\Omega_{r_0}} |\partial_{x_1}\tilde{u}_1^{(2)}|\,dx \le C.$$

So, we obtain $\int_{Y'} |\nabla w_1|^2 dx \le C$.

For case $i = 2$. Instead of (2.44), we have

$$\mathcal{T}_{22}^{11} = m^2\kappa_0^2\frac{x_1^{2m-2}x_2^2}{\delta^3(x_1)}, \quad \mathcal{T}_{21}^{12} = -\frac{m^2\kappa_0^2(\lambda+\mu)}{\mu}\frac{x_1^{2m-2}x_2^2}{\delta^3(x_1)}$$

to obtain $\int_{Y'} |\nabla w_2|^2 dx \le C$, by the same way as in case $i = 1$. □

Next, we prove that Lemma 2.2 is also true.

Proof For $i = 1, 2$, by (3.9) and (3.17), we have

$$\int_{\widehat{\Omega}_s(z_1)} |\mathcal{L}_{\lambda,\mu} u_i|^2 \, dx \leq \begin{cases} Cs|z_1|^{m-4}, \ \varepsilon^{\frac{1}{m}} < |z_1| \leq \frac{r}{4} \ \text{ and } \ 0 < s < \frac{2|z_1|}{3}, \\ Cs\varepsilon^{1-\frac{4}{m}}, \quad |z_1| \leq \varepsilon^{\frac{1}{m}} \ \text{ and } \ 0 < s < \varepsilon^{\frac{1}{m}}. \end{cases}$$

With step-length

$$\begin{cases} 2C_0|z_1|^m, & \varepsilon^{\frac{1}{m}} < |z_1| \leq \frac{r}{4}, \\ 2C_0\varepsilon, & |z_1| \leq \varepsilon^{\frac{1}{m}}, \end{cases}$$

after

$$k = \begin{cases} \left[\frac{1}{4C_0|z_1|^{m-1}}\right], & \varepsilon^{\frac{1}{m}} < |z_1| \leq \frac{r}{4}, \\ \left[\frac{1}{4C_0\varepsilon^{\frac{m-1}{m}}}\right], & |z_1| \leq \varepsilon^{\frac{1}{m}} \end{cases}$$

iterations, as in Lemma 2.2, using (2.30), we have

$$\int_{\widehat{\Omega}_{\delta(z_1)}(z_1)} |\nabla w_i|^2 \, dx \leq \begin{cases} C|z_1|^{4m-4}, & \varepsilon^{\frac{1}{m}} < |z_1| \leq \frac{r}{4}, \\ C\varepsilon^{4-\frac{4}{m}}, & |z_1| \leq \varepsilon^{\frac{1}{m}}. \end{cases} \tag{3.18}$$

Then Lemma 2.2 is proved. □

Finally, we prove that Proposition 1.3 is true in this case.

Proof For $i = 1, 2$, by (3.9) and (3.17), we have

$$\delta^2 \left|\mathcal{L}_{\lambda,\mu} u_i\right| \leq \begin{cases} |z_1|^{2m-2}, & \varepsilon^{\frac{1}{m}} < |z_1| \leq \frac{r}{4}, \\ \varepsilon^{2-\frac{2}{m}}, & |z_1| \leq \varepsilon^{\frac{1}{m}}. \end{cases} \quad \text{in } \Omega_\delta(z_1),$$

combining with (3.18), we deduce from (2.50) that

$$|\nabla w_i(x_2, z_1)| \leq C, \quad \forall -\frac{\varepsilon}{2} + h_2(z_1) < x_2 < \frac{\varepsilon}{2} + h_1(z_1).$$

We show that Proposition 1.3 holds in this case. □

Proof (Proof of Theorem 3.1) For the reader's convenience, we only list the key differences. For case $i = 1$. By using (3.2) and Proposition 1.3, it follows that

$$\left|\int_{\Omega_{\frac{r}{2}}} \mathrm{II} \, dx\right| = \left|\int_{\Omega_{\frac{r}{2}}} 2(\mathbb{C}\nabla u_1, \nabla w_1) dx\right| \leq C \int_{\Omega_{\frac{r}{2}}} \frac{1}{\delta(x_1)} dx \leq C,$$

and (2.55) is still true with no change.

Realling (2.56) and (2.57) for the definition of term I, (2.58) becomes

$$\partial_{x_1}\bar{u}_1^{(1)}\partial_{x_2}\bar{u}_1^{(1)} = -m\kappa_0 \frac{x_1^{m-1}x_2}{\delta^3(x_1)}, \quad \left|\partial_{x_2}\bar{u}_1^{(1)}\right|^2 = \frac{1}{\delta^2(x_1)},$$

$$\partial_{x_2}\bar{u}_1^{(1)}\partial_{x_2}\tilde{u}_1^{(2)} = \frac{m\kappa_0(\lambda+\mu)}{(\lambda+2\mu)} \frac{x_1^{m-1}x_2}{\delta^3(x_1)}, \tag{3.19}$$

expect for these three terms above, all the other terms can be controlled by $\frac{C}{\delta(x_1)}$, by using (3.2). It is known that

$$\int_{\Omega_{\frac{r}{2}}} \frac{1}{\delta(x_1)}\,dx \le C. \tag{3.20}$$

So, estimate (2.60) still holds. Because $\frac{x_1^{m-1}x_2}{\delta^3(x_1)}$ is an odd function of x_2, it follows that

$$\int_{\Omega_{\frac{r}{2}}} \partial_{x_1}\bar{u}_1^{(1)}\partial_{x_2}\bar{u}_1^{(1)}\,dx = \int_{\Omega_{\frac{r}{2}}} \partial_{x_2}\bar{u}_1^{(1)}\partial_{x_2}\tilde{u}_1^{(2)}\,dx = \int_{\Omega_{\frac{r}{2}}} \frac{x_1^{m-1}x_2}{\delta^3(x_1)}\,dx = 0, \tag{3.21}$$

combining with (3.19), we obtain that (2.62) and (2.63) still hold.

Now divide

$$\mathrm{I}_{12} = \mathrm{I}_{12}^1 + \mathrm{I}_{12}^2 + \mathrm{I}_{12}^3 + \mathrm{I}_{12}^4,$$

as (2.64) in proof of Theorem 2.1. First, by (3.19), we note that (2.65) and (2.66) still hold. Then in view of (3.20), estimate (2.67) still holds. (2.68) becomes

$$\int_{\Omega_{\frac{r}{2}}} \mathrm{I}_{12}^2 = \mu \int_{|x_1|<\frac{r}{2}} \frac{1}{\delta(x_1)}\,dx_1 + O(1) = 2\mu\frac{\pi}{m\sin\frac{\pi}{m}}\frac{1}{\kappa_0^{\frac{1}{m}}}\frac{1}{\varepsilon^{1-\frac{1}{m}}} + O(1).$$

So,

$$\int_{\Omega_{\frac{r}{2}}} \mathrm{I}\,dx = \int_{\Omega_{\frac{r}{2}}} \mathrm{I}_{12}dx + O(1) = 2\mu\frac{\pi}{m\sin\frac{\pi}{m}}\frac{1}{\kappa_0^{\frac{1}{m}}}\frac{1}{\varepsilon^{1-\frac{1}{m}}} + O(1).$$

Therefore, instead of (2.71), we obtain

$$\mathcal{E}_1 = 2\mu\frac{\pi}{m\sin\frac{\pi}{m}}\frac{1}{\kappa_0^{\frac{1}{m}}}\frac{1}{\varepsilon^{1-\frac{1}{m}}} + O(1).$$

For case $i = 2$. By (3.11), we still have (2.72) in proof of Theorem 2.1. Recalling (2.73) and (2.74), we now calculate

$$\mathrm{I} = (\mathbb{C}\nabla u_2, \nabla u_2) = \mathrm{I}_{11} + \mathrm{I}_{12} + \mathrm{I}_{21} + \mathrm{I}_{22}.$$

In term I, except these three terms

$$\partial_{x_1}\bar{u}_2^{(2)}\partial_{x_2}\bar{u}_2^{(2)} = -m\kappa_0\frac{x_1^{m-1}x_2}{\delta^3(x_1)}, \quad \left|\partial_{x_2}\bar{u}_2^{(2)}\right|^2 = \frac{1}{\delta^2(x_1)},$$
$$\partial_{x_2}\bar{u}_2^{(2)}\partial_{x_2}\tilde{u}_2^{(1)} = \frac{m\kappa_0(\lambda+\mu)}{\mu}\frac{x_1^{m-1}x_2}{\delta^3(x_1)},$$

by (3.11) all the other terms can be controlled by $\frac{C}{\delta(x_1)}$. In view of (3.21), it follows that

$$\int_{\Omega_{\frac{r}{2}}} \partial_{x_1}\bar{u}_2^{(2)}\partial_{x_2}\bar{u}_2^{(2)}\,dx = \int_{\Omega_{\frac{r}{2}}} \partial_{x_2}\bar{u}_2^{(2)}\partial_{x_2}\tilde{u}_2^{(1)}\,dx = \int_{\Omega_{\frac{r}{2}}} \frac{x_1^{m-1}x_2}{\delta^3(x_1)}\,dx = 0.$$

Thus, (2.77) becomes

$$\mathcal{E}_2 = \int_{\Omega_{\frac{r}{2}}} \mathrm{I}_{22}\,dx + O(1) = \frac{2(\lambda+2\mu)\pi}{m\sin\frac{\pi}{m}}\frac{1}{\kappa_0^{\frac{1}{m}}}\frac{1}{\varepsilon^{1-\frac{1}{m}}} + O(1).$$

The proof of Theorem 3.1 is completed. □

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On a Fractional Nirenberg Problem Involving the Square Root of the Laplacian on $\mathbb{S}^3$

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Abstract
In this paper, we are devoted to establishing the compactness and existence results of the solutions to the fractional Nirenberg problem for $n = 3$, $\sigma = 1/2$, when the prescribing σ-curvature function satisfies the $(n - 2\sigma)$-flatness condition near its critical points. The compactness results are new and optimal. In addition, we obtain a degree-counting formula of all solutions. From our results, we can know where blow up occur. Moreover, for any finite distinct points, the sequence of solutions that blow up precisely at these points can be constructed. We extend the results of Li (Commun Pure Appl Math 49:541–597, 1996) from the local problem to nonlocal cases.

Keywords Fractional Laplacian · Nirenberg problem · Blow up analysis

Mathematics Subject Classification 35B38 · 35B44 · 35J20

1 Introduction

Great attention has been focused on the study of fractional and nonlocal operators of elliptic type, both for pure mathematical research and in view of concrete real-world applications. This type of operator arises in a quite natural way in many different contexts, such as, the thin obstacle problem, optimization, phase transitions, mini-

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mal surfaces, materials science, water waves, population dynamics, geophysical fluid dynamics, and mathematical finance. For more details and applications, see [3, 7, 8, 14, 18, 20, 21, 29, 35] and references therein.

In this paper, we are concerned with the Nirenberg's problem in the fractional setting which constitutes in itself a branch in geometric analysis. We first introduce the Nirenberg problem. Let $(\mathbb{S}^n, g_0)$ be the standard n-sphere. The Nirenberg problem is the following: which function K on $\mathbb{S}^2$ is the Gauss curvature of a metric g on $\mathbb{S}^2$ conformally equivalent to g_0? If we write $g = e^v g_0$, this problem is equivalent to finding a function v on $\mathbb{S}^2$ to solving

$$-\Delta_{g_0} v + 1 = K(x)e^{2v} \quad \text{on } \mathbb{S}^2, \tag{1.1}$$

where Δ_{g_0} denotes the Laplace–Beltrami operator associated with the metric g_0.

Naturally one may ask a similar question in higher dimensional case, namely which function K on $\mathbb{S}^n$ $(n \geq 3)$ is the scalar curvature of a metric g on $\mathbb{S}^n$ conformally equivalent to g_0? If we write $g = v^{4/(n-2)} g_0$, this problem is equivalent to finding a function v on $\mathbb{S}^n$ which satisfies the following equation:

$$-\Delta_{g_0} v + c(n) R_0 v = c(n) K(x) v^{\frac{n+2}{n-2}} \quad \text{on } \mathbb{S}^n, \tag{1.2}$$

where $c(n) = (n-2)/(4(n-1))$, $R_0 = n(n-1)$ is the scalar curvature of g_0.

It is well known that a necessary condition for solving (1.1) or (1.2) is that K should be positive somewhere. Kazdan and Warner [22] obtained another necessary condition for the existence of solutions by exploiting the center dilation conformal transformations of $\mathbb{S}^n$.

The first significant result on the Nirenberg problem was made by Koutroufiotis [23], which established the existence of the solutions to (1.1) by assuming that K is an antipodally symmetric function which close to 1. Morse [30] proved the existence of antipodally symmetric solutions to (1.1) for all antipodally symmetric functions K which are positive somewhere. Later on, Chang and Yang [11] further extended this existence result to the case of K without any symmetry assumption. In addition, Bahri and Coron [6] gave a sufficient condition for existence of the solutions to (1.2) in dimension $n = 3$ by assuming that $K(x)$ has only nondegenerate critical points. As for the compactness of all solutions in dimensions $n = 2, 3$, Chang-Gursky-Yang [10], Han [19], and Schoen and Zhang [34] proved that a sequence of solutions cannot blow up at more than one point.

Li [26, 27] established the compactness and existence results for (1.2) by characterizing the flatness order of $K(x)$ near its critical points with $(*)_\beta$ conditions. More precisely, the cases of $\beta > n-2$ and $\beta = n-2$ are given in [26] and [27], respectively. In these two papers, the compactness result is very different from the previous low-dimensional case. In fact, when $n = 2$ or $n = 3$, a sequence of solutions to the Nirenberg problem cannot blow up at more than one point. However, if $n > 3$, there could be blow up at many points, which considerably complicates the study of the problem.

The linear operators defined on left-hand side of (1.1) and (1.2) are called the conformal Laplacian associated to the metric g_0 and are denoted as $P_1^{g_0}$. For any Riemannian manifold (M, g), let R_g be the scalar curvature of (M, g), and the conformal Laplacian be defined as $P_1^g = -\Delta_g + \frac{n-2}{4(n-1)} R_g$. The Paneitz operator P_2^g is another conformal invariant operator, which was discovered by Paneitz [31]. Graham–Jenne–Mason–Sparling [16] generalized the operators P_1^g and P_2^g to a sequence of integer order conformally covariant elliptic operators P_k^g for $k \in \{1, 2, \ldots\}$ if n is odd, and $k \in \{1, \ldots, n/2\}$ if n is even. Furthermore, Peterson [32] constructed an intrinsically defined conformally covariant pseudo-differential operator of arbitrary real number order. Graham and Zworski [17] introduced a mesomorphic family of conformally invariant operators on the conformal infinity of asymptotically hyperbolic manifolds. Chang and González [9] proved that the operator P_σ^g of non-integer order $\sigma \in (0, n/2)$ can be defined as a Dirichlet-to-Neumann operator of a conformally compact Einstein manifold by using localization method in [8]. This lead naturally to a fractional order curvature $R_\sigma^g := P_\sigma^g(1)$, which will be called σ-curvature in this paper. The fractional operators P_σ^g and their associated fractional order curvatures $P_\sigma^g(1)$ have been the subject of many studies, for instance, see [1, 2, 12, 13, 20, 21, 28].

As in the Nirenberg problem associated to P_1^g, the question of prescribing σ-curvature can be formulate as fractional Nirenberg problem as follows: which function K on $\mathbb{S}^n$ is the σ-curvature of a metric g on $\mathbb{S}^n$ conformally equivalent to g_0? If we denote $g = v^{4/(n-2\sigma)} g_0$, this problem can be expressed as finding the solution of the following nonlinear equation with critical exponent:

$$P_\sigma^g(v) = c(n, \sigma) K v^{\frac{n+2\sigma}{n-2\sigma}} \quad \text{on } \mathbb{S}^n, \tag{1.3}$$

where $c(n, \sigma) = \Gamma(\frac{n}{2} + \sigma)/\Gamma(\frac{n}{2} - \sigma)$, K is a function defined on $\mathbb{S}^n$,

$$P_\sigma^g = \frac{\Gamma(B + \frac{1}{2} + \sigma)}{\Gamma(B + \frac{1}{2} - \sigma)}, \quad B = \sqrt{-\Delta_{g_0} + \left(\frac{n-1}{2}\right)^2},$$

and Γ is the Gamma function. In what follows, P_σ^g is simply written as P_σ.

Let $K \in C^{1,1}(\mathbb{S}^n)$ be a positive function and β is a positive constant, we say that K satisfies the flatness condition $(*)_\beta$ if for every critical point ξ_0 of K, in some geodesic normal coordinates $\{y_1, \ldots, y_n\}$ centered at ξ_0, there exists a small neighborhood $\mathcal{O}$ of 0 and $a_j(\xi_0) \neq 0$, $\sum_{j=1}^n a_j(\xi_0) \neq 0$, such that

$$K(y) = K(0) + \sum_{j=1}^n a_j(\xi_0)|y_j|^\beta + R(y) \quad \text{in } \mathcal{O},$$

where

$$\sum_{s=0}^{[\beta]} |\nabla^s R(y)||y|^{-\beta+s} \to 0 \quad \text{as } y \to 0,$$

here ∇^s denotes all possible derivatives of order s and $[\beta]$ is the integer part of β.

For $0 < \sigma < 1$, Jin-Li-Xiong [20, 21] proved the existence of the solutions to (1.3) and derived some compactness properties when K satisfies the $(*)_\beta$ condition with the flatness order $\beta \in (n - 2\sigma, n)$, by using the approach based on approximation of the solutions to (1.3) by a blow up subcritical method. Since their conclusions are valid only when the flatness order $\beta > n - 2\sigma$, some very interesting functions K are excluded. In fact, note that an important class of functions, which is worth including in the results of existence and compactness for (1.3), are the Morse functions with only nondegenerate critical points. Such functions satisfy the $(*)_2$ condition.

By using a self-contained approach, the description of lack of compactness and the existence results of the solutions to (1.3) were given by Abdelhedi-Chtioui-Hajaiej [2] when $\beta \in (1, n - 2\sigma]$, and by Chtioui and Abdelhedi [13] when $\beta \in [n - 2\sigma, n)$. However, under the assumption of the flatness order $\beta = n - 2\sigma$, which is called the critical flatness condition in this paper, the precise compactness result and the degree-counting formula of the solutions to (1.3) are unknown. Therefore, it is natural to study the compactness results when the prescribing curvature function K satisfies the critical flatness condition. When $\sigma = 1$ and K satisfy the critical flatness condition, namely $\beta = n - 2$, the compactness and existence results of the solutions to (1.2) was obtained by Li [27].

What we consider here is the case when the prescribing σ-curvature function satisfies the critical flatness order $\beta = n - 2\sigma = 2$, which include an important class of functions, for instance the Morse functions. In addition, we can establish the optimal compactness result and give a degree-counting formula of all solutions to (1.3) in this case. In this paper, we always consider Eq. (1.3) under assumption of

$$n = 3 \quad \text{and} \quad \sigma = 1/2.$$

From our results, we show that a sequence of solutions to (1.3) can blow up at more than one point and for any finite distinct points on $\mathbb{S}^3$, we can construct a sequence of solutions to (1.3) that blow up precisely at these points.

We now explain the reason why only the case of $n = 3$, $\sigma = 1/2$ is considered here under the critical flatness condition $\beta = n - 2\sigma = 2$. Because only when $\sigma \in (0, 1)$, the extension formula (see (1.5) below) can be used in the process of further characterizing the behavior of the blow up point. For the high-dimensional case, i.e., $n - 2\sigma = 2$, $\sigma = 1 + m/2$, $m \in \mathbb{N}_+$, we can establish the same compactness and existence results as in this paper by using the integral equation method, see [24] for more details.

Before state our results, we introduce some definitions and notations.

For $\sigma \in (0, 1)$, the fractional Laplacian is a nonlocal pseudo-differential operator, taking the form:

$$\begin{aligned}(-\Delta)^\sigma u(x) :&= C(n, \sigma) P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2\sigma}} \, dy \\ &= C(n, \sigma) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{n+2\sigma}} \, dy, \quad x \in \mathbb{R}^n,\end{aligned} \tag{1.4}$$

where $B_\varepsilon(x)$ is the ball centered at $x \in \mathbb{R}^n$ with radius ε. Here $P.V.$ is a commonly used abbreviation for "in the principal value sense" and $C(n,\sigma)$ is a dimensional constant that depends on n and σ, precisely given by

$$C(n,\sigma) := \Big(\int_{\mathbb{R}^n} \frac{1-\cos\zeta_1}{|\zeta|^{n+2\sigma}}\mathrm{d}\zeta\Big)^{-1}$$

with $\zeta = (\zeta_1, \zeta')$, $\zeta' \in \mathbb{R}^{n-1}$.

The singular integral given in (1.4) can be written as a weighted second-order differential quotient as follows (see [15, Lemma 3.2]):

$$(-\Delta)^\sigma u(x) := -\frac{1}{2}C(n,\sigma) int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2\sigma}}\,\mathrm{d}y, \quad x \in \mathbb{R}^n.$$

This operator is well defined in $\mathscr{S}$, the Schwartz space of rapidly decreasing C^∞ function in $\mathbb{R}^n$, and it can be equivalently defined in terms of the Fourier transform:

$$(-\Delta)^\sigma u(x) := \mathscr{F}^{-1}(|\xi|^{2\sigma}(\mathscr{F}u)(\xi))(x), \quad x \in \mathbb{R}^n.$$

where $\mathscr{F}$ denotes the Fourier transform operator.

Let $\dot{H}^\sigma(\mathbb{R}^n)$ denote the closure of the set $C_c^\infty(\mathbb{R}^n)$ of compactly supported smooth functions under the norm

$$\|u\|_{\dot{H}^\sigma(\mathbb{R}^n)} = \||\xi|^\sigma \mathscr{F}(u)(\xi)\|_{L^2(\mathbb{R}^n)}.$$

For any $u \in \dot{H}^\sigma(\mathbb{R}^n)$, we set

$$U(x,t) = \mathcal{P}_\sigma[u] := \int_{\mathbb{R}^n} \mathcal{P}_\sigma(x-\xi,t)u(\xi)\mathrm{d}\xi, \quad (x,t) \in \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0,\infty), \tag{1.5}$$

where

$$\mathcal{P}_\sigma(x,t) = \beta(n,\sigma)\frac{t^{2\sigma}}{(|x|^2+t^2)^{(n+2\sigma)/2}},$$

with a constant $\beta(n,\sigma)$ such that $\int_{\mathbb{R}^n} \mathcal{P}_\sigma(x,1)\mathrm{d}x = 1$. Let us denote that for any open set $D \subset \mathbb{R}^{n+1}_+$, the space $L^2(t^{1-2\sigma}, D)$ is the Banach space endowed with the norm

$$\|V\|_{L^2(t^{1-2\sigma},D)} := \Big(\int_D t^{1-2\sigma}V^2\,\mathrm{d}X\Big)^{1/2} < \infty,$$

for any $V \in L^2(t^{1-2\sigma}, D)$. Then the above $U(x,t) \in L^2(t^{1-2\sigma}, K)$ for any compact set K in $\overline{\mathbb{R}^{n+1}_+}$, $\nabla U(x,t) \in L^2(t^{1-2\sigma}, \mathbb{R}^{n+1}_+)$ and $U(x,t) \in C^\infty(\mathbb{R}^{n+1}_+)$.

By the celebrated work by Caffarelli and Silvestre (see [8]), one can find that $U(x,t)$ satisfies

$$\operatorname{div}(t^{1-2\sigma}\nabla U)=0 \quad \text{in } \mathbb{R}^{n+1}_+,$$

$$\|\nabla U\|_{L^2(t^{1-2\sigma},\mathbb{R}^{n+1}_+)}=N_\sigma\|u\|_{\dot{H}^\sigma(\mathbb{R}^n)},$$

and

$$-\lim_{t\to 0} t^{1-2\sigma}\partial_t U(x,t)=N_\sigma(-\Delta)^\sigma u(x) \quad \text{in } \mathbb{R}^n$$

in the distribution sense, where $N_\sigma=2^{1-2\sigma}\Gamma(1-\sigma)/\Gamma(\sigma)$. Here one refer $U(x,t)=\mathcal{P}_\sigma[u]$ in (1.5) as the extension of $u\in\dot{H}^\sigma(\mathbb{R}^n)$.

Let $\Omega\subset\mathbb{R}^n$ $(n\geq 3)$ be a domain, $\tau_i\geq 0$, $i=1,2,\ldots$, satisfy $\lim_{i\to\infty}\tau_i=0$, $p_i=(n+2\sigma)/(n-2\sigma)-\tau_i$, and $K_i\in C^{1,1}(\Omega)$ satisfy, for some constants A_1, $A_2>0$,

$$1/A_1\leq K_i(x)\leq A_1 \quad \text{for all } x\in\Omega, \quad \|K_i\|_{C^{1,1}(\Omega)}\leq A_2. \tag{1.6}$$

Let $u_i\in L^\infty(\Omega)\cap\dot{H}^\sigma(\mathbb{R}^n)$ with $u_i\geq 0$ in $\mathbb{R}^n$ satisfy

$$(-\Delta)^\sigma u_i=c(n,\sigma)K_i u_i^{p_i} \quad \text{in } \Omega, \tag{1.7}$$

where $c(n,\sigma)$ is as in (1.3).

Definition 1.1 Suppose that $\{K_i\}$ satisfies (1.6) and $\{u_i\}$ satisfies (1.7). A point $\overline{y}\in\Omega$ is called a blow up point of $\{u_i\}$ if there exists a sequence y_i tending to $\overline{y}$ such that $u_i(y_i)\to\infty$.

Definition 1.2 A blow up point $\overline{y}\in\Omega$ is called an isolated blow up point of $\{u_i\}$ if there exist $0<\overline{r}<\operatorname{dist}(\overline{y},\Omega)$, $\overline{C}>0$, and a sequence y_i tending to $\bar{y}$, such that y_i is a local maximum point of u_i, $u_i(y_i)\to\infty$ and

$$u_i(y)\leq\overline{C}|y-y_i|^{-2\sigma(p_i-1)} \quad \text{for all } \ y\in B_{\overline{r}}(y_i). \tag{1.8}$$

Let $y_i\to\overline{y}$ be an isolated blow up point of $\{u_i\}$, and define, for $r>0$,

$$\overline{u}_i(r):=\frac{1}{|\partial B_r(y_i)|}\int_{\partial B_r(y_i)}u_i \quad \text{and} \quad \overline{w}_i(r):=r^{2\sigma/(p_i-1)}\overline{u}_i(r). \tag{1.9}$$

Definition 1.3 A point $y_i\to\overline{y}\in\Omega$ is called an isolated simple blow up point if $y_i\to\overline{y}$ is an isolated blow up point such that for some $\rho>0$ (independent of i), $\overline{w}_i$ has precisely one critical point in $(0,\rho)$ for large i.

For $K \in C^2(\mathbb{S}^3)$, we introduce the following notation:

$$\begin{aligned}
\mathscr{K} &= \{q \in \mathbb{S}^3 : \nabla_{g_0} K(q) = 0\}, \\
\mathscr{K}^+ &= \{q \in \mathbb{S}^3 : \nabla_{g_0} K(q) = 0,\ \Delta_{g_0} K(q) > 0\}, \\
\mathscr{K}^- &= \{q \in \mathbb{S}^3 : \nabla_{g_0} K(q) = 0,\ \Delta_{g_0} K(q) < 0\}, \\
\mathscr{M}_K &= \{v \in C^2(\mathbb{S}^3) : v \text{ satisfies}(1.3)\}.
\end{aligned} \tag{1.10}$$

For any $k \in \mathbb{N}_+$ distinct points $q^{(1)}, \ldots, q^{(k)} \in \mathscr{K} \backslash \mathscr{K}^+$, the following $k \times k$ real symmetric matrix M is defined by, for $i, j = 1, \ldots, k$,

$$\begin{aligned}
M_{ii} &= -\frac{\Delta_{g_0} K(q^{(i)})}{K(q^{(i)})^3}, \\
M_{ij} &= -6 \frac{G_{q^{(i)}}(q^{(j)})}{K(q^{(i)}) K(q^{(j)})}, \quad i \neq j,
\end{aligned} \tag{1.11}$$

where

$$G_{q^{(i)}}(q^{(j)}) = \frac{1}{1 - \cos d(q^{(i)}, q^{(j)})} \tag{1.12}$$

is the Green's function of P_σ, $\sigma = 1/2$, on $\mathbb{S}^3$, and $d(\cdot, \cdot)$ denotes the geodesic distance. Let $\mu(M)$ denote the smallest eigenvalue of M, and when $k = 1$,

$$\mu(M) = M = -\frac{\Delta_{g_0} K(q^{(1)})}{K(q^{(1)})^3}.$$

Now we are going to present our first result about characterization of blow up behavior of the solutions, which is:

Theorem 1.1 *Let $K \in C^2\left(\mathbb{S}^3\right)$ be a positive function and $\mathscr{K}, \mathscr{K}^-, \mathscr{K}^+$ be as in* (1.10). *Let $\sigma = 1/2$, p_i satisfy $p_i \leq 2$, $p_i \to 2$, $\tau_i = 2 - p_i$, $K_i \in C^2(\mathbb{S}^3)$ satisfy $K_i \to K$ in $C^2(\mathbb{S}^3)$, and $v_i \in C^2(\mathbb{S}^3)$ satisfy*

$$P_\sigma(v_i) = K_i v_i^{p_i}, \quad v_i > 0 \quad \text{on } \mathbb{S}^3, \tag{1.13}$$

and

$$\lim_{i \to \infty} \max_{\mathbb{S}^3} v_i = \infty.$$

Then there exists a constant $\delta^* > 0$ depending only on $\min_{\mathbb{S}^3} K$ and $\|K\|_{C^2(\mathbb{S}^3)}$, such that after passing to a subsequence,

(i) *$\{v_i\}$ (still denote the subsequence by $\{v_i\}$) has only isolated simple blow up points $q^{(1)}, \ldots, q^{(k)} \in \mathscr{K} \backslash \mathscr{K}^+$ $(k \geq 1)$ with $|q^{(j)} - q^{(\ell)}| \geq \delta^*$, $\forall\, j \neq \ell$, and $\mu(M(q^{(1)}, \ldots, q^{(k)})) \geq 0$. Furthermore, $q^{(1)}, \ldots, q^{(k)} \in \mathscr{K}^-$ if $k \geq 2$.*

(ii) *Let $q^{(1)}, \ldots, q^{(k)}$ be as in (i), and $q_i^{(j)}$ be the local maximum of v_i with $q_i^{(j)} \to q^{(j)}$, we have*

$$\lambda_j := K(q^{(j)})^{-1} \lim_{i\to\infty} v_i(q_i^{(1)})(v_i(q_i^{(j)}))^{-1} \in (0,\infty), \tag{1.14}$$

$$\mu^{(j)} := \lim_{i\to\infty} \tau_i v_i(q_i^{(j)})^2 \in [0,\infty). \tag{1.15}$$

(iii) *Let $\lambda_j, \mu^{(j)}$, $j = 1, \ldots, k$ be as in (ii), then when $k = 1$,*

$$\mu^{(1)} = -4\frac{\Delta_{g_0} K(q^{(1)})}{K(q^{(1)})^3}, \tag{1.16}$$

when $k \geq 2$,

$$\sum_{\ell=1}^{k} M_{\ell j}(q^{(1)}, \ldots, q^{(k)})\lambda_\ell = \lambda_j \mu^{(j)}, \quad \forall\, j : 1 \leq j \leq k. \tag{1.17}$$

(iv) $\mu^{(j)} \in (0,\infty)$, $\forall\, j = 1, \ldots, k$, *if and only if* $\mu(M(q^{(1)}, \ldots, q^{(k)})) > 0$.

In what follows, we define

$$\begin{aligned} \mathscr{A} = \{K \in C^2(\mathbb{S}^3) : K > 0 \text{ on } \mathbb{S}^3,\ \Delta_{g_0} K \neq 0 \text{ on } \mathscr{K}, \\ \mu(M(q^{(1)}, \ldots, q^{(k)})) \neq 0,\ \forall\, q^{(1)}, \ldots, q^{(k)} \in \mathscr{K}^-,\ k \geq 2\}, \end{aligned} \tag{1.18}$$

and

$$C^2(\mathbb{S}^3)^+ := \{K \in C^2(\mathbb{S}^3) : K > 0 \text{ on } \mathbb{S}^3\}. \tag{1.19}$$

It is obvious that $\mathscr{A}$ is open in $C^2(\mathbb{S}^3)$ and $\mathscr{A}$ is dense in $C^2(\mathbb{S}^3)^+$ with respect to the C^2 norm.

We will introduce an integer-valued continuous function Index: $\mathscr{A} \to \mathbb{Z}$, which has an explicit formula for $K \in \mathscr{A}$ being a Morse function.

Definition 1.4 We define Index: $\mathscr{A} \to \mathbb{Z}$ by the following properties:

(i) For any Morse function $K \in \mathscr{A}$ with $\mathscr{K}^- = \{q^{(1)}, \ldots, q^{(s)}\}$, we define

$$\text{Index}(K) = -1 + \sum_{k=1}^{s} \sum_{\substack{\mu(M(q^{(i_1)}, \ldots, q^{(i_k)}))>0 \\ 1\leq i_1<\cdots<i_k\leq s}} (-1)^{k-1+\sum_{j=1}^{k} i(q^{(i_j)})},$$

where $i(q^{(i_j)})$ denotes the Morse index of K at $q^{(i_j)}$.

(ii) Index : $\mathscr{A} \to \mathbb{Z}$ is continuous with respect to the $C^2(\mathbb{S}^3)$ norm of $\mathscr{A}$ and hence is locally constant.

Remark 1.1 The existence and uniqueness of the Index mapping follows from Theorem 1.2 and the proof of Theorem 1.3 below.

Our second result is about the compactness of the solutions when $K \in \mathscr{A}$, which is:

Theorem 1.2 *Let $\sigma = 1/2$, $\mathscr{A}$ be as in (1.18) and $K \in \mathscr{A}$. Then there exists a constant $C = C(K) > 0$, such that for any $K_i \to K$ in $C^2(\mathbb{S}^3)$, and any $v_i \in \mathscr{M}_{K_i}$, we have*

$$1/C \le \liminf_{i\to\infty}(\min_{\mathbb{S}^3} v_i) \le \limsup_{i\to\infty}(\max_{\mathbb{S}^3} v_i) \le C. \tag{1.20}$$

Furthermore, for any $\alpha \in (0,1)$, there exists a constant $R = R(K,\alpha) > 0$, such that for any $v \in \mathscr{M}_K$, we have

$$1/R < v(x) < R, \quad \forall x \in \mathbb{S}^3 \quad \textit{and} \quad \|v\|_{C^{2,\alpha}(\mathbb{S}^3)} < R,$$

where $\mathscr{M}_{K_i}$ and $\mathscr{M}_K$ are as in (1.10).

For any given $0 < \alpha < 1$, $R > 0$, we define

$$\mathscr{O}_R := \{v \in C^{2,\alpha}(\mathbb{S}^3) : 1/R < v < R,\ \|v\|_{C^{2,\alpha}(\mathbb{S}^3)} < R\}. \tag{1.21}$$

Our third result is about degree-counting formula and the existence of the solutions to (1.3), which is:

Theorem 1.3 *Let $\sigma = 1/2$, $\mathscr{A}$ be as in (1.18), $K \in \mathscr{A}$ and $\mathrm{Index}(K)$ be as in Definition 1.4. Then for any $\alpha \in (0,1)$, there exists a constant $R_0 = R_0(K,\alpha)$, such that for all $R > R_0$, we have*

$$\deg_{C^{2,\alpha}}(v - P_\sigma^{-1}(Kv^2), \mathscr{O}_R, 0) = \mathrm{Index}(K), \tag{1.22}$$

where $\deg_{C^{2,\alpha}}$ denotes the Leray-Schauder degree in $C^{2,\alpha}(\mathbb{S}^3)$.

Furthermore, if $\mathrm{Index}(K) \neq 0$, then (1.3) has at least one solution.

Remark 1.2 It follows from Theorem 1.1 that when $K \in \mathscr{A}$, the solutions to (1.3) belong to $\mathscr{O}_R$ for some $R > 0$. We call the left-hand side of (1.22) the total degree of the solutions to the fractional equation. From Theorem 1.3, the total degree is $\mathrm{Index}(K)$.

For any finite subset $\mathcal{R} \subset \mathbb{S}^3$, we use $\sharp\mathcal{R}$ to denote the number of elements in the set $\mathcal{R}$. Let us now state a corollary of Theorem 1.3, which is:

Corollary 1.1 *Let $\sigma = 1/2$, $\mathscr{A}$ be as in (1.18) and $K \in \mathscr{A}$ be a Morse function satisfying $\sharp\mathscr{K}^- \le 1$ or for any distinct $P, Q \in \mathscr{K}^-$,*

$$\Delta_{g_0}K(P)\Delta_{g_0}K(Q) < 9K(P)K(Q). \tag{1.23}$$

Then for any $\alpha \in (0, 1)$, *there exists a constant* $C = C(K, \alpha) > 0$, *such that for all solutions* v *of* (1.3), *we have*

$$1/C < v(x) < C, \quad \forall x \in \mathbb{S}^3, \quad \|v\|_{C^{2,\alpha}(\mathbb{S}^3)} < C,$$

and for all $R \geq C$,

$$\deg_{C^{2,\alpha}}(v - P_\sigma^{-1}(Kv^2), \mathscr{O}_R, 0) = \mathrm{Index}(K) = -1 + \sum_{\substack{\nabla_{g_0} K(q_0)=0 \\ \Delta_{g_0} K(q_0)<0}} (-1)^{i(q_0)},$$

where $i(q_0)$ *denotes the Morse index of* K *at* q_0.

Furthermore, if

$$\sum_{\substack{\nabla_{g_0} K(q_0)=0 \\ \Delta_{g_0} K(q_0)<0}} (-1)^{i(q_0)} \neq 1,$$

then (1.3) *has at least one solution.*

Our fourth result is about the blow up behavior of the solutions when the σ-curvature function $K \notin \mathscr{A}$, which is:

Theorem 1.4 *Let* $\sigma = 1/2$, $\mathscr{A}$ *be as in* (1.18) *and* $C^2(\mathbb{S}^3)^+$ *be as in* (1.19). *Then for any* $K \in C^2(\mathbb{S}^3)^+ \backslash \mathscr{A} = \partial \mathscr{A}$, *there exists* $K_i \to K$ *in* $C^2(\mathbb{S}^3)$ *and* $v_i \in \mathscr{M}_{K_i}$, *such that*

$$\lim_{i\to\infty} (\max_{\mathbb{S}^3} v_i) = \infty, \quad \lim_{i\to\infty} (\min_{\mathbb{S}^3} v_i) = 0, \tag{1.24}$$

where $\mathscr{M}_{K_i}$ *is as in* (1.10).

From Remark 1.2, the total degree of solutions to (1.3) strongly depend on the sign of the smallest eigenvalue of $M(q^{(1)}, \ldots, q^{(k)})$, which plays a role in counting the total degree of solutions and in the compactness result. In fact, the points $q^{(1)}, \ldots, q^{(k)}$ for which $\mu(M(q^{(1)}, \ldots, q^{(k)}))$ is positive characterize the so-called asymptotic in the theory of critical points at infinity developed by Bahri [4, 6]. For instance, considering a continuous family of functions K_t $(0 \leq t \leq 1)$, the total degree changes when the smallest eigenvalue of $M_t(q^{(1)}, \ldots, q^{(k)})$ crosses zero while it remains unchanged when other eigenvalues cross zero.

It follows from Theorem 1.4 that when $K \notin \mathscr{A}$, the solutions to (1.3) may blow up. A natural question is where the blow up occur. The following results present the accurate location of the blow up.

For any $K \in C^2(\mathbb{S}^3)^+$, we first define

$$\begin{aligned}\mathscr{H}(K) = \Big\{&(q^{(1)}, \ldots, q^{(k)}) : k \geq 1,\ q^{(j)} \in \mathscr{K} \backslash \mathscr{K}^+,\ \forall j : 1 \leq j \leq k, \\ &q^{(j)} \neq q^{(\ell)},\ \forall j \neq \ell,\ \mu(M(q^{(1)}, \ldots, q^{(k)})) = 0\Big\}.\end{aligned} \tag{1.25}$$

Our fifth result is about the location of blowing up when $K \notin \mathscr{A}$, which is:

Theorem 1.5 *Let $\sigma = 1/2$, $\mathscr{A}$ be as in (1.18) and $C^2(\mathbb{S}^3)^+$ be as in (1.19). For a given function $K \in C^2(\mathbb{S}^3)^+ \backslash \mathscr{A}$, we have the following results:*

(i) *For any $K_i \to K$ in $C^2(\mathbb{S}^3)$, and $v_i \in \mathscr{M}_{K_i}$ with $\max_{\mathbb{S}^3} v_i \to \infty$, then for some $(q^{(1)}, \ldots, q^{(k)}) \in \mathscr{H}(K)$, $k \in \mathbb{N}_+$, $\{v_i\}$ (after passing to a subsequence) blows up at precisely those k points.*
(ii) *For any $(q^{(1)}, \ldots, q^{(k)}) \in \mathscr{H}(K)$, $k \in \mathbb{N}_+$, there exists $K_i \to K$ in $C^2(\mathbb{S}^3)$, $v_i \in \mathscr{M}_{K_i}$, such that $\{v_i\}$ blows up at precisely those k points.*

Corollary 1.2 *Let $\sigma = 1/2$. For any $k \in \mathbb{N}_+$ distinct points $q^{(1)}, \ldots, q^{(k)} \in \mathbb{S}^3$, there exists a sequence of Morse functions $\{K_i\} \subset \mathscr{A}$, such that for some $v_i \in \mathscr{M}_{K_i}$, $\{v_i\}$ blows up at precisely the k points.*

Using those compactness results (Theorems 1.2 through 1.5), we can completely characterize blow up of a sequence of solutions to (1.3) when $n = 3$, $\sigma = 1/2$. In particular, these results provide an optimal compactness characterization, that is, for any blow up solution to (1.3), we can know exactly where the blowing up occurs, on the other hand, for given any finite number of points, we can construct a sequence of solutions to (1.3) that blow up precisely at these points.

When further characterizing the blow up behavior of the solution to (1.3) (see Sect. 3 below), we mainly use the Pohozaev type identity (see Proposition 2.1 below) and its property (see Proposition 2.2 below) to judge the sign of the Laplace of the prescribing curvature function at the blow up point. Due to the limitation of the form of the Pohozaev type identity, the method in this paper is only effective for the case $n-2\sigma = 2$. In a forthcoming paper, we deal with the higher order case, i.e., $n = 2\sigma+2$ for any $1 < \sigma < n/2$.

The paper is organized as follows: In Sect. 2, we recall some known results on blow up analysis of the fractional Nirenberg problem obtained by Jin-Li-Xiong [20].

In Sect. 3, our main task is to prove Theorems 1.1 and 1.2. By using the method of subcritical approximation, we obtain Theorem 1.1, which further characterizes the blow up points for solutions to (1.3). More precisely, we consider the subcritical equation with $\tau > 0$ small:

$$P_\sigma v = K v^{2-\tau}, \quad v > 0 \quad \text{on } \mathbb{S}^3. \tag{1.26}$$

Then we use Theorem 1.1 and some results in [20] to prove Theorem 1.2.

Section 4 is devoted to proving the Theorems 1.3, 1.4, and 1.5. Firstly, we give the definition of $\Sigma_\tau = \Sigma_\tau(\overline{P}_1, \ldots, \overline{P}_k)$, for $\overline{P}_1, \ldots, \overline{P}_k \in \mathscr{K}^-$ with $\mu(M(\overline{P}_1, \ldots, \overline{P}_k)) > 0$. Then by using Theorem 1.1 and some results in [20], we obtain that for $\tau > 0$ very small, the solutions to (1.26) either stay bounded or stay in one of the Σ_τ (see Proposition 4.1 below). Furthermore, we obtain the H^σ topological degree of the solutions to (1.26) on Σ_τ (see Theorem 4.1 below). It follows from the above results that for all $0 < \tau < 2$, the H^σ total degree of the solutions to (1.26) is equal to -1 (see Proposition 4.6 below). Then we can conclude that H^σ topological degree of those solutions to (1.26) which remain bounded as τ tends to zero is equal to Index(K). Some well-known results in degree theory imply that the H^σ degree contribution above is equal to the $C^{2,\alpha}$ topological degree of those bounded solutions to (1.26). Thus, we prove Theorem 1.3. Furthermore, we complete the proof of Theorem 1.4 by using the degree-counting formula and perturbing the function K near its critical point. In the end, using Theorem 1.1 and the idea of the proof of Theorem 1.4, we prove Theorem 1.5.

In the Appendix, we provide some useful technical results and elementary estimates.

Finally, we make some conventions on notation. Let $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, +\infty)$. For $X = (x, t) \in \mathbb{R}^{n+1}$ and $R \geq 0$, the symbol $\mathcal{B}_R(X)$ denotes the balls in $\mathbb{R}^{n+1}$ with radius R and center X, and $\mathcal{B}^+_R(X) := \mathcal{B}_R(X) \cap \mathbb{R}^{n+1}_+$. The symbol $B_R(x)$ denotes the ball in $\mathbb{R}^n$ with radius R and center x. We also write $\mathcal{B}_R$, $\mathcal{B}^+_R$, B_R for $\mathcal{B}_R(0)$, $\mathcal{B}^+_R(0)$, $B_R(0)$, respectively. We always denote by C a positive constant which is independent of the main parameters, but it may vary from line to line.

2 Quick Review of Some Known Facts

In this section, we review some results about the blow up analysis of the fractional Nirenberg problem obtained in Jin-Li-Xiong [20].

Let $\sigma \in (0, 1)$, $u_i \in C^2(\Omega) \cap \dot{H}^\sigma(\mathbb{R}^n)$ with $u_i \geq 0$ in $\mathbb{R}^n$ satisfy (1.7) with K_i satisfying (1.6). Let $U_i(x, t)$ be the extension of u_i as in (1.5), we have

$$\begin{cases} \operatorname{div}\left(t^{1-2\sigma}\nabla U_i\right) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ -\lim_{t\to 0} t^{1-2\sigma}\partial_t U_i(x, t) = c_0 K_i(x) U_i(x, 0)^{p_i} & \text{for any } x \in \Omega, \end{cases} \tag{2.1}$$

where $c_0 = 2^{1-2\sigma} c(n, \sigma)\Gamma(1 - \sigma)/\Gamma(\sigma)$.

We say that $U \in H(|t|^{1-2\sigma}, D)$ if $U \in L^2(|t|^{1-2\sigma}, D)$, and its weak derivatives ∇U exist and belong to $L^2(|t|^{1-2\sigma}, D)$. The norm of U in $H(|t|^{1-2\sigma}, D)$ is given by

$$\|U\|_{H(|t|^{1-2\sigma},D)} := \left(\int_D |t|^{1-2\sigma} U^2 \, \mathrm{d}X + \int_D |t|^{1-2\sigma} |\nabla U|^2 \, \mathrm{d}X\right)^{1/2}.$$

In the following, for a domain $D \subset \mathbb{R}^{n+1}$ with boundary ∂D, we denote by $\partial' D$ the interior of $\overline{D} \cap \partial\mathbb{R}^{n+1}_+$ in $\mathbb{R}^n = \partial\mathbb{R}^{n+1}_+$, and we set $\partial'' D = \partial D \backslash \partial' D$.

Proposition 2.1 (Pohozaev type identity, [20, Proposition 4.7]) *Suppose that $K \in C^1(B_{2R})$. Let $U \in H(t^{1-2\sigma}, \mathcal{B}^+_{2R})$ with $U \geq 0$ in $\mathcal{B}^+_{2R}$ be a weak solution of*

$$\begin{cases} \operatorname{div}\left(t^{1-2\sigma}\nabla U\right) = 0 & \text{in } \mathcal{B}^+_{2R}, \\ -\lim_{t\to 0} t^{1-2\sigma}\partial_t U(x, t) = K(x) U^p(x, 0) & \text{on } \partial'\mathcal{B}^+_{2R}, \end{cases}$$

where $p > 0$. Then

$$\int_{\partial'\mathcal{B}^+_R} B'(X, U, \nabla U, R, \sigma) + \int_{\partial''\mathcal{B}^+_R} t^{1-2\sigma} B''(X, U, \nabla U, R, \sigma) = 0,$$

where

$$B'(X,U,\nabla U,R,\sigma)=\frac{n-2\sigma}{2}KU^{p+1}+\langle X,\nabla U\rangle KU^{p},$$
$$B''(X,U,\nabla U,R,\sigma)=\frac{n-2\sigma}{2}U\frac{\partial U}{\partial\nu}-\frac{R}{2}|\nabla U|^2+R\Big|\frac{\partial U}{\partial\nu}\Big|^2. \tag{2.2}$$

Proposition 2.2 *Let $\mathcal{M}\in\mathbb{R}$ and $\alpha(X)$ be some differentiable function near the origin with $\alpha(0)=0$. Then for $U(X)=|X|^{2\sigma-n}+\mathcal{M}+\alpha(X)$, we have*

$$\lim_{\delta\to 0}\int_{\partial''\mathcal{B}_\delta^+}t^{1-2\sigma}B''(X,U,\nabla U,\delta,\sigma)=-\frac{(n-2\sigma)^2}{2}\mathcal{M}|\mathbb{S}^{n-1}|\frac{\mathrm{B}(n/2,1-\sigma)}{2}, \tag{2.3}$$

where $\mathrm{B}(\cdot,\cdot)$ *is the Beta function.*

Proof Since $U(X)=|X|^{2\sigma-n}+\mathcal{M}+\alpha(X)$, we have

$$\nabla U(X)=(2\sigma-n)\delta^{2\sigma-n-2}X+\nabla\alpha(X)\quad\text{on }\partial''\mathcal{B}_\delta^+,$$

and

$$\frac{\partial U}{\partial\nu}=\nabla U\cdot\nu=(2\sigma-n)\delta^{2\sigma-n-1}+\frac{\nabla\alpha(X)\cdot X}{\delta}\quad\text{on }\partial''\mathcal{B}_\delta^+.$$

It follows that

$$|\nabla U|^2=(2\sigma-n)^2\delta^{4\sigma-2n-2}+2(2\sigma-n)\delta^{2\sigma-n-2}\nabla\alpha(X)\cdot X+|\nabla\alpha(X)|^2,$$

and

$$\Big|\frac{\partial U}{\partial\nu}\Big|^2=(2\sigma-n)^2\delta^{4\sigma-2n-2}+2(2\sigma-n)\delta^{2\sigma-n-2}\nabla\alpha(X)\cdot X+|\nabla\alpha(X)\cdot X|^2\delta^{-2}$$

on $\partial''\mathcal{B}_\delta^+$. Substituting the above results into (2.2), we can easily obtain

$$\begin{aligned}&\lim_{\delta\to 0}\int_{\partial''\mathcal{B}_\delta^+}t^{1-2\sigma}B''(X,U,\nabla U,\delta,\sigma)\\&=\lim_{\delta\to 0}-\frac{(2\sigma-n)^2}{2}\mathcal{M}\delta^{2\sigma-n-1}\int_{\partial''\mathcal{B}_\delta^+}t^{1-2\sigma}\\&=-\frac{(2\sigma-n)^2}{2}\mathcal{M}\int_{\partial''\mathcal{B}_1^+}s^{1-2\sigma}\\&=-\frac{(n-2\sigma)^2}{4}\mathcal{M}|\mathbb{S}^{n-1}|\mathrm{B}(n/2,1-\sigma)|.\end{aligned}$$

Proposition 2.2 follows from the above. ◻

Proposition 2.3 ([20, Lemma 4.10]) *Suppose that for all $\varepsilon \in (0,1)$, $U \in H(t^{1-2\sigma}, \mathcal{B}_1^+ \backslash \overline{\mathcal{B}_\varepsilon^+})$ with $U > 0$ in $\mathcal{B}_1^+ \backslash \overline{\mathcal{B}_\varepsilon^+}$ is a weak solution of*

$$\begin{cases} \operatorname{div}\left(t^{1-2\sigma}\nabla U\right) = 0 & \text{in } \mathcal{B}_1^+ \backslash \overline{\mathcal{B}_\varepsilon^+}, \\ -\lim_{t\to 0} t^{1-2\sigma}\partial_t U(x,t) = 0 & \text{on } B_1 \backslash \overline{B_\varepsilon^+}. \end{cases}$$

Then

$$U(X) = \mathcal{A}|X|^{2\sigma-n} + \mathcal{W}(X),$$

where $\mathcal{A}$ is a nonnegative constant and $\mathcal{W} \in H(t^{1-2\sigma}, \mathcal{B}_1^+)$ satisfies

$$\begin{cases} \operatorname{div}\left(t^{1-2\sigma}\nabla \mathcal{W}\right) = 0, & \text{in } \mathcal{B}_1^+, \\ -\lim_{t\to 0} t^{1-2\sigma}\partial_t \mathcal{W}(x,t) = 0 & \text{on } B_1. \end{cases}$$

Proposition 2.4 ([20, Lemma 4.3]) *Suppose that $u_i \in C^2(\Omega) \cap \dot{H}^\sigma(\mathbb{R}^n)$ with $u_i \geq 0$ in $\mathbb{R}^n$ satisfies (1.7) with K_i satisfying (1.6), and $y_i \to 0$ is an isolated blow up point of $\{u_i\}$, i.e., for some positive constants A_3 and $\overline{r}$ independent of i,*

$$|y - y_i|^{2\sigma/(p_i-1)} u_i(y) \leq A_3 \quad \text{for all } y \in B_{\overline{r}} \subset \Omega. \tag{2.4}$$

Denote $U_i = \mathcal{P}_\sigma[u_i]$ and $Y_i = (y_i, 0)$. Then for any $0 < r < \overline{r}/3$, we have the following Harnack inequality:

$$\sup_{\mathcal{B}_{2r}^+(Y_i)\backslash\overline{\mathcal{B}_{r/2}^+(Y_i)}} U_i \leq C \inf_{\mathcal{B}_{2r}^+(Y_i)\backslash\overline{\mathcal{B}_{r/2}^+(Y_i)}} U_i,$$

where C is a positive constant depending only on n, σ, A_3, $\overline{r}$, and $\sup_i \|K_i\|_{L^\infty(B_r(y_i))}$.

Proposition 2.5 ([20, Proposition 4.4]) *Under the hypotheses of Proposition 2.4, then for any $R_i \to \infty$ and $\varepsilon_i \to 0^+$, we have, after passing to a subsequence (still denoted as $\{u_i\}$, $\{y_i\}$, etc.),*

$$\|m_i^{-1} u_i(m_i^{-(p_i-1)/2\sigma} \cdot + y_i) - (1 + k_i|\cdot|^2)^{(2\sigma-n)/2}\|_{C^2(B_{2R_i}(0))} \leq \varepsilon_i,$$
$$R_i m_i^{-(p_i-1)/2\sigma} \to 0 \quad \text{as } i \to \infty,$$

where $m_i = u_i(y_i)$ and $k_i = K_i(y_i)^{1/\sigma}/4$.

Proposition 2.6 ([20, Lemma 4.8 and Proposition 4.9]) *Under the hypotheses of Proposition 2.4, and in addition that $y_i \to 0$ is also an isolated simple blow up point with constant ρ, i.e., $\overline{w}_i$ (see (1.9) for definition) has precisely one critical point in $(0, \rho)$ for large i. Then we have*

$$\tau_i = O(u_i(y_i)^{-2/(n-2\sigma)+o(1)}) \quad \text{and} \quad u_i(y_i)^{\tau_i} = 1 + o(1).$$

Moreover,

$$u_i(y) \le C u_i(y_i)^{-1}|y - y_i|^{2\sigma - n} \quad \textit{for all } |y - y_i| \le 1.$$

Proposition 2.7 ([20, Lemma 4.11]) *Under the hypotheses of Proposition 2.4, we have*

$$\int_{|y-y_i|\le r_i} |y - y_i|^s u_i(y)^{p_i+1} = \begin{cases} O(u_i(y_i)^{-2s/(n-2\sigma)}), & -n < s < n, \\ O(u_i(y_i)^{-2n/(n-2\sigma)} \log u_i(y_i)), & s = n, \\ o(u_i(y_i)^{-2n/(n-2\sigma)}), & s > n, \end{cases}$$

and

$$\int_{r_i<|y-y_i|\le 1} |y - y_i|^s u_i(y)^{p_i+1} = \begin{cases} o(u_i(y_i)^{-2s/(n-2\sigma)}), & -n < s < n, \\ O(u_i(y_i)^{-2n/(n-2\sigma)} \log u_i(y_i)), & s = n, \\ O(u_i(y_i)^{-2n/(n-2\sigma)}), & s > n. \end{cases}$$

3 Compactness of Solutions and Characterization of Blow Up Behavior

In this section, our main task is to prove Theorems 1.1 and 1.2. We first give the proof of Theorem 1.1, which further characterizes the blow up points for solutions to (1.3) and plays a key role in proving Theorem 1.2. Recall the definitions of the matrix M given in (1.11) and its smallest eigenvalue $\mu(M)$.

Proof *(Proof of Theorem 1.1)* From Jin-Li-Xiong [20, Theorem 5.3], there exists a constant $\delta^* > 0$ depending only on $\min_{\mathbb{S}^3} K$ and $\|K\|_{C^2(\mathbb{S}^3)}$ such that $\{v_i\}$ has only isolated simple blow up points $q^{(1)}, \dots, q^{(k)} \in \mathscr{K}$ $(k \ge 1)$ with $|q^{(j)} - q^{(\ell)}| \ge \delta^*$ $(j \ne \ell)$.

Under the stereographic projection F with $q^{(j)}$ being the south pole:

$$F : \mathbb{R}^3 \to \mathbb{S}^3 \backslash \{-q^{(j)}\}, \quad y \mapsto \left(\frac{2y}{1 + |y|^2}, \frac{|y|^2 - 1}{|y|^2 + 1} \right),$$

the Eq. (1.13) is transformed to

$$(-\Delta)^\sigma u_i(y) = \widetilde{K}_i(y) H_i(y)^{\tau_i} u_i(y)^{p_i}, \quad y \in \mathbb{R}^3, \tag{3.1}$$

where

$$u_i(y) = \left(\frac{2}{1 + |y|^2} \right) v_i(F(y)), \quad \widetilde{K}_i(y) = K_i(F(y)), \quad H_i(y) = \frac{2}{1 + |y|^2}. \tag{3.2}$$

Let $y^{(\ell)} := F^{-1}(q^{(\ell)})$, $\ell = 1, \dots, k$, and $y_i^{(\ell)} \to y^{(\ell)}$ be the local maximum point of u_i as in Definition 1.2. It is easy to see that $y^{(j)} = 0$ from the definition of F. Since

0 is an isolated simple blow up point of u_i. Let $U_i(Y)$, $Y := (y,t) \in \mathbb{R}^4_+$, be the extension of $u_i(y)$ and satisfy

$$\begin{cases} \operatorname{div}\left(t^{1-2\sigma}\nabla U_i\right) = 0 & \text{in } \mathbb{R}^4_+, \\ -\lim_{t\to 0} t^{1-2\sigma}\partial_t U_i(y,t) = \widetilde{K}_i(y)H_i(y)^{\tau_i}u_i(y)^{p_i} & \text{for } y \in \mathbb{R}^3. \end{cases} \tag{3.3}$$

Let $Y_i^{(j)} := (y_i^{(j)}, 0)$, then Propositions 2.4, 2.6, 2.3, and elliptic theory together imply that

$$U_i(Y_i^{(j)})U_i(Y) \to \mathcal{H}^{(j)}(Y) := \mathcal{A}^{(j)}|Y|^{-2} + \mathcal{W}^{(j)}(Y) \quad \text{in } C^2_{\mathrm{loc}}(\overline{\mathbb{R}^4_+}\backslash\{\cup_{\ell=1}^k Y^{(\ell)}\}), \tag{3.4}$$

where $\mathcal{A}^{(j)}$ is a nonnegative constant and $\mathcal{W}^{(j)}(Y)$ satisfies

$$\begin{cases} \operatorname{div}(t^{1-2\sigma}\nabla\mathcal{W}^{(j)}) = 0 & \text{in } \mathbb{R}^4_+, \\ -\lim_{t\to 0} t^{1-2\sigma}\partial_t\mathcal{W}^{(j)}(y,t) = 0 & \text{for } y \in \mathbb{R}^3\backslash\{\cup_{\ell\neq j} y^{(\ell)}\}. \end{cases} \tag{3.5}$$

It follows from the maximum principle and the Harnack inequality that

$$\mathcal{W}^{(j)}(Y) \equiv 0 \quad \text{if } k = 1, \quad \mathcal{W}^{(j)}(Y) > 0 \quad \text{if } k \geq 2. \tag{3.6}$$

Let's next calculate $\mathcal{A}^{(j)}$. Multiplying (3.3) by $U_i(Y_i^{(j)})$ and integrating by parts on $\mathcal{B}_1^+$ leads to

$$\begin{aligned} 0 &= \int_{\mathcal{B}_1^+} U_i(Y_i^{(j)})\operatorname{div}(\nabla U_i) \\ &= \int_{B_1} u_i(y_i^{(j)})\widetilde{K}_i(y)H_i(y)^{\tau_i}u_i(y)^{p_i} + \int_{\partial''\mathcal{B}_1^+}\frac{\partial}{\partial\nu}(U_i(Y_i^{(j)})U_i) =: \mathcal{I}_1 + \mathcal{I}_2. \end{aligned} \tag{3.7}$$

Let R_i be given in Proposition 2.5, and

$$m_{ij} := u_i(y_i^{(j)}), \quad r_i := R_i(m_{ij})^{-(p_i-1)}. \tag{3.8}$$

For $\mathcal{I}_1$, from Propositions 2.5 and 2.6 we have

$$\begin{aligned} \mathcal{I}_1 &= \int_{|y-y_i^{(j)}|\leq r_i} m_{ij}\widetilde{K}_i(y)u_i(y)^{p_i} + \int_{\{|y-y_i^{(j)}|>r_i\}\cap\{|y|<1\}} m_{ij}\widetilde{K}_i(y)H_i(y)^{\tau_i}u_i(y)^{p_i} \\ &\quad + O\Big(\tau_i\int_{|y-y_i^{(j)}|\leq r_i} m_{ij}\widetilde{K}_i(y)u_i(y)^{p_i}\Big) \\ &= m_{ij}^{2\tau_i}\int_{|x|\leq R_i}\big(\widetilde{K}_i(0)+O(|y|)\big)\big(m_{ij}^{-1}u_i(m_{ij}^{-(p_i-1)}x + y_i^{(j)})\big)^{p_i} \\ &= 2\pi|\mathbb{S}^2|K_i(q^{(j)})^{-2} + o(1). \end{aligned} \tag{3.9}$$

For $\mathcal{I}_2$, it follows from (3.4) and (3.5) that

$$\lim_{i\to\infty}\mathcal{I}_2=\int_{\partial''\mathcal{B}_1^+}\frac{\partial}{\partial\nu}(\mathcal{A}^{(j)}|Y|^{-2}+\mathcal{W}^{(j)}(Y))=\int_{\partial''\mathcal{B}_1^+}-2\mathcal{A}^{(j)}=-\frac{\pi|\mathbb{S}^2|}{2}\mathcal{A}^{(j)}. \tag{3.10}$$

By (3.7), (3.9), and (3.10), we conclude that $\mathcal{A}^{(j)}=4K(q^{(j)})^{-2}$.

From (3.4), we have

$$\begin{aligned}U_i(Y_i^{(j)})U_i(Y)&\to\mathcal{H}^{(j)}(Y)\\&:=4K(q^{(j)})^{-2}|Y|^{-2}+\mathcal{W}^{(j)}(Y)\quad\text{in }C^2_{\mathrm{loc}}(\overline{\mathbb{R}^4_+}\backslash\{\cup_{\ell=1}^kY^{(\ell)}\}),\end{aligned}$$

and

$$\begin{aligned}u_i(y_i^{(j)})u_i(y)\to&h^{(j)}(y)\\&:=4K(q^{(j)})^{-2}|y|^{-2}+W^{(j)}(y)\quad\text{in }C^2_{\mathrm{loc}}(\mathbb{R}^3\backslash\{\cup_{\ell=1}^ky^{(\ell)}\}),\end{aligned}\tag{3.11}$$

where $W^{(j)}(y):=\mathcal{W}^{(j)}(y,0)$.

By (3.2) and $y_i^{(j)}\to0$ as $i\to\infty$, we have

$$\lim_{i\to\infty}v_i(q_i^{(j)})v_i(q)=\frac{1}{4}\lim_{i\to\infty}(1+|y|^2)u_i(y_i^{(j)})u_i(y),$$

combining with (3.11), it easy to see that for $q\neq q^{(j)}$ and close to $q^{(j)}$,

$$\lim_{i\to\infty}v_i(q_i^{(j)})v_i(q)=2G_{q^{(j)}}(q)K(q^{(j)})^{-2}+\widetilde{W}^{(j)}(q)\quad\text{in }C^2_{\mathrm{loc}}(\mathbb{S}^3\backslash\{\cup_{\ell=1}^kq^{(\ell)}\}), \tag{3.12}$$

where $\widetilde{W}^{(j)}(q)$ is some regular function on $\mathbb{S}^3\backslash\cup_{\ell\neq j}\{q^{(\ell)}\}$ satisfying $P_\sigma\widetilde{W}^{(j)}=0$, and $G_{q^{(j)}}(q)$ is the Green function defined as in (1.12).

When $k\geq2$, taking into account the contribution of all the poles, we deduce

$$\lim_{i\to\infty}v_i(q_i^{(j)})v_i(q)=2\frac{G_{q^{(j)}}(q)}{K(q^{(j)})^2}+2\sum_{\ell\neq j}\lim_{i\to\infty}\frac{v_i(q_i^{(j)})}{v_i(q_i^{(\ell)})}\frac{G_{q^{(\ell)}}(q)}{K(q^{(\ell)})^2}\quad\text{in }C^2_{\mathrm{loc}}(\mathbb{S}^3\backslash\{\cup_{\ell=1}^kq^{(\ell)}\}). \tag{3.13}$$

In fact, subtracting all the poles from the limit function, we obtain a regular function $\widetilde{W}_0:\mathbb{S}^3\to\mathbb{R}$ such that $P_\sigma\widetilde{W}_0=0$ on $\mathbb{S}^3$, so it must be $\widetilde{W}_0\equiv0$.

Using (3.13), we have, for $|y|>0$ small,

$$h^{(j)}(y)=\frac{4}{K(q^{(j)})^2|y|^2}+8\sum_{\ell\neq j}\lim_{i\to\infty}\frac{v_i(q_i^{(j)})}{v_i(q_i^{(\ell)})}\frac{G_{q^{(\ell)}}(q^{(j)})}{K(q^{(\ell)})^2}+O(|y|). \tag{3.14}$$

The conclusion obtained from the above is easy to see that (1.15) is true and (1.14) can be obtained from Proposition 2.6. We have proved Part (ii).

Before stating the result to be proved, we give the following estimates (3.15) and (3.16). Using [20, Lemmas 4.13 and 4.14], we obtain

$$|\nabla K_i(y_i^{(j)})| = O(u_i(y_i^{(j)})^{-1}), \quad \tau_i = O(u_i(y_i^{(j)})^{-2}), \tag{3.15}$$

and from Propositions 2.5, 2.6, and 2.7, we get, for sufficiently small $\delta > 0$,

$$\begin{aligned}
&\sum_{j=1}^{3}\left|\int_{B_\delta} x_j u_i(y+y_i^{(j)})^{p_i+1}\right| = o(u_i(y_i^{(j)})^{-1}),\\
&\sum_{j\neq\ell}\left|\int_{B_\delta} x_j x_\ell u_i(y+y_i^{(j)})^{p_i+1}\right| = o(u_i(y_i^{(j)})^{-2}),\\
&\int_{\partial B_\delta} u_i(y+y_i^{(j)})^{p_i+1} = O(u_i(y_i^{(j)})^{-p_i-1}),\\
&\lim_{i\to\infty} u_i(y_i^{(j)})^2\int_{B_\delta}|y|^2 u_i(y+y_i^{(j)})^{p_i+1} = 6\pi|\mathbb{S}^2|K(q^{(j)})^{-5}.
\end{aligned}\tag{3.16}$$

Now we give the proof only for the last formula in (3.16). Let m_{ij} and r_i be as in (3.8). Applying Propositions 2.5, 2.6, and 2.7, we have

$$\begin{aligned}
&m_{ij}^2\int_{|y|\le\delta}|y|^2 u_i(y+y_i^{(j)})^{p_i+1}\\
&= m_{ij}^2\int_{|y|\le r_i}|y|^2 u_i(y+y_i^{(j)})^{p_i+1} + m_{ij}^2\int_{r_i<|y|\le\delta}|y|^2 u_i(y+y_i^{(j)})^{p_i+1}\\
&= m_{ij}^{4(2-p_i)}\int_{|x|\le R_i}|x|^2\Big(m_{ij}^{-1}u_i(m_{ij}^{-(p_i-1)}x+y_i^{(j)})\Big)^{p_i+1}\\
&\quad + m_{ij}^2\int_{r_i\le|x-y_i^{(j)}|\le\delta}|x-y_i^{(j)}|^2 u_i(x)^{p_i+1}\\
&= 6\pi|\mathbb{S}^2|K(q^{(j)})^{-5}+o(1).
\end{aligned}$$

For any $0<\delta<1$, combining (3.16) with Proposition 2.7, we can obtain

$$\tau_i^2\int_{B_\delta}\widetilde{K}_i(y+y_i^{(j)})H_i(y+y_i^{(j)})^{\tau_i}u_i(y+y_i^{(j)})^{p_i+1} \le Cu_i(y_i^{(j)})^{-4} = o(u_i(y_i^{(j)})^{-2}),$$

$$\tau_i\int_{B_\delta}\langle y,\nabla(\widetilde{K}_i(y+y_i^{(j)})H_i(y+y_i^{(j)})^{\tau_i})\rangle u_i(y+y_i^{(j)})^{p_i+1} = o(u_i(y_i^{(j)})^{-2}),$$

and

$$\tau_i \int_{\partial B_\delta} \widetilde{K}_i(y+y_i^{(j)}) H_i(y+y_i^{(j)})^{\tau_i} u_i(y+y_i^{(j)})^{p_i+1} = o(u_i(y_i^{(j)})^{-2}).$$

Then using Proposition 2.7 again, we have

$$\begin{aligned} &\frac{\tau_i}{3} \int_{B_\delta} \widetilde{K}_i(y+y_i^{(j)})(H_i(y+y_i^{(j)})^{\tau_i} - 1) u_i(y+y_i^{(j)})^{p_i+1} \\ &\leq C\tau_i^2 \int_{B_\delta} u_i(y)^{p_i+1} = o(u_i(y_i^{(j)})^{-2}). \end{aligned} \tag{3.17}$$

The above estimates, Proposition 2.1, and (3.16) yield, for any $0<\delta<1$,

$$\begin{aligned} &\int_{\partial' \mathcal{B}_\delta^+} B'(Y, U_i(Y+Y_i^{(j)}), \nabla U_i(Y+Y_i^{(j)}), \delta, \sigma) \\ &= \int_{B_\delta} \widetilde{K}_i(y+y_i^{(j)}) H_i(y+y_i^{(j)})^{\tau_i} u_i(y+y_i^{(j)})^{p_i+1} \\ &\quad + \int_{B_\delta} \langle y, \nabla u_i(y+y_i^{(j)})\rangle \widetilde{K}_i(y+y_i^{(j)}) H_i(y+y_i^{(j)})^{\tau_i} u_i(y+y_i^{(j)})^{p_i} \\ &= -\frac{\tau_i}{3} \int_{B_\delta} \widetilde{K}_i(y+y_i^{(j)}) H_i(y+y_i^{(j)})^{\tau_i} u_i(y+y_i^{(j)})^{p_i+1} \\ &\quad - \frac{1}{3} \int_{B_\delta} \langle y, \nabla(\widetilde{K}_i(y+y_i^{(j)}) H_i(y+y_i^{(j)})^{\tau_i})\rangle u_i(y+y_i^{(j)})^{p_i+1} \\ &\quad + \frac{\delta}{3} \int_{\partial B_\delta} \widetilde{K}_i(y+y_i^{(j)}) H_i(y+y_i^{(j)})^{\tau_i} u_i(y+y_i^{(j)})^{p_i+1} \\ &\quad + o(u_i(y_i^{(j)})^{-2}) \\ &=: \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + o(u_i(y_i^{(j)})^{-2}), \end{aligned} \tag{3.18}$$

where in the first equality, we take advantage of the fact that the Taylor expansion:

$$\frac{1}{p_i+1} = \frac{1}{3-\tau_i} = \frac{1}{3(1-\tau_i/3)} = \frac{1}{3}\Big(1 + \frac{\tau_i}{3} + O(\tau_i^2)\Big).$$

It follows from (3.16), (3.17), Propositions 2.5, and 2.7 that

$$\begin{aligned} \mathcal{J}_1 &= -\frac{\tau_i}{3} \int_{B_\delta} \widetilde{K}_i(y+y_i^{(j)}) u_i(y+y_i^{(j)})^{p_i+1} + o(m_{ij}^{-2}) \\ &= -\frac{\tau_i}{3} \frac{2^3}{K(q^{(j)})^2} \int_{\mathbb{R}^3} \frac{1}{(1+|z|^2)^3} + o(m_{ij}^{-2}) \\ &= -\frac{\pi|\mathbb{S}^2|}{6} \frac{\tau_i}{K(q^{(j)})^2} + o(m_{ij}^{-2}). \end{aligned} \tag{3.19}$$

Applying Proposition 2.7 and (3.16), we conclude that

$$
\begin{aligned}
\mathcal{J}_2 =& -\frac{1}{3}\int_{B_\delta}\langle y, \nabla\big(\widetilde{K}_i(y+y_i^{(j)})H_i(y+y_i^{(j)})^{\tau_i}\big)\rangle u_i(y+y_i^{(j)})^{p_i+1} + o(m_{ij}^{-2})\\
=& -\frac{1}{3}\int_{B_\delta}\langle y, \nabla(\widetilde{K}_i(y+y_i^{(j)}))H_i(y+y_i^{(j)})^{\tau_i}\rangle u_i(y+y_i^{(j)})^{p_i+1}\\
& -\frac{1}{3}\int_{B_\delta}\langle y, \widetilde{K}_i(y+y_i^{(j)})\nabla(H_i(y+y_i^{(j)})^{\tau_i})\rangle u_i(y+y_i^{(j)})^{p_i+1} + o(m_{ij}^{-2})\\
=& -\frac{1}{3}\sum_{\ell}\int_{B_\delta} y_\ell \frac{\partial \widetilde{K}_i}{\partial y_\ell}(y+y_i^{(j)})u_i(y+y_i^{(j)})^{p_i+1} + o(m_{ij}^{-2})\\
=& -\frac{1}{3}\int_{B_\delta} y\cdot\nabla\widetilde{K}_i(y_i^{(j)})u_i(y+y_i^{(j)})^{p_i+1}\\
& -\frac{1}{3}\sum_{\ell,m}\int_{B_\delta} y_\ell y_m \frac{\partial^2 \widetilde{K}_i}{\partial y_\ell \partial y_m}(y_i^{(j)})u_i(y+y_i^{(j)})^{p_i+1} + o(m_{ij}^{-2})\\
=& -\frac{1}{9}\Delta\widetilde{K}(0)\int_{B_\delta}|y|^2 u_i(y+y_i^{(j)})^{p_i+1} + o(m_{ij}^{-2})\\
=& -\frac{4}{9}\Delta_{g_0}K(q^{(j)})\int_{B_\delta}|y|^2 u_i(y+y_i^{(j)})^{p_i+1} + o(m_{ij}^{-2}),
\end{aligned} \tag{3.20}
$$

where we used the definition of the Laplace–Beltrami operator in the last equality. By (3.16), we get

$$
\mathcal{J}_3 = \frac{\delta}{3}\int_{\partial B_\delta}\widetilde{K}_i(y+y_i^{(j)})H_i(y+y_i^{(j)})^{\tau_i}u_i(y+y_i^{(j)})^{p_i+1} = o(m_{ij}^{-2}). \tag{3.21}
$$

It follows from Proposition 2.1 and (3.18)–(3.21) that

$$
\begin{aligned}
&\int_{\partial''\mathcal{B}_\delta^+} B''(Y, U_i(Y+Y_i^{(j)}), \nabla U_i(Y+Y_i^{(j)}), \delta, \sigma)\\
&= -\int_{\partial'\mathcal{B}_\delta^+} B'(Y, U_i(Y+Y_i^{(j)}), \nabla U_i(Y+Y_i^{(j)}), \delta, \sigma)\\
&= \frac{\pi|\mathbb{S}|^2}{6}\frac{\tau_i}{K(q^{(j)})^2} + \frac{4}{9}\Delta_{g_0}K(q^{(j)})\int_{B_\delta}|y|^2 u_i(y+y_i^{(j)})^{p_i+1} + o(m_{ij}^{-2}).
\end{aligned} \tag{3.22}
$$

By (3.2) and the definition of $\mu^{(j)}$, we have

$$
\mu^{(j)} = \lim_{i\to\infty}\tau_i v_i(q_i^{(j)})^2 = \lim_{i\to\infty}\frac{1}{4}\tau_i u_i(y_i^{(j)})^2.
$$

Thus, multiplying (3.22) by $U_i(Y_i^{(j)})^2$ and sending i to ∞, and using Proposition 2.1 and (3.16), we conclude that

$$\begin{aligned}&\int_{\partial''\mathcal{B}_\delta^+} B''(Y, \mathcal{H}^{(j)}(Y+Y_i^{(j)}), \nabla\mathcal{H}^{(j)}(Y+Y_i^{(j)}), \delta, \sigma)\\&= \frac{8\pi|\mathbb{S}^2|\Delta_{g_0}K(q^{(j)})}{3K(q^{(j)})^5} + \frac{2\pi|\mathbb{S}^2|\mu^{(j)}}{3K(q^{(j)})^2}.\end{aligned}$$

Let $\delta \to 0$, it follows from Proposition 2.2 that

$$\frac{8\pi|\mathbb{S}^2|\Delta_{g_0}K(q^{(j)})}{3K(q^{(j)})^5} + \frac{2\pi|\mathbb{S}^2|\mu^{(j)}}{3K(q^{(j)})^2} = -\frac{2\pi|\mathbb{S}^2|W^{(j)}(0)}{K(q^{(j)})^2}. \tag{3.23}$$

Consequently, we have $q^{(j)} \in \mathscr{K}\backslash\mathscr{K}^+$, $\forall\, 1 \le j \le k$, and when $k \ge 2$, $q^{(j)} \in \mathscr{K}^-$, $\forall\, 1 \le j \le k$.

It is easy to see that (1.16) follows from (3.6) and (3.23) when $k = 1$. When $k \ge 2$, by (3.14) we have

$$W^{(j)}(0) = 8\sum_{\ell\neq j}\frac{\lambda_\ell}{\lambda_j}\frac{G_{q^{(\ell)}}(q^{(j)})}{K(q^{(j)})K(q^{(\ell)})}, \quad \forall\, 1 \le j \le k. \tag{3.24}$$

Substituting (3.24) into (3.23), we get

$$-6\sum_{\ell\neq j}\frac{G_{q^{(\ell)}}(q^{(j)})}{K(q^{(j)})K(q^{(\ell)})}\lambda_\ell - \frac{\Delta_{g_0}K(q^{(j)})}{K(q^{(j)})^3}\lambda_j = \frac{1}{4}\lambda_j\mu^{(j)}.$$

We have established (1.17) and, thus, verified Part (iii).

We claim that there exists some

$$\eta = (\eta_1, \ldots, \eta_k) \neq 0 \quad \text{with} \quad \eta_\ell \ge 0,\ \forall\, \ell = 1, \ldots, k, \tag{3.25}$$

such that

$$\sum_{\ell=1}^k M_{\ell j}(q^{(1)}, \ldots, q^{(k)})\eta_\ell = \mu(M)\eta_j, \quad \forall\, j = 1, \ldots, k.$$

Indeed, choose $\Lambda > \max_i M_{ii}$, then the matrix $\Lambda I - M$ is a positive matrix, that is, each entry is positive, where I denotes the unit matrix. The claim follows from the Perron-Frobenius Theorem.

Multiplying (1.17) by η_j and summing over j, and using Part (ii) and (3.25), we have

$$\mu(M)\sum_j \lambda_j\eta_j = \sum_{\ell,j} M_{\ell j}\lambda_\ell\eta_j = \frac{1}{4}\sum_j \lambda_j\eta_j\mu^{(j)} \ge 0. \tag{3.26}$$

It follows that $\mu(M) \geq 0$. We have verified Part (i). Part (iv) follows from Parts (i)–(iii). The proof of Theorem 1.1 is completed. □

Using Theorem 1.1, we can give the proof of Theorem 1.2.

Proof *(Proof of Theorem 1.2)* We first prove the upper bounds in (1.20). Suppose this assertion of the theorem is false. Then we can find that there exists $K_i \to K$ in $C^2(\mathbb{S}^3)$ such that $\max_{\mathbb{S}^3} v_i \to \infty$ for some $v_i \in \mathscr{M}_{K_i}$. Theorem 1.1 shows that $\{v_i\}$ has only isolated simple blow up points $\{q^{(1)}, \ldots, q^{(k)}\}$. It follows from [20, Theorem 5.5] that $k > 1$. Using Part (i) of Theorem 1.1, we obtain $q^{(i)}, \ldots, q^{(k)} \in \mathscr{K}^-$.

Applying Theorem 1.1 with $\tau_i = 0$, we deduce that $q^{(1)}, \ldots, q^{(k)} \in \mathscr{K}^-$ and for all $1 \leq j \leq k$, $\sum_{\ell=1}^{k} M_{\ell j}\lambda_\ell = 0$, where $\lambda_\ell > 0$, $\ell = 1, \ldots, k$.

Analysis similar to that in the proof of Theorem 1.1 shows that $\mu(M)$ has at least one nonnegative eigenvector $\eta = (\eta_1, \ldots, \eta_k)$ as in (3.25), then we have

$$\mu(M) \sum_j \lambda_j \eta_j = \sum_{\ell, j} M_{\ell j}\lambda_\ell \eta_j = 0.$$

It follows that $\mu(M) = 0$. This leads to a contradiction with $K \in \mathscr{A}$. Then by the Harnack inequality in [20, Lemma 4.3] and Schauder estimates in [20, Theorem 2.11], we complete the proof of Theorem 1.2. □

4 The Existence Results on $\mathbb{S}^3$

In this section, we first prove Theorem 1.3, which is about the degree-counting formula and the existence of the solutions. Before that, we prove that as $\tau \to 0^+$, the solutions to the subcritical equation (see (4.1) below) either stay bounded and converge to the solutions to critical equations (1.3) in C^2 norm or become unbounded and blow up at finite points.

Then by using Theorem 1.3 and perturbing the prescribing function near its critical point, we can know exactly where the blow up occur when $K \notin \mathscr{A}$. From Theorem 1.1 and the proof of Theorem 1.4, we show Theorem 1.5 holds.

4.1 On the Case of Subcritical Equations

In this subsection, we consider the following subcritical equation:

$$P_\sigma v = K v^{2-\tau} \quad \text{on } \mathbb{S}^3, \tag{4.1}$$

where $\sigma = 1/2$, $K \in C^2(\mathbb{S}^3)$ and $\tau > 0$.

Denote the $H^\sigma(\mathbb{S}^3)$ inner product and norm by

$$\langle u, v \rangle = \int_{\mathbb{S}^n} (P_\sigma u)v, \quad \|u\|_\sigma = \sqrt{\langle u, u \rangle}.$$

The Euler-Lagrange functional associated with (4.1) is

$$I_\tau(v) = \frac{1}{2}\int_{\mathbb{S}^3}(P_\sigma v)v - \frac{1}{3-\tau}\int_{\mathbb{S}^3} K|v|^{3-\tau}, \quad \forall\, v \in H^\sigma(\mathbb{S}^3). \tag{4.2}$$

Definition 4.1 Let $K \in C^2(\mathbb{S}^3)$, $\mathscr{K}^-$ be as in (1.10) and $k \in \mathbb{N}_+$. Let $\overline{P}_1, \ldots, \overline{P}_k \in \mathscr{K}^-$ be the critical points of K with $\mu(M(\overline{P}_1, \ldots, \overline{P}_k)) > 0$ and $\varepsilon_0 > 0$ be sufficiently small. Define

$$\begin{aligned}\Omega_{\varepsilon_0} =&\Omega_{\varepsilon_0}(\overline{P}_1, \ldots, \overline{P}_k)\\ =&\Big\{(\alpha, t, P) \in \mathbb{R}_+^k \times \mathbb{R}_+^k \times (\mathbb{S}^3)^k : |\alpha_i - 1/K(P_i)| < \varepsilon_0,\\ &t_i > 1/\varepsilon_0,\ |P_i - \overline{P}_i| < \varepsilon_0, 1 \le i \le k\Big\}.\end{aligned}$$

It is well known that for $P \in \mathbb{S}^3$ and $t > 0$,

$$\delta_{P,t}(x) := \frac{t}{1 + \frac{t^2-1}{2}(1 - \cos d(x, P))}, \quad \forall\, x \in \mathbb{S}^3 \tag{4.3}$$

is a family of positive solutions to

$$P_\sigma v = v^2, \quad v > 0 \quad \text{on } \mathbb{S}^3, \tag{4.4}$$

where $d(\cdot, \cdot)$ is the distance induced by the standard metric of $\mathbb{S}^3$.

Lemma 4.1 *Let ε_0 be sufficiently small and $\Omega_{\varepsilon_0} = \Omega_{\varepsilon_0}(\overline{P}_1, \ldots, \overline{P}_k)$ be as in Definition 4.1. For any $u \in H^\sigma(\mathbb{S}^3)$ satisfying the inequality*

$$\Big\| u - \sum_{i=1}^k \widetilde{\alpha}_i \delta_{\widetilde{P}_i, \widetilde{t}_i} \Big\|_\sigma < \frac{\varepsilon_0}{2}$$

for some $(\widetilde{\alpha}, \widetilde{t}, \widetilde{P}) \in \Omega_{\varepsilon_0/2}$, then there exists a unique $(\alpha, t, P) \in \Omega_{\varepsilon_0}$ such that

$$u = \sum_{i=1}^k \alpha_i \delta_{P_i, t_i} + v,$$

with v satisfies

$$\langle v, \delta_{P_i,t_i}\rangle = \left\langle v, \frac{\partial \delta_{P_i,t_i}}{\partial P_i^{(\ell)}}\right\rangle = \left\langle v, \frac{\partial \delta_{P_i,t_i}}{\partial t_i}\right\rangle = 0, \tag{4.5}$$

where $\frac{\partial}{\partial P_i^{(\ell)}}$ denotes the corresponding derivatives.

Proof The proof of this lemma is similar to [4, Proposition 5.2], [5, pp. 348–350] and [25, Proposition 4.1], for reader's convenience, we give it here.

We argue by contradiction. Given $u \in H^\sigma(\mathbb{S}^3)$ satisfying $\|u - \sum_{i=1}^k \widetilde{\alpha}_i \delta_{\widetilde{P}_i, \widetilde{t}_i}\|_\sigma < \varepsilon_0/2$ for some $(\widetilde{\alpha}, \widetilde{t}, \widetilde{P}) \in \Omega_{\varepsilon_0/2}$. Suppose Proposition is not true. Then there exists $(\alpha, t, P), (\bar{\alpha}, \bar{t}, \bar{P}) \in \Omega_{\varepsilon_0}$,

$$(\alpha, t, P) \neq (\bar{\alpha}, \bar{t}, \bar{P}),$$

however, both (α, t, P) and $(\bar{\alpha}, \bar{t}, \bar{P})$ are minimum to

$$\min_{(\alpha, t, P) \in \Omega_{\varepsilon_0}} \left\| u - \sum_{i=1}^k \alpha_i \delta_{P_i, t_i} \right\|_\sigma .$$

Set

$$\begin{aligned} u &= \sum_{i=1}^k \alpha_i \delta_{P_i, t_i} + v, \\ u &= \sum_{i=1}^k \bar{\alpha}_i \delta_{\bar{P}_i, \bar{t}_i} + \bar{v}. \end{aligned}$$

Denote $\delta_i = \delta_{P_i, t_i}$ and $\bar{\delta}_i = \delta_{\bar{P}_i, \bar{t}_i}$. We then have

$$0 = \langle v, \delta_i \rangle = \left\langle v, \frac{\partial \delta_i}{\partial t_i} \right\rangle, \tag{4.6}$$

$$0 = \langle \bar{v}, \bar{\delta}_i \rangle = \left\langle \bar{v}, \frac{\partial \bar{\delta}_i}{\partial \bar{t}_i} \right\rangle, \tag{4.7}$$

$$0 = \left\langle v, \frac{\partial \delta_i}{\partial P_i^{(\ell)}} \right\rangle, \quad \ell = 1, \ldots, n, \tag{4.8}$$

$$0 = \left\langle \bar{v}, \frac{\partial \bar{\delta}_i}{\partial \bar{P}_i^{(\ell)}} \right\rangle, \quad \ell = 1, \ldots, n. \tag{4.9}$$

Similar to the proof of Lemma 4.2 in [25], we have

$$\begin{aligned} &\frac{t_i}{\bar{t}_i} = 1 + o(1), \\ &t_i \bar{t}_i |P_i - \bar{P}_i|^2 = o(1), \\ &|\alpha_i - \bar{\alpha}_i| = o(1), \end{aligned} \tag{4.10}$$

where $o(1) \to 0$ as $\varepsilon_0 \to 0$. Let $a_i = \bar{t}_i |P_i - \bar{P}_i|$, $\eta_i = \bar{t}_i / t_i - 1$, $\mu_i = \alpha_i - \bar{\alpha}_i$, where $|P_i - \bar{P}_i|$ represents the distance between two points P_i and $\bar{P}_i$ after through a stereographic projection. We know that $a_i = o(1)$, $\eta_i = o(1)$, $\mu_i = o(1)$.

Using (4.6) and (4.7) we have

$$\sum_{j=1}^{k}\int_{\mathbb{S}^n}(\alpha_j\delta_j-\bar{\alpha}_j\bar{\delta}_j)P_\sigma\delta_i=\int_{\mathbb{S}^n}\bar{v}P_\sigma(\delta_i-\bar{\delta}_i). \tag{4.11}$$

Notice that

$$\begin{aligned}&\int_{\mathbb{S}^n}(\alpha_j\delta_j-\bar{\alpha}_j\bar{\delta}_j)P_\sigma\delta_i\\&=(\alpha_j-\bar{\alpha}_j)\int_{\mathbb{S}^n}\delta_jP_\sigma\delta_i+\bar{\alpha}_j\int_{\mathbb{S}^n}(\delta_j-\bar{\delta}_j)P_\sigma\delta_i\\&=(\alpha_j-\bar{\alpha}_j)\int_{\mathbb{S}^n}\delta_jP_\sigma\delta_i+\bar{\alpha}_j\int_{\mathbb{S}^n}(\delta_j-\bar{\delta}_j)\delta_i^2.\end{aligned} \tag{4.12}$$

Expressions (4.11) and (4.12) yield

$$\begin{aligned}&\mu_i\|\delta_i\|_\sigma^2+\bar{\alpha}_i\int_{\mathbb{S}^n}(\delta_i-\bar{\delta}_i)\delta_i^2\\&=-\sum_{j\neq i}\left(\mu_j\int_{\mathbb{S}^n}\delta_jP_\sigma\delta_i+\bar{\alpha}_j\int_{\mathbb{S}^n}(\delta_j-\bar{\delta}_j)\delta_i^2\right)+o(1)\|\delta_i-\bar{\delta}_i\|_\sigma,\end{aligned} \tag{4.13}$$

where we have used the fact that $\|\bar{v}\|_\sigma=o(1)$ and the Cauchy-Schwarz inequality.

Using sterographic projection, we have

$$\int_{\mathbb{S}^n}(\delta_j-\bar{\delta}_j)\delta_i^2=\int_{\mathbb{R}^n}(\psi_j-\bar{\psi}_j)\psi_i^2, \tag{4.14}$$

where

$$\psi_j(x)=\frac{2t_j}{1+t_j^2|x-x_j|^2}\quad\text{and}\quad\bar{\psi}_j(x)=\frac{2\bar{t}_j}{1+\bar{t}_j^2|x-\bar{x}_j|^2}.$$

By direct calculation, we see that

$$|\psi_j(x)-\bar{\psi}_j(x)|\le C(|\eta_j|+|a_j|)\psi_j(x). \tag{4.15}$$

By (4.13), (4.14), (4.15), we obtain

$$\mu_i\|\delta_i\|_\sigma^2+\bar{\alpha}_i\int_{\mathbb{S}^n}(\delta_i-\bar{\delta}_i)\delta_i^2=o(1)\sum_j(|\eta_j|+|a_j|+|\mu_j|)+o(1)\|\delta_i-\bar{\delta}_i\|_\sigma. \tag{4.16}$$

It is easy to check that

$$\begin{aligned} \int_{\mathbb{S}^n} (\delta_i - \bar{\delta}_i)\delta_i^2 = o(1)(|a_i| + |\eta_i|), \\ \|\delta_i - \bar{\delta}_i\|_\sigma \le C(|a_i| + |\eta_i|). \end{aligned} \tag{4.17}$$

Using (4.16) and (4.17) we have

$$\mu_i = o(1)\sum_j (|\eta_j| + |a_j| + |\mu_j|). \tag{4.18}$$

Similarly, we have

$$\eta_i \le o(1)\sum_j (|\eta_j| + |a_j| + |\mu_j|), \tag{4.19}$$

and

$$a_i = o(1)\sum_j (|\eta_j| + |a_j| + |\mu_j|). \tag{4.20}$$

From (4.18), (4.19), and (4.20) we deduce that for ε_0 small enough we have

$$\eta_i = 0, \quad a_i = 0, \quad \mu_i = 0.$$

This is a contradiction, we finish the proof. □

We denote the set of $v \in H^\sigma(\mathbb{S}^3)$ satisfying (4.5) by $E_{P,t}$. In what follows, we work in some orthonormal basis near $\{\overline{P}_1, \dots, \overline{P}_k\}$.

Definition 4.2 Let A be sufficiently large, $\varepsilon_0, \nu_0 > 0$ be sufficiently small, $k \in \mathbb{N}_+$, and $\Omega_{\varepsilon_0/2} = \Omega_{\varepsilon_0/2}(\overline{P}_1, \dots, \overline{P}_k)$ be as in Definition 4.1. Define

$$\begin{aligned} &\Sigma_\tau(\overline{P}_1, \dots, \overline{P}_k) \\ &= \Big\{(\alpha, t, P, v) \in \Omega_{\varepsilon_0/2} \times H^\sigma(\mathbb{S}^3) : \\ &\quad |P_i - \overline{P}_i| < \tau^{1/2}|\log\tau|,\ A^{-1}\tau^{-1/2} < t_i < A\tau^{-1/2},\ v \in E_{P,t},\ \|v\| < \nu_0\Big\}. \end{aligned} \tag{4.21}$$

Without confusion we use the same notation for

$$\Sigma_\tau(\overline{P}_1, \dots, \overline{P}_k) = \left\{u = \sum_{i=1}^k \alpha_i\delta_{P_i,t_i} + v : (\alpha, t, P, v) \in \Sigma_\tau\right\} \subset H^\sigma(\mathbb{S}^3).$$

Remark 4.1 Due to Theorem 1.2, we only need to prove Theorem 1.3 for $K \in \mathscr{A}$ being a Morse function. Once this is achieved, we also prove that the Index as in Definition 1.4 is well defined on $\mathscr{A}$.

Using blow up analysis, we first give the necessary conditions on blowing up solutions to (4.1) when τ tends to 0.

Proposition 4.1 *Let $\sigma = 1/2$, $K \in \mathscr{A}$ be a Morse function and $\mathscr{K}^-$ be as in (1.10). Then for any $\alpha \in (0, 1)$, there exists some positive constants $\varepsilon_0, \nu_0 \ll 1$, and $A, R \gg 1$ depending only on K, such that when $\tau > 0$ is sufficiently small, for all u satisfying $u \in H^\sigma(\mathbb{S}^3)$, $u > 0$, $I'_\tau(u) = 0$, we have*

$$u \in \mathscr{O}_R \cup \{\cup_{k\geq 1} \cup_{\overline{P}_1,\ldots,\overline{P}_k \in \mathscr{K}^-, \mu(M(\overline{P}_1,\ldots,\overline{P}_k))>0} \Sigma_\tau(\overline{P}_1, \ldots, \overline{P}_k)\},$$

where $I'_\tau(u)$ is as in (4.1), $\mathscr{O}_R$ is as in (1.21) and $\Sigma_\tau(\overline{P}_1, \ldots, \overline{P}_k)$ is as in (4.21).

Proof For any $\tau > 0$ sufficiently small, let $u_\tau \in H^\sigma(\mathbb{S}^3)$, $u_\tau > 0$ be a critical point of $I_\tau(u)$. If u_τ is uniformly bounded, then there exists a $R > 0$ such that $u_\tau \in \mathscr{O}_R$, and the proof is now completed. If not, there exists $\tau_i \to 0$ such that $u_{\tau_i} \to \infty$. It follows from Theorem 1.1 and $K \in \mathscr{A}$ that there exists a constant $\delta^* > 0$ such that $\{u_{\tau_i}\}$ has only isolated simple blow up points $q^{(1)}, \ldots, q^{(k)} \in \mathscr{K}^-$, with $|q^{(j)} - q^{(\ell)}| \geq \delta^*$, $\forall j \neq \ell$, and $\mu(q^{(1)}, \ldots, q^{(k)}) > 0$. Then Proposition 4.1 can be deduced from Propositions 2.3, 2.4, 2.6, and elliptic theory. $\square$

Now we are going to show that if $K \in \mathscr{A}$ is a Morse function, one can construct solutions highly concentrating at arbitrary points $q^{(1)}, \ldots, q^{(k)} \in \mathscr{K}^-$ provided $\mu(M(q^{(1)}, \ldots, q^{(k)})) > 0$.

Theorem 4.1 *Let $\sigma = 1/2$, $K \in \mathscr{A}$ be a Morse function and $\mathscr{K}^-$ be as in (1.10). Let $\tau, \varepsilon_0, \nu_0 > 0$ be sufficiently small, $A > 0$ be sufficiently large and $k \in \mathbb{N}_+$. Then for any $\overline{P}_1, \ldots, \overline{P}_k \in \mathscr{K}^-$ satisfying $\mu(M(\overline{P}_1, \ldots, \overline{P}_k)) > 0$, we have*

$$\deg_{H^\sigma}\left(u - P_\sigma^{-1}(K|u|^{1-\tau}u), \Sigma_\tau(\overline{P}_1, \ldots, \overline{P}_k), 0\right) = (-1)^{k+\sum_{j=1}^k i(\overline{P}_j)}, \tag{4.22}$$

where $\deg_{H^\sigma}$ denotes the Leray-Schauder degree in $H^\sigma(\mathbb{S}^3)$, and $i(\overline{P}_j)$ is the Morse index of K at $\overline{P}_j$.

The following conclusion is needed for proving Theorem 4.1.

Proposition 4.2 *Under the assumptions of the Theorem 4.1, in addition that $\Sigma_\tau(\overline{P}_1, \ldots, \overline{P}_k)$ is as in (4.22) and $E_{P,t}$ is as in (4.5) for the given (α, t, P). Then there exists a unique minimizer $\overline{v} = \overline{v}_\tau(\alpha, t, P) \in E_{P,t}$ of $I_\tau(\sum_{i=1}^k \alpha_i \delta_{P_i,t_i} + v)$ with respect to $\{v \in E_{P,t} : \|v\|_\sigma < \nu_0\}$. Furthermore, there exists a constant C independent of τ such that*

$$\|\overline{v}\|_\sigma \leq C \sum_{i=1}^k |\nabla K(P_i)|\tau^{1/2} + C\tau|\log\tau| \leq C\tau|\log\tau|.$$

Proof For $(\alpha, t, P, v) \in \Sigma_\tau(\overline{P}_1, \dots, \overline{P}_k)$, which is simply written as Σ_τ, it follows from (4.5) that

$$\begin{aligned} &I_\tau\Big(\sum_{i=1}^{k}\alpha_i\delta_{P_i,t_i}+v\Big)\\ &=\frac{1}{2}\sum_{i=1}^{k}\alpha_i^2\int_{\mathbb{S}^3}\delta_{P_i,t_i}^3+\frac{1}{2}\sum_{j\neq i}\alpha_i\alpha_j\int_{\mathbb{S}^3}\delta_{P_i,t_i}^2\delta_{P_j,t_j}+\frac{1}{2}\int_{\mathbb{S}^3}(P_\sigma v)v \\ &\quad+\frac{1}{3-\tau}\int_{\mathbb{S}^3}K\Big|\sum_{i=1}^{k}\alpha_i\delta_{P_i,t_i}+v\Big|^{3-\tau}. \end{aligned} \tag{4.23}$$

Using Lemma A.1 and (A.16), we have,

$$\begin{aligned} &I_\tau\Big(\sum_{i=1}^{k}\alpha_i\delta_{P_i,t_i}+v\Big)\\ &=\frac{|\mathbb{S}^3|}{2}\sum_{i=1}^{k}\alpha_i^2+\sum_{i<j}\alpha_i\alpha_j\int_{\mathbb{S}^3}\delta_{P_i,t_i}^2\delta_{P_j,t_j}\\ &\quad-\frac{1}{3-\tau}\int_{\mathbb{S}^3}K\Big(\sum_{i=1}^{k}\alpha_i\delta_{P_i,t_i}\Big)^{3-\tau}-\int_{\mathbb{S}^3}K\Big(\sum_{i=1}^{k}\alpha_i\delta_{P_i,t_i}\Big)^{2-\tau}v\\ &\quad+\frac{1}{2}\int_{\mathbb{S}^3}(P_\sigma v)v-\frac{2-\tau}{2}\int_{\mathbb{S}^3}\Big(\sum_{i=1}^{k}\alpha_i\delta_{P_i,t_i}\Big)^{1-\tau}v^2+V(\tau,\alpha,t,P,v), \end{aligned}$$

where $|V(\tau, \alpha, t, P, v)| \le C\|v\|_\sigma^{3-\tau}$ and C depends only on K, ν_0, and A.

For $\varphi, v \in E_{P,t}$, set

$$f_\tau(v)=-\int_{\mathbb{S}^3}K\Big(\sum_{i=1}^{k}\alpha_i\delta_{P_i,t_i}\Big)^{2-\tau}v, \tag{4.24}$$

$$Q_\tau(v,\varphi)=\frac{1}{2}\int_{\mathbb{S}^3}(P_\sigma v)\varphi-\frac{2-\tau}{2}\int_{\mathbb{S}^3}K\Big(\sum_{i=1}^{k}\alpha_i\delta_{P_i,t_i}\Big)^{1-\tau}v\varphi, \tag{4.25}$$

$$Q_0(v,\varphi)=\frac{1}{2}\int_{\mathbb{S}^3}(P_\sigma v)\varphi-\int_{\mathbb{S}^3}\Big(\sum_{i=1}^{k}\alpha_i\delta_{P_i,t_i}\Big)v\varphi. \tag{4.26}$$

A direct calculation gives, for all $\varphi \in E_{P,t}$,

$$I_\tau'\Big(\sum_{i=1}^{k}\alpha_i\delta_{P_i,t_i}+v\Big)\varphi=f_\tau(\varphi)+2Q_\tau(v,\varphi)+\langle V_v(\tau,\alpha,t,P,v),\varphi\rangle, \tag{4.27}$$

where V_v is some function satisfying $\|V_v(\tau,\alpha,t,P,v)\|_\sigma \le C\|v\|_\sigma^{2-\tau}$.

Since f_τ is a continuous linear functional over $E_{P,t}$, there exists a unique $\widetilde{f}_\tau \in E_{P,t}$ such that

$$f_\tau(\varphi) = \langle \widetilde{f}_\tau, \varphi\rangle, \quad \forall \varphi \in E_{P,t}. \tag{4.28}$$

It is proved in [28] that there exists a constant $\delta_0 > 0$ (independent of τ) such that

$$Q_0(v,v) \ge \delta_0\|v\|_\sigma^2, \quad \forall (\alpha,t,P,v) \in \Sigma_\tau.$$

We choose ε_0 sufficiently small from the beginning. Using some elementary estimates as in Appendix, we have, for $\tau > 0$ small,

$$Q_\tau(v,v) \ge \frac{\delta_0}{2}\|v\|_\sigma^2, \quad \forall (\alpha,t,P,v) \in \Sigma_\tau. \tag{4.29}$$

Thus, there exists a unique symmetric continuous and coercive operator $\widetilde{Q}_\tau$ from $E_{P,t}$ onto itself such that, for any $\varphi \in E_{P,t}$,

$$Q_\tau(v,\varphi) = \langle \widetilde{Q}_\tau v, \varphi\rangle. \tag{4.30}$$

Using these notations, (4.27), (4.28), and (4.30), we have

$$I'_\tau\Big(\sum_{i=1}^k \alpha_i\delta_{P_i,t_i} + v\Big) = \widetilde{f}_\tau + 2\widetilde{Q}_\tau v + V_v(\tau,\alpha,t,P,v). \tag{4.31}$$

There is an equivalence between the existence of minimizer $\overline{v}_\tau$ and

$$\widetilde{f}_\tau + 2\widetilde{Q}_\tau v + V_v(\tau,\alpha,t,P,v) = 0, \quad v \in E_{P,t}. \tag{4.32}$$

As in [28, 33], by the implicit function theorem, there exists a unique $v_\tau \in E_{P,t}$ with $\|v\|_\sigma < \nu_0$ satisfying (4.32) and

$$\|\overline{v}\|_\sigma \le C\|\widetilde{f}_\tau\|_\sigma. \tag{4.33}$$

Thus, we only need to estimate $\|\widetilde{f_\tau}\|_\sigma$. From Lemma A.3, (A.19), and (A.24), we can obtain

$$\begin{aligned}
f_\tau(v) =& -\int_{\mathbb{S}^3} K\Big(\sum_{i=1}^k \alpha_i^{2-\tau}\delta_{P_i,t_i}^{2-\tau}\Big)v + O\Big(\sum_{i\neq j}\int_{\mathbb{S}^3}\delta_{P_i,t_i}^{1-\tau}\delta_{P_j,t_j}|v|\Big)\\
=& -\int_{\mathbb{S}^3}(K-K(P_i))\sum_{i=1}^k \alpha_i^{2-\tau}\delta_{P_i,t_i}^2 v + O\Big(\sum_{i=1}^k\int_{\mathbb{S}^3}|\delta_{P_i,t_i}^{2-\tau}-\delta_{P_i,t_i}^2||v|\Big)\\
&+ O\Big(\sum_{i\neq j}\|\delta_{P_i,t_i}^{1-\tau}\delta_{P_j,t_j}\|_{L^{3/2}(\mathbb{S}^3)}\|v\|_\sigma\Big)\\
=& O\Big(\sum_{i=1}^k|\nabla_{g_0}K(P_i)|\int_{\mathbb{S}^3}|P-P_i|\delta_{P_i,t_i}^2|v|\Big) + O\Big(\sum_{i=1}^k\int_{\mathbb{S}^3}|P-P_i|^2\delta_{P_i,t_i}^2|v|\Big)\\
&+ O(\tau|\log\tau|\|v\|_\sigma),
\end{aligned}$$

where $|P-P_i|$ represents the distance between two points P and P_i after through a stereographic projection.

Using (A.24) again, we have, for all $(\alpha, t, P, v) \in \Sigma_\tau$,

$$\begin{aligned}
|f_\tau(v)| \leq& C\Big\{\tau^{1/2}\sum_{i=1}^k|\nabla K(P_i)| + \tau + \tau|\log\tau|\Big\}\|v\|_\sigma\\
\leq& C\tau|\log\tau|\|v\|_\sigma,
\end{aligned} \tag{4.34}$$

this, combining (4.28) and (4.33), we complete the proof. □

Proposition 4.3 *Under the assumptions of Theorem 4.1, in addition that $\Sigma_\tau(\overline{P}_1, \ldots, \overline{P}_k)$ is as in (4.22). Then for any $(\alpha, t, P, v) \in \Sigma_\tau(\overline{P}_1, \ldots, \overline{P}_k)$, there exists $V_{\alpha_i}(\tau, \alpha, t, P, v)$ such that*

$$\frac{\partial}{\partial\alpha_i}I_\tau\Big(\sum_{j=1}^k\alpha_j\delta_{P_j,t_j}+v\Big) = -|\mathbb{S}^3|\beta_i + V_{\alpha_i}(\tau,\alpha,t,P,v),$$

where $\beta_i := \alpha_i - 1/K(P_i)$, $i = 1, \ldots, k$. Furthermore, let $\overline{v}$ be as in Proposition 4.2, then

$$\frac{\partial}{\partial\alpha_i}I_\tau\Big(\sum_{j=1}^k\alpha_j\delta_{P_j,t_j}+\overline{v}\Big) = -|\mathbb{S}^3|\beta_i + O(|\beta|^2+\tau|\log\tau|).$$

Proof from (A.16), (A.18), Lemmas A.2, A.3, and (A.19), we have

$$
\begin{aligned}
&\frac{\partial}{\partial \alpha_i} I_\tau\Big(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v\Big) \\
&= \alpha_i \int_{\mathbb{S}^3} \delta_{P_i,t_i}^3 + \frac{1}{2}\sum_{j\neq i} \alpha_j \int_{\mathbb{S}^3} \delta_{P_j,t_j}^2 \delta_{P_i,t_i} \\
&\quad - \int_{\mathbb{S}^3} K\Big|\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v\Big|^{1-\tau}\Big(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v\Big)\delta_{P_i,t_i} \\
&= |\mathbb{S}^3|\alpha_i - \int_{\mathbb{S}^3} K\Big(\sum_{j=1}^k \alpha_j^{2-\tau}\delta_{P_j,t_j}^{2-\tau}\Big)\delta_{P_i,t_i} - (2-\tau)\int_{\mathbb{S}^3} K\Big(\alpha_i^{1-\tau}\delta_{P_i,t_i}^{1-\tau}\Big)\delta_{P_i,t_i} v \\
&\quad + O(\tau|\log\tau|) + O(\tau) + O(\|v\|_\sigma^{2-\tau}).
\end{aligned}
$$

By using (A.24), we obtain

$$
\begin{aligned}
\int_{\mathbb{S}^3} K\alpha_i^2 \delta_{P_i,t_i}^{3-\tau} &= \int_{\mathbb{S}^3} K(P_i)\alpha_i^2 \delta_{P_i,t_i}^{3-\tau} - \int_{\mathbb{S}^3} (K(P) - K(P_i))\alpha_i^2 \delta_{P_i,t_i}^{3-\tau} \\
&= \int_{\mathbb{S}^3} K(P_i)\alpha_i^2 \delta_{P_i,t_i}^{3-\tau} + O(\tau).
\end{aligned} \tag{4.35}
$$

Similarly, by (4.5), (A.22), (4.21), and (A.24), we have

$$
\begin{aligned}
&\int_{\mathbb{S}^3} K\alpha_i \delta_{P_i,t_i}^{2-\tau} v \\
&= \int_{\mathbb{S}^3} K(P_i)\alpha_i \delta_{P_i,t_i}^2 v + \int_{\mathbb{S}^3} (K(P) - K(P_i))\alpha_i \delta_{P_i,t_i}^2 v + O\left(\tau|\log\tau|\|v\|_\sigma\right) \\
&= O(\tau|\log\tau|) + O(\|v\|_\sigma^2).
\end{aligned} \tag{4.36}
$$

It follows from the fact $|\alpha_i^{2-\tau} - \alpha_i^2| = O(\tau)$, (4.35), (4.36), (A.2), and (A.23) that

$$
\begin{aligned}
&\frac{\partial}{\partial \alpha_i} I_\tau\Big(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v\Big) \\
&= |\mathbb{S}^3|\alpha_i - \int_{\mathbb{S}^3} K\alpha_i^2 \delta_{P_i,t_i}^{3-\tau} - 2\int_{\mathbb{S}^3} K\alpha_i \delta_{P_i,t_i}^{2-\tau} v + O(\tau|\log\tau|) + O(\|v\|_\sigma^{2-\tau}) \\
&= -\beta_i \int_{\mathbb{S}^3} \delta_{P_i,t_i}^3 + O(|\beta|^2) + O(\tau|\log\tau|) + O(\|v\|_\sigma^{2-\tau}) \\
&= -|\mathbb{S}^3|\beta_i + O(|\beta|^2) + O(\tau|\log\tau|) + O(\|v\|_\sigma^{2-\tau}).
\end{aligned}
$$

Hence

$$\frac{\partial}{\partial \alpha_i} I_\tau\Big(\sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} + v\Big) = -|\mathbb{S}^3|\beta_i + V_{\alpha_i}(\tau, \alpha, t, P, v), \tag{4.37}$$

where

$$V_{\alpha_i}(\tau, \alpha, t, P, v) = O(|\beta|^2) + O(\tau|\log\tau|) + O(\|v\|_\sigma^{2-\tau}).$$

Combining with Proposition 4.2, we get

$$\begin{aligned}\frac{\partial}{\partial \alpha_i} I_\tau\Big(\sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} + \overline{v}\Big) &= -|\mathbb{S}^3|\beta_i + V_{\alpha_i}(\tau, \alpha, t, P, \overline{v}) \\ &= -|\mathbb{S}^3|\beta_i + O(|\beta|^2 + \tau|\log\tau|). \end{aligned}\tag{4.38}$$

Proposition 4.3 follows from the above. □

Proposition 4.4 *Under the assumptions of Theorem 4.1, in addition that $\Sigma_\tau(\overline{P}_1, \ldots, \overline{P}_k)$ is as in (4.22). Then for any $(\alpha, t, P, v) \in \Sigma_\tau(\overline{P}_1, \ldots, \overline{P}_k)$, there exists $V_{t_i}(\tau, \alpha, t, P, v)$ such that*

$$\begin{aligned}&\frac{\partial}{\partial t_i} I_\tau\Big(\sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} + v\Big) \\ &= \frac{\Gamma_3}{K(P_i)^2}\frac{\tau}{t_i} + \frac{\Gamma_4 \Delta_{g_0} K(P_i)}{K(P_i)^3}\frac{1}{t_i^3} + \sum_{j\neq i}\frac{\Gamma_5 G_{P_i}(P_j)}{K(P_i)K(P_j)}\frac{1}{t_i^2 t_j} + V_{t_i}(\tau, \alpha, t, P, v),\end{aligned}$$

where

$$V_{t_i}(\tau, \alpha, t, P, v) = O(|\beta|\tau^{3/2}) + O(\tau\|v\|_\sigma) + O(\tau^{1/2}\|v\|_\sigma^{2-\tau}) + O(\tau^{3/2}|\log\tau|),$$

$\Gamma_3, \Gamma_4, \Gamma_5$ are positive constants, and $G_{P_i}(P_j)$ is as in (1.12).

Proof Using (4.23), Lemma A.2, Hölder inequality, and Sobolev embedding, we have,

$$\begin{aligned}&\frac{\partial}{\partial t_i} I_\tau\Big(\sum_{j=1}^{k} \alpha_j \delta_{P_j,t_j} + v\Big) \\ &= \frac{1}{2}\sum_{j\neq i}\alpha_i\alpha_j \frac{\partial}{\partial t_i}\int_{\mathbb{S}^3} \delta_{P_j,t_j}\delta_{P_i,t_i}^2 - \int_{\mathbb{S}^3} K\Big(\sum_{j=1}^{k}\alpha_j\delta_{P_j,t_j}\Big)^{2-\tau}\alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial t_i} \\ &\quad - (2-\tau)\int_{\mathbb{S}^3} K\Big(\sum_{j=1}^{k}\alpha_j\delta_{P_j,t_j}\Big)^{1-\tau} v\alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial t_i} + O\Big(\|v\|_\sigma^{2-\tau}\Big\|\frac{\partial \delta_{P_i,t_i}}{\partial t_i}\Big\|_\sigma\Big).\end{aligned}\tag{4.39}$$

By (4.5), we have

$$\begin{aligned}\int_{\mathbb{S}^3}\delta_{P_i,t_i}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}v&=\frac{1}{2}\int_{\mathbb{S}^3}v\frac{\partial}{\partial t_i}(P_\sigma\delta_{P_i,t_i})\\&=\frac{1}{2}\frac{\partial}{\partial t_i}\langle v,\delta_{P_i,t_i}\rangle\\&=\frac{1}{2}\Big\langle v,\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\Big\rangle\\&=0.\end{aligned}\tag{4.40}$$

It follows from (4.40), (A.22), (A.17), and (A.24) that

$$\begin{aligned}&\Big|\int_{\mathbb{S}^3}K\delta_{P_i,t_i}^{1-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}v\Big|\\&=\Big|\int_{\mathbb{S}^3}(K-K(P_i))\delta_{P_i,t_i}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}v+\int_{\mathbb{S}^3}K(\delta_{P_i,t_i}^{1-\tau}-\delta_{P_i,t_i})\frac{\partial\delta_{P_i,t_i}}{\partial t_i}v\Big|\\&\le C(\tau^{1/2}|\log\tau|)\\&\int_{\mathbb{S}^3}|P-P_i|\delta_{P_i,t_i}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}v+O\Big(\|\delta_{P_i,t_i}^{1-\tau}-\delta_{P_i,t_i}\|_{L^3(\mathbb{S}^3)}\Big\|\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\Big\|_\sigma\|v\|_\sigma\Big)\\&\le C(\tau^{1/2}|\log\tau|)O\Big(\big\||P-P_i|\delta_{P_i,t_i}\big\|_{L^3(\mathbb{S}^3)}\Big\|\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\Big\|_\sigma\|v\|_\sigma\Big)+O\big(\tau^{3/2}\|v\|_\sigma\big)\\&\le C\tau\|v\|_\sigma,\end{aligned}$$

this, and (A.20) yields

$$\begin{aligned}&\Big|\int_{\mathbb{S}^3}K\Big(\sum_{j=1}^k\alpha_i\delta_{P_j,t_j}\Big)^{1-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}v\Big|\\&\le\Big|\int_{\mathbb{S}^3}K\alpha_i^{1-\tau}\delta_{P_i,t_i}^{1-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}v\Big|+C\sum_{j\neq i}\int_{\mathbb{S}^3}\delta_{P_j,t_j}^{1-\tau}\Big|\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\Big||v|\\&\le C\tau\|v\|_\sigma+O\Big(\sum_{j\neq i}\Big\|\delta_{P_j,t_j}^{1-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\Big\|_{L^{3/2}(\mathbb{S}^3)}\|v\|_\sigma\Big)\\&\le C\tau\|v\|_\sigma.\end{aligned}\tag{4.41}$$

Using (4.41) and Lemma A.3, we obtain

$$\begin{aligned}&\frac{\partial}{\partial t_i}I_\tau\Big(\sum_{j=1}^k\alpha_j\delta_{P_j,t_j}+v\Big)\\&=\frac{1}{2}\sum_{j\neq i}\alpha_i\alpha_j\frac{\partial}{\partial t_i}\int_{\mathbb{S}^3}\delta_{P_j,t_j}\delta_{P_i,t_i}^2-\int_{\mathbb{S}^3}K\Big(\sum_{j=1}^k\alpha_j\delta_{P_j,t_j}\Big)^{2-\tau}\alpha_i\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\end{aligned}$$

$$
\begin{aligned}
&+ O(\tau\|v\|_\sigma) + O(\tau^{1/2}\|v\|_\sigma^{2-\tau})\\
=& \frac{1}{2}\sum_{j\neq i}\alpha_i\alpha_j\frac{\partial}{\partial t_i}\int_{\mathbb{S}^3}\delta_{P_j,t_j}\delta_{P_i,t_i}^2 - \int_{\mathbb{S}^3}K\alpha_i^3\delta_{P_i,t_i}^{2-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\\
&-\sum_{j\neq i}\int_{\mathbb{S}^3}\alpha_i K\alpha_j^2\delta_{P_j,t_j}^{2-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\\
&+ O(\tau\|v\|_\sigma) + O(\tau^{1/2}\|v\|_\sigma^{2-\tau}) + O(\tau^{3/2}|\log\tau|).
\end{aligned}
$$

We have used the following facts. By (A.19), (A.20), (A.17), and (A.28), we have

$$
\begin{aligned}
&\sum_{j\neq\ell}\int_{\mathbb{S}^3}\delta_{P_j,t_j}^{1-\tau}\delta_{P_\ell,t_\ell}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\\
&=\int_{\mathbb{S}^3}\Big(\sum_{j=i,\ell\neq i}\delta_{P_i,t_i}^{1-\tau}\delta_{P_\ell,t_\ell}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\\
&\quad+\sum_{j\neq i,\ell=i}\delta_{P_j,t_j}^{1-\tau}\delta_{P_i,t_i}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}+\sum_{j\neq\ell\neq i}\delta_{P_j,t_j}^{1-\tau}\delta_{P_\ell,t_\ell}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\Big)\\
&=O\Big(\sum_{\ell\neq i}\frac{\partial}{\partial t_i}\int_{\mathbb{S}^3}\delta_{P_\ell,t_\ell}\delta_{P_i,t_i}^{2-\tau}\Big)+O\Big(\sum_{j\neq i}\Big\|\delta_{P_j,t_j}^{1-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\Big\|_{L^{3/2}(\mathbb{S}^3)}\Big)\\
&\quad+O\Big(\sum_{j\neq\ell\neq i}\Big\|\delta_{P_j,t_j}^{1-\tau}\delta_{P_\ell,t_\ell}\Big\|_{L^{3/2}(\mathbb{S}^3)}\Big\|\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\Big\|_\sigma\Big)\\
&=O(\tau^{3/2}|\log\tau|).
\end{aligned}
$$

Using (A.5) and (A.29), we have

$$
\begin{aligned}
&\int_{\mathbb{S}^3}K\delta_{P_j,t_j}^{2-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}=\frac{\partial}{\partial t_i}\int_{\mathbb{S}^3}K\delta_{P_j,t_j}^{2-\tau}\delta_{P_i,t_i}\\
&=K(P_j)\frac{\partial}{\partial t_i}\int_{\mathbb{S}^3}\delta_{P_j,t_j}^2\delta_{P_i,t_i}\\
&\quad+O(\tau^{5/2}|\log\tau|)+O\Big(\frac{\partial}{\partial t_i}\int_{\mathbb{S}^3}(K(P)-K(P_j))\delta_{P_j,t_j}^{2-\tau}\delta_{P_i,t_i}\Big)\\
&=K(P_j)\frac{\partial}{\partial t_i}\int_{\mathbb{S}^3}\delta_{P_j,t_j}^2\delta_{P_i,t_i}+O(\tau^{5/2}|\log\tau|)+O(\tau^2).
\end{aligned}\tag{4.42}
$$

By (A.25), we have

$$
\begin{aligned}
&-\int_{\mathbb{S}^3}K\alpha_i^3\delta_{P_i,t_i}^{2-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}=-\frac{1}{3-\tau}\alpha_i^3K(P_i)\frac{\partial}{\partial t_i}\int_{\mathbb{S}^3}\delta_{P_i,t_i}^{3-\tau}\\
&-\frac{2}{3(3-\tau)}\Delta_{g_0}K(P_i)\frac{\partial}{\partial t_i}\int_{\mathbb{S}^3}|P-P_i|^2\alpha_i^3\delta_{P_i,t_i}^{3-\tau}+O(\tau^2).
\end{aligned}\tag{4.43}
$$

Let

$$\mathscr{E} = O(\tau\|v\|_\sigma) + O(\tau^{1/2}\|v\|_\sigma^{2-\tau}) + O(\tau^{3/2}|\log\tau|), \tag{4.44}$$

then, by (A.7), (A.8), and (4.43), we have

$$\begin{aligned}
&\frac{\partial}{\partial t_i} I_\tau\Big(\sum_{j=1}^{k}\alpha_j\delta_{P_j,t_j}+v\Big)\\
&=\frac{1}{2}\sum_{j\neq i}\alpha_i\alpha_j\frac{\partial}{\partial t_i}\int_{\mathbb{S}^3}\delta_{P_j,t_j}\delta_{P_i,t_i}^2-\int_{\mathbb{S}^3}K\alpha_i^3\delta_{P_i,t_i}^{2-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial t_i}\\
&\quad-\alpha_i\sum_{j\neq i}K(P_j)\alpha_j^2\frac{\partial}{\partial t_i}\int_{\mathbb{S}^3}\delta_{P_j,t_j}^2\delta_{P_i,t_i}+\mathscr{E}\\
&=\sum_{j\neq i}\Big\{\frac{1}{2}\alpha_i\alpha_j-\alpha_i\alpha_j^2K(P_j)\Big\}\frac{\partial}{\partial t_i}\int_{\mathbb{S}^3}\delta_{P_j,t_j}\delta_{P_i,t_i}^2\\
&\quad-\frac{1}{3-\tau}\alpha_i^3K(P_i)\frac{\partial}{\partial t_i}\int_{\mathbb{S}^3}\delta_{P_i,t_i}^{3-\tau}\\
&\quad-\frac{2}{3(3-\tau)}\alpha_i^3\Delta_{g_0}K(P_i)\frac{\partial}{\partial t_i}\int_{\mathbb{S}^3}|P-P_i|^2\delta_{P_i,t_i}^{3-\tau}+O(\tau^{5/2})+\mathscr{E}\\
&=-\frac{1}{2}\sum_{j\neq i}\frac{1}{K(P_i)K(P_j)}\frac{\partial}{\partial t_i}\int_{\mathbb{S}^3}\delta_{P_j,t_j}\delta_{P_i,t_i}^2-\frac{1}{3}\frac{1}{K(P_i)^2}\frac{\partial}{\partial t_i}\int_{\mathbb{S}^3}\delta_{P_i,t_i}^{3-\tau}\\
&\quad-\frac{2}{9}\frac{\Delta_{g_0}K(P_i)}{K(P_i)^3}\frac{\partial}{\partial t_i}\int_{\mathbb{S}^3}|P-P_i|^2\delta_{P_i,t_i}^{3-\tau}+\mathscr{E}+O(|\beta|\tau^{3/2}).
\end{aligned}$$

It follows from (A.3), (A.7), and (A.8) that

$$\begin{aligned}
&\frac{\partial}{\partial t_i} I_\tau\Big(\sum_{j=1}^{k}\alpha_j\delta_{P_j,t_j}+v\Big)\\
&=\frac{\Gamma_3}{K(P_i)^2}\frac{\tau}{t_i}+\frac{\Gamma_4\Delta_{g_0}K(P_i)}{K(P_i)^3}\frac{1}{t_i^3}\\
&\quad+\sum_{j\neq i}\frac{\Gamma_5G_{P_i}(P_j)}{K(P_i)K(P_j)}\frac{1}{t_i^2t_j}+V_{t_i}(\tau,\alpha,t,P,v),
\end{aligned}\tag{4.45}$$

where

$$V_{t_i}(\tau,\alpha,t,P,v)=O(|\beta|\tau^{3/2})+O(\tau\|v\|_\sigma)+O(\tau^{1/2}\|v\|_\sigma^{2-\tau})+O(\tau^{3/2}|\log\tau|),$$

$$\Gamma_3=\frac{4}{3}\pi|\mathbb{S}^2|,\quad \Gamma_4=\frac{2}{3}\pi|\mathbb{S}^2|,\quad \Gamma_5=2\pi|\mathbb{S}^2|.$$

Proposition 4.4 follows from the above. □

Proposition 4.5 *Under the assumptions of Theorem 4.1, in addition that* $\Sigma_\tau(\overline{P}_1, \ldots, \overline{P}_k)$ *is as in (4.22). Then for any* $(\alpha, t, P, v) \in \Sigma_\tau(\overline{P}_1, \ldots, \overline{P}_k)$*, there exists a constant* $\nu_1 > 0$ *independent of* τ *and a vector* $V_{P_i}(\tau, \alpha, t, P, v)$*, such that*

$$\frac{\partial}{\partial P_i} I_\tau\Big(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v\Big) = -\Gamma_6 \nabla_{g_0} K(P_i) + V_{P_i}(\tau, \alpha, t, P, v),$$

where $\Gamma_6 \geq \nu_1 > 0$ *is a constant, and*

$$V_{P_i}(\tau, \alpha, t, P, v) = O(\tau^{1/2}) + O(\|v\|_\sigma) + O(\tau^{-1/2}\|v\|_\sigma^{2-\tau}).$$

Proof Using Lemma A.2, we have

$$\begin{aligned}
&\frac{\partial}{\partial P_i} I_\tau\Big(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v\Big) \\
&= \frac{1}{2}\sum_{j\neq i} \alpha_i\alpha_j \frac{\partial}{\partial P_i}\int_{\mathbb{S}^3} \alpha_j \delta_{P_j,t_j}^2 \delta_{P_i,t_i} \\
&\quad - \int_{\mathbb{S}^3} K\Big|\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v\Big|^{1-\tau}\Big(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j} + v\Big)\alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \\
&= \frac{1}{2}\sum_{j\neq i} \alpha_i\alpha_j \frac{\partial}{\partial P_i}\int_{\mathbb{S}^3} \alpha_j \delta_{P_j,t_j}^2 \delta_{P_i,t_i} \\
&\quad - \int_{\mathbb{S}^3} K\Big(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j}\Big)^{2-\tau}\alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \\
&\quad - (2-\tau)\int_{\mathbb{S}^3} K\Big(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j}\Big)^{1-\tau} v\alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \\
&\quad + O\Big(\|v\|_\sigma^{2-\tau}\Big\|\frac{\partial \delta_{P_i,t_i}}{\partial P_i}\Big\|_\sigma\Big).
\end{aligned} \tag{4.46}$$

By (4.5), we have

$$\int_{\mathbb{S}^3} \delta_{P_i,t_i}\frac{\partial \delta_{P_i,t_i}}{\partial P_i} v = \frac{1}{2}\frac{\partial}{\partial P_i}\int_{\mathbb{S}^3} \delta_{P_i,t_i}^2 v = \frac{1}{2}\Big\langle \frac{\partial \delta_{P_i,t_i}}{\partial P_i}, v\Big\rangle = 0. \tag{4.47}$$

It follows from (A.26), (A.30), (4.47), and (A.22) that

$$\begin{aligned}
&\int_{\mathbb{S}^3} K\Big(\sum_{j=1}^k \alpha_j \delta_{P_j,t_j}\Big)^{1-\tau} v\alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \\
&= \int_{\mathbb{S}^3} K(\alpha_i^{1-\tau}\delta_{P_i,t_i}^{1-\tau})\alpha_i \frac{\partial \delta_{P_i,t_i}}{\partial P_i} v + O\Big(\sum_{j\neq i}\int_{\mathbb{S}^3} \delta_{P_j,t_j}^{1-\tau}\Big|\frac{\partial \delta_{P_i,t_i}}{\partial P_i}\Big||v|\Big)
\end{aligned}$$

$$
\begin{aligned}
&= K(P_i)\alpha_i^{2-\tau}\int_{\mathbb{S}^3}\delta_{P_i,t_i}^{1-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial P_i}v + O(\|v\|_\sigma)\\
&= O(\|v\|_\sigma).
\end{aligned}
\tag{4.48}
$$

Then Lemma A.3, (4.48), (A.16), (A.27) and (A.28) yields

$$
\begin{aligned}
&\frac{\partial}{\partial P_i}I_\tau\Big(\sum_{j=1}^k\alpha_j\delta_{P_j,t_j}+v\Big)\\
&= \frac{1}{2}\sum_{j\neq i}\alpha_i\alpha_j\frac{\partial}{\partial P_i}\int_{\mathbb{S}^3}\delta_{P_j,t_j}^2\delta_{P_i,t_i}\\
&\quad -\alpha_i\int_{\mathbb{S}^3}K(\alpha_i\delta_{P_i,t_i})^{2-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial P_i}-\sum_{j\neq i}\alpha_i\int_{\mathbb{S}^3}K(\alpha_j\delta_{P_j,t_j})^{2-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial P_i}\\
&\quad + O(\tau^{1/2}|\ln\tau|)+O(\|v\|_\sigma)+O\left(\tau^{-1/2}\|v\|_\sigma^{2-\tau}\right)\\
&= O\Big(\sum_{j\neq i}\frac{\partial}{\partial P_i}\int_{\mathbb{S}^3}\delta_{P_j,t_j}^2\delta_{P_i,t_i}\Big)\\
&\quad -\alpha_i^3\int_{\mathbb{S}^3}K\delta_{P_i,t_i}^{2-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial P_i}+O\Big(\tau\Big|\int_{\mathbb{S}^3}\delta_{P_i,t_i}^{2-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial P_i}\Big|\Big)\\
&\quad + O\Big(\sum_{j\neq i}\Big|\int_{\mathbb{S}^3}\delta_{P_j,t_j}^{2-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial P_i}\Big|\Big)+O(\|v\|_\sigma)+O(\tau^{-1/2}\|v\|_\sigma^{2-\tau})+O(\tau^{1/2}|\ln\tau|))\\
&= -\alpha_i^3\int_{\mathbb{S}^3}K\delta_{P_i,t_i}^{2-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial P_i}+O(\tau^{1/2}|\ln\tau|)+O(\|v\|_\sigma)+O(\tau^{-1/2}\|v\|_\sigma^{2-\tau}).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\frac{\partial}{\partial P_i}I_\tau\Big(\sum_{j=1}^k\alpha_j\delta_{P_j,t_j}+v\Big)\\
&\quad = -\alpha_i^3\int_{\mathbb{S}^3}(K(P)-K(P_i))\delta_{P_i,t_i}^{2-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial P_i}-\alpha_i^3\int_{\mathbb{S}^3}K(P_i)\delta_{P_i,t_i}^{2-\tau}\frac{\partial\delta_{P_i,t_i}}{\partial P_i}\\
&\qquad + O(\tau^{1/2}|\ln\tau|)+O\left(\|v\|_\sigma\right)+O\left(\tau^{-1/2}\|v\|_\sigma^{2-\tau}\right)\\
&\quad = -\Gamma_6\nabla_{g_0}K(P_i)+V_{P_i}(\tau,\alpha,t,P,v),
\end{aligned}
$$

where

$$
\Gamma_6(\tau,\alpha,t,P,v)\geq\nu_1>0\quad\text{with }\nu_1\text{ independent of }\tau,
\tag{4.49}
$$

and

$$
V_{P_i}(\tau,\alpha,t,P,v)=O(\tau^{1/2}|\ln\tau|)+O\left(\|v\|_\sigma\right)+O\left(\tau^{-1/2}\|v\|_\sigma^{2-\tau}\right).
\tag{4.50}
$$

The existence of ν_1 is proved below. In fact, let P_i be the south pole and make a stereographic projection F to the equatorial plane of $\mathbb{S}^3$ with $y = (y^{(1)}, y^{(2)}, y^{(3)})$ as the stereographic projection coordinates, let $\widetilde{K} = K(F(y))$ and $|J_F| := (2/(1 + |y|^2))^3$. Then we have $F(0) = P_i$ and

$$\begin{aligned}
&\alpha_i^3 \int_{\mathbb{S}^3} (K(P) - K(P_i))\delta_{P_i,t_i}^{2-\tau} \frac{\partial \delta_{P_i,t_i}}{\partial P_i} \\
&= \alpha_i^3 \int_{\mathbb{R}^3} t_i y(\widetilde{K}(y) - \widetilde{K}(0))\omega_{0,t_i}^4 (|J_F|^{1/3}\omega_{0,t_i}^{-1})^\tau \\
&=: \mathcal{L} = (\mathcal{L}^{(1)}, \mathcal{L}^{(2)}, \mathcal{L}^{(3)}),
\end{aligned}$$

where $\omega_{0,t_i} := 2t_i/(1 + t_i^2|y|^2)$ is the solution of

$$(-\Delta)^{1/2}\omega_{0,t_i} = \omega_{0,t_i}^2 \quad \text{on } \mathbb{R}^3.$$

For $j = 1, 2, 3$, we have

$$\begin{aligned}
\mathcal{L}^{(j)} &= -\alpha_i^3 \int_{\mathbb{R}^3} t_i y^{(j)}(\widetilde{K}(y) - \widetilde{K}(0))\omega_{0,t_i}^4 (|J_F|^{1/3}\omega_{0,t_i}^{-1})^\tau \\
&= \alpha_i^3 \int_{\mathbb{R}^3} t_i y^{(j)}(\nabla \widetilde{K}(0) \cdot y + O(|y|^2))\omega_{0,t_i}^4 (|J_F|^{1/3}\omega_{0,t_i}^{-1})^\tau \\
&= \frac{1}{3}\alpha_i^3 \frac{\partial \widetilde{K}}{\partial y^{(j)}}(0) \int_{\mathbb{R}^3} t_i |y|^2 \omega_{0,y_i}^4 (|J_F|^{1/3}\omega_{0,t_i}^{-1})^\tau + O(\tau^{1/2}),
\end{aligned}$$

thus,

$$\begin{aligned}
\mathcal{L} =& \nabla \widetilde{K}(0) \Big\{ \frac{\alpha_i^3}{3} \int_{\mathbb{R}^3} t_i |y|^2 \omega_{0,t_i}^4 (|J_F|^{1/3}\omega_{0,t_i}^{-1})^\tau + O(\tau^{1/2}) \Big\} \\
&= \nabla_{g_0} K(P_i) \frac{2\alpha_i^3}{3} \Big\{ \int_{\mathbb{R}^3} t_i |y|^2 \omega_{0,t_i}^4 (|J_F|^{1/3}\omega_{0,t_i}^{-1})^\tau + O(\tau^{1/2}) \Big\}.
\end{aligned}$$

It follows from $t_i^{-\tau} \le (|J_F|^{1/3}\omega_{0,t_i}^{-1})^\tau \le t_i^\tau$ that

$$\int_{\mathbb{R}^3} t_i |y|^2 \omega_{0,t_i}^4 (|J_F|^{1/3}\omega_{0,t_i}^{-1})^\tau \ge t_i^{-\tau} \int_{\mathbb{R}^3} t_i |y|^2 \omega_{0,t_i}^4 \to C_0 \int_{\mathbb{R}^3} \frac{|y|^2}{(1 + |y|^2)^3},$$

as $\tau \to 0$, where $C_0 > 0$ is a constant. This ensures the existence of ν_1. We have proved Proposition 4.5. □

We now apply Propositions 4.2, 4.3, 4.4, 4.5 and construct a family of homotopy Id+compact operators to obtain the degree-counting formula of the solutions to the subcritical equation (4.1) on $\Sigma_\tau(\overline{P}_1, \dots, \overline{P}_k)$.

Proof of Theorem 4.1 Given $\tau > 0$ and $K \in \mathscr{A}$, let $\mathscr{K}^-$ be as in (1.10) and $\Sigma_\tau(\overline{P}_1, \dots, \overline{P}_k)$ be as in (4.21) for the given $\overline{P}_1, \dots, \overline{P}_k \in \mathscr{K}^-$.

For $u = \sum_{i=1}^{k} \alpha_i \delta_{P_i,t_i} + v \in \Sigma_\tau(\overline{P}_1, \ldots, \overline{P}_k)$, we have

$$T_u H^\sigma(\mathbb{S}^3) = E_{P,t} \bigoplus \operatorname{span}\left\{\delta_{P_i,t_i}, \frac{\partial \delta_{P_i,t_i}}{\partial t_i}, \frac{\partial \delta_{P_i,t_i}}{\partial P_i}\right\}.$$

Since $I'_\tau(u) \in T_u H^\sigma(\mathbb{S}^3)$, there exist $\xi \in E_{P,t}$, $\eta \in \operatorname{span}\left\{\delta_{P_i,t_i}, \frac{\partial \delta_{P_i,t_i}}{\partial t_i}, \frac{\partial \delta_{P_i,t_i}}{\partial P_i}\right\}$ such that

$$I'_\tau(u) = \xi + \eta.$$

From (4.27), we obtain, for all $\varphi \in E_{P,t}$,

$$\langle \xi, \varphi \rangle = I'_\tau(u)\varphi = f_\tau(\varphi) + 2Q_\tau(v, \varphi) + \langle V_v(\tau, \alpha, t, P, v), \varphi \rangle, \tag{4.51}$$

where $\|V_v(\tau, \alpha, t, P, v)\|_\sigma \le C\|v\|_\sigma^{2-\tau}$. Replacing φ by v in (4.51) and using (4.29), we have

$$\|\xi\|_\sigma \ge \delta_0\|v\|_\sigma - \|f_\tau\|_\sigma - O(\|v\|_\sigma^{2-\tau}) \ge \frac{\delta_0}{2}\|v\|_\sigma - \|f_\tau\|,$$

where δ_0 is as in (4.29).

Let $\beta = (\beta_1, \ldots, \beta_k)$, $\beta_i = \alpha_i - 1/K(P_i)$ be as in Proposition 4.3, we define

$$\widehat{\Sigma}_\tau = \left\{u = \sum_{i=1}^{k} \alpha_i \delta_{P_i,t_i} + v \in \Sigma_\tau(\overline{P}_1, \ldots, \overline{P}_k) : \|v\|_\sigma < \tau|\log\tau|^3,\ |\beta| < \tau|\log\tau|^2\right\}.$$

It follows from Proposition 4.2 and (4.38) that

$$I'_\tau(u) \ne 0, \quad \forall\, u \in \Sigma_\tau(\overline{P}_1, \ldots, \overline{P}_k)\backslash\widehat{\Sigma}_\tau.$$

For $u = \sum_{i=1}^{k} \alpha_i \delta_{P_i,t_i} + v \in \widehat{\Sigma}_\tau$, by using (4.23), we have

$$\begin{aligned}
\left\langle \eta, \delta_{P_i,t_i} \right\rangle &= I'_\tau(u)\delta_{P_i,t_i} \\
&= \alpha_i \int_{\mathbb{S}^3} \delta^3_{P_i,t_i} + \frac{1}{2}\sum_{j\ne i}\alpha_j \int_{\mathbb{S}^3} \delta^2_{P_i,t_i}\delta_{P_j,t_j} \\
&\quad - \int_{\mathbb{S}^3} K\left|\sum_{j=1}^{k}\alpha_j\delta_{P_j,t_j} + v\right|^{1-\tau}\Big(\sum_{j=1}^{k}\alpha_j\delta_{P_j,t_j} + v\Big)\delta_{P_i,t_i} \\
&= \frac{\partial}{\partial\alpha_i} I_\tau\Big(\sum_{j=1}^{k}\alpha_j\delta_{P_j,t_j} + v\Big) \\
&= -|\mathbb{S}^3|\beta_i + V_{\alpha_i}(\tau, \alpha, t, P, v),
\end{aligned} \tag{4.52}$$

where V_{α_i} satisfying

$$
\begin{aligned}
V_{\alpha_i}(\tau,\alpha,t,P,v) = \ & O(|\beta|^2) + O(\tau|\log\tau|) + O(\|v\|_\sigma^{2-\tau}) \\
\le \ & C\left(|\beta|^2 + \tau|\log\tau|\right).
\end{aligned}
$$

It follows from (4.39) and (4.45) that

$$
\begin{aligned}
\left\langle \eta, \frac{\partial \delta_{P_i,t_i}}{\partial t_i}\right\rangle &= I'_\tau(u)\,\frac{\partial \delta_{P_i,t_i}}{\partial t_i} \\
&= \frac{1}{2}\sum_{j\neq i}\alpha_j \frac{\partial}{\partial t_i}\int_{\mathbb{S}^3}\delta^2_{P_j,t_j}\delta_{P_i,t_i} \\
&\quad - \int_{\mathbb{S}^3} K\Big|\sum_{j=1}^k \alpha_j\delta_{P_j,t_j}+v\Big|^{1-\tau}\Big(\sum_{j=1}^k \alpha_j\delta_{P_j,t_j}+v\Big)\frac{\partial \delta_{P_i,t_i}}{\partial t_i} \\
&= \frac{1}{\alpha_i}\frac{\partial}{\partial t_i} I_\tau\Big(\sum_{j=1}^k \alpha_j\delta_{P_j,t_j}+v\Big) \\
&= \frac{1}{\alpha_i}\Big\{\frac{\Gamma_3}{K(P_i)^2}\frac{\tau}{t_i} + \frac{\Gamma_4\Delta_{g_0}K(P_i)}{K(P_i)^3}\frac{1}{t_i^3} \\
&\quad + \sum_{j\neq i}\frac{\Gamma_5 G_{P_i}(P_j)}{K(P_i)K(P_j)}\frac{1}{t_i^2 t_j} + V_{t_i}(\tau,\alpha,t,P,v)\Big\},
\end{aligned}
\tag{4.53}
$$

where $|V_{t_i}(\tau,\alpha,t,P,v)| = O(\tau^{3/2}|\log\tau|)$.

Applying (4.46) and (4.50), we obtain

$$
\begin{aligned}
\left\langle \eta, \frac{\partial \delta_{P_i,t_i}}{\partial P_i}\right\rangle &= I'_\tau(u)\frac{\partial \delta_{P_i,t_i}}{\partial P_i} \\
&= \frac{1}{\alpha_i}\frac{\partial}{\partial P_i} I_\tau\Big(\sum_{i=1}^k \alpha_i\delta_{P_i,t_i}+v\Big) \\
&= \frac{1}{\alpha_i}\left\{-\Gamma_6\nabla_{g_0}K(P_i) + V_{P_i}(\tau,\alpha,t,P,v)\right\},
\end{aligned}
\tag{4.54}
$$

with V_{P_i} satisfying $\left|V_{P_i}(\tau,\alpha,t,P,v)\right| \le C\tau^{1/2}|\ln\tau|$.

Under the conditions (4.51)–(4.54) stated above, we define a family of operators on $\widetilde{\Sigma}_\tau$ as follows: for $u=\sum_{i=1}^k \alpha_i\delta_{P_i,t_i}+v \in \widehat{\Sigma}_\tau$ given above,

$$
X_\theta(u) := \xi_\theta(u) + \eta_\theta(u), \quad 0\le\theta\le 1,
$$

where for any $\varphi \in E_{P,t}$,

$$
\langle \xi_\theta, \varphi\rangle := \theta f_\tau(\varphi) + (1-\theta)\langle v,\phi\rangle + 2\theta Q_\tau(\varphi,v) + \theta\langle V_v(\tau,\alpha,t,P,v),\varphi\rangle, \tag{4.55}
$$

and

$$
\begin{aligned}
\langle \eta_\theta, \delta_{P_i,t_i}\rangle :=& -2\pi|\mathbb{S}^3|\left\{\alpha_i - \frac{\theta}{K(P_i)} - \frac{(1-\theta)}{K(\overline{P}_i)}\right\} + \theta V_{\alpha_i}(\tau,\alpha,t,P,v),\\
\left\langle \eta_\theta, \frac{\partial\delta_{P_i,t_i}}{\partial t_i}\right\rangle :=& \left\{\frac{1}{\alpha_i} + (1-\theta)\right\}\left\{\frac{\Gamma_3}{K(P_i(\theta))^2}\frac{\tau}{t_i} + \frac{\Gamma_4\Delta_{g_0}K(P_i(\theta))}{K(P_i(\theta))^3}\frac{1}{t_i^3}\right.\\
&\left. + \sum_{j\neq i}\frac{\Gamma_5 G_{P_i(\theta)}(P_j(\theta))}{K(P_i(\theta))K(P_j(\theta))}\frac{1}{t_i^2 t_j}\right\} + \frac{\theta}{\alpha_i}V_{t_i}(\tau,\alpha,t,P,v),\\
\left\langle \eta_\theta, \frac{\partial\delta_{P_i,t_i}}{\partial P_i}\right\rangle :=& -\left\{(1-\theta) + \frac{\theta}{\alpha_i}\Gamma_6\right\}\nabla_{g_0}K(P_i) + \frac{\theta}{\alpha_i}V_{P_i}(\tau,\alpha,t,P,v),
\end{aligned}
\tag{4.56}
$$

where $P_i(\theta)$ is the short geodesic trajectory on $\mathbb{S}^3$ with $P_i(0) = \overline{P}_i$, $P_i(1) = P_i$.

Obviously, $X_1 = I'_\tau(u) = \xi + \eta$. From Sobolev compact embedding theorem and the explicit forms of V_v, V_{α_i}, V_{t_i}, V_{P_i}, we conclude that $I'_\tau(u)$ is of the form Id+compact on $\widehat{\Sigma}_\tau$. Since $\Omega_{\varepsilon_0/2}$ in the definition of $\widehat{\Sigma}_\tau$ is a finite dimensional submanifold of $H^\sigma(\mathbb{S}^3)$, we easily obtain from (4.55) and (4.56) that X_θ $(0 \le \theta \le 1)$ is the form Id+compact. Furthermore, we have $X_\theta \neq 0$ on $\partial\widehat{\Sigma}_\tau$, $\forall\, 0 \le \theta \le 1$. In fact, for a given $u = \sum_{i=1}^k \alpha_i\delta_{P_i,t_i} + v \in \partial\widehat{\Sigma}_\tau$, we obtain $\xi \neq 0$ by using (4.51) and (4.34). When $\theta = 0$, $\xi_0 = v \neq 0$. It follows from (4.55) that $\xi_\theta \neq 0$, $\forall\, 0 < \theta < 1$.

By the homotopy invariance of the Leray–Schauder degree, we have

$$
\deg_{H^\sigma}(X_1, \widehat{\Sigma}_\tau, 0) = \deg_{H^\sigma}(X_0, \widehat{\Sigma}_\tau, 0). \tag{4.57}
$$

From (4.55) and (4.56), we can obtain, for $u = \sum_{i=1}^k \alpha_i\delta_{P_i,t_i} + v \in \widehat{\Sigma}_\tau$,

$$
X_0(u) = \xi_0(u) + \eta_0(u),
$$

where $\xi_0 \in E_{P,t}$, $\eta_0 \in \operatorname{span}\left\{\delta_{P_i,t_i}, \frac{\partial\delta_{P_i,t_i}}{\partial t_i}, \frac{\partial\delta_{P_i,t_i}}{\partial P_i}\right\}$ satisfy

$$
\begin{aligned}
\langle \xi_0, \varphi\rangle &= \langle v, \varphi\rangle,\\
\langle \eta_0, \delta_{P_i,t_i}\rangle &= -2\pi|\mathbb{S}^3|(\alpha_i - K(\overline{P}_i)^{-1}),\\
\left\langle \eta_0, \frac{\partial\delta_{P_i,t_i}}{\partial t_i}\right\rangle &= \frac{\Gamma_3}{K(\overline{P}_i)^2}\frac{\tau}{t_i} + \frac{\Gamma_4\Delta_{g_0}K(\overline{P}_i)}{K(\overline{P}_i)^3}\frac{1}{t_i^3} + \sum_{j\neq i}\frac{\Gamma_5 G_{\overline{P}_i}(\overline{P}_j)}{K(\overline{P}_i)K(\overline{P}_j)}\frac{1}{t_i^2 t_j},\\
\left\langle \eta_0, \frac{\partial\delta_{P_i,t_i}}{\partial P_i}\right\rangle &= -\nabla_{g_0}K(P_i).
\end{aligned}
\tag{4.58}
$$

Recalling the definition of $M(\overline{P}_1, \ldots, \overline{P}_k)$, which is simply written as (M_{ij}). By (4.59), we can easily get

$$
X_0(u) = 0 \quad \text{on } \widehat{\Sigma}_\tau,
$$

if and only if

$$\begin{aligned}
&\alpha_i = K(\overline{P}_i)^{-1}, \quad P_i = \overline{P}_i, \quad v = 0, \\
&\frac{4}{K(P_i)^2}\frac{\tau}{t_i} - \Big(M_{ii}\frac{1}{t_i^3} + \sum_{j=1}^{k} M_{ij}\frac{1}{t_i^2 t_j}\Big) = 0.
\end{aligned} \tag{4.59}$$

For any $(s_1, \ldots, s_k) \in \mathbb{R}^k$, $s_i > 0$, $i = 1, \ldots, k$, we define

$$F(s_1, \ldots, s_k) := -\sum_{j=1}^{k}\Big(\frac{4\tau}{K(\overline{P}_j)^2}\log s_j\Big) + \frac{1}{2}\sum_{i=1}^{k}\Big(M_{ii}s_i^2 + \sum_{j=1}^{k} M_{ij}s_i s_j\Big),$$

and for $t_i = s_i^{-1}$,

$$\widehat{F}(t_1, \ldots, t_k) := F(s_1, \ldots, s_k).$$

The derivative with respect to t_i is

$$\frac{\partial \widehat{F}}{\partial t_i}(t_1, \ldots, t_k) = \frac{4}{K(\overline{P}_i)^2}\frac{\tau}{t_i} - \Big(M_{ii}\frac{1}{t_i^3} + \sum_{j=1}^{k} M_{ij}\frac{1}{t_i^2 t_j}\Big),$$

combining this and (4.58), we have

$$\Big\langle \eta_0, \frac{\partial \delta_{P_i,t_i}}{\partial t_i}\Big\rangle = \frac{\pi|\mathbb{S}^2|}{3}\frac{\partial \widehat{F}}{\partial t_i}(t_1, \ldots, t_k).$$

It is obvious that $\nabla\widehat{F}(t_1, \ldots, t_k) = 0$ if and only if $\nabla F(s_1, \ldots, s_k) = 0$. A trivial verification shows that $F(s_1, \ldots, s_k)$ is a strictly convex function, and having a unique critical point in the first quadrant. It follows that $\widehat{F}(t_1, \ldots, t_k)$ has unique critical point in the first quadrant with Morse index zero. Hence X_0 has precisely one nondegenerate zero in $\widehat{\Sigma}_\tau$. Furthermore, by (4.59) we can easily obtain

$$\deg_{H^\sigma}(X_0, \widehat{\Sigma}_\tau, 0) = (-1)^{k+\sum_{i=1}^{k} i(\overline{P}_i)}. \tag{4.60}$$

Combining (4.60) and (4.57), we complete the proof of Theorem 4.1. □

Recall the definition of $\mathscr{O}_R$ in (1.21). For $\delta > 0$ suitably small, define

$$\mathscr{O}_{R,\delta} := \{u \in H^\sigma(\mathbb{S}^3) : \inf_{\omega\in\mathscr{O}_R} \|u-\omega\|_\sigma < \delta\}. \tag{4.61}$$

Proposition 4.6 *Let $\sigma = 1/2$, $K \in \mathscr{A}$ be a Morse function and $0 < \tau_0 \leq \tau \leq 4/(n-2\sigma) - \tau_0$. Then there exists some constants $C_0 > 0$, $\delta_0 > 0$ depending only on τ_0, $\min_{\mathbb{S}^3} K$, and the modulo of the continuity of K, such that*

$$\{u \in H^\sigma(\mathbb{S}^3) : u > 0 \ \ a.e., \ I'_\tau(u) = 0\} \subset \mathscr{O}_{C_0,\delta_0}. \tag{4.62}$$

Furthermore, we have $I'_\tau(u) \neq 0$ on $\partial \mathscr{O}_{C_0,\delta_0}$ and

$$\deg_{H^\sigma}\left(u - P_\sigma^{-1}(K|u|^{1-\tau}u), \mathscr{O}_{C_0,\delta_0}, 0\right) = -1. \tag{4.63}$$

Proof From Proposition 4.1, we know that for $\tau > 0$ small there exists some suitable value of ν_0, A, R such that u satisfying $u \in H^\sigma(\mathbb{S}^3)$, $u > 0$, $a.e.$, $I'_\tau(u) = 0$ are either in $\mathscr{O}_R$ or in some $\Sigma_\tau(q^{(1)}, \ldots, q^{(k)})$. Combining (4.21), (4.5), (A.1), and (A.16), we conclude that there exists some positive constants C_0 and δ_0 such that (4.62) holds.

For $K^*(x) = x^{(4)} + 2$, $x = (x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}) \in \mathbb{S}^3 \subset \mathbb{R}^4$ and $t \in (0,1)$, we consider $K_t = tK + (1-t)K^*$. By the homotopy invariance of the Leray-Schauder degree, we only need to establish (4.63) for K^* and τ very small. It is easy to see that $K^* \in \mathscr{A}$ is a Morse function. The proof of (4.63) is straightforward by the Kazdan-Warner condition and Theorem 4.1. □

4.2 The Proof of the Theorems 1.3 and 1.4

Using Theorem 4.1 and Proposition 4.6, we next prove Theorem 1.3.

Proof *(Proof of Theorem 1.3)* Using Theorem 1.2 and the homotopy invariance of the Leray-Schauder degree, for $\tau > 0$ sufficiently small, we obtain that there exist a constant R such that,

$$\deg_{C^{2,\alpha}}(u - P_\sigma^{-1}(Ku^2), \mathscr{O}_R, 0) = \deg_{C^{2,\alpha}}(u - P_\sigma^{-1}(K|u|^{1-\tau}u), \mathscr{O}_R, 0). \tag{4.64}$$

For $C_0 \gg R$, $0 < \delta_1 \ll \delta_0$, and τ_0 be given by Proposition 4.6. Using (4.63), Proposition 4.1, (4.22), and the excision property of the degree, we have

$$\deg_{H^\sigma}(u - P_\sigma^{-1}(K|u|^{1-\tau}u), \mathscr{O}_{R,\delta_1}, 0) = \mathrm{Index}(K). \tag{4.65}$$

As in the proof of Proposition 4.6, one can check that there are no critical points of I_τ in $\overline{\mathscr{O}_{R,\delta_1}} \backslash \mathscr{O}_R$. Using the same proof idea as Li [26, Theorem B.2], we can easily get

$$\deg_{C^{2,\alpha}}(u - P_\sigma^{-1}K(|u|^{1-\tau}u), \mathscr{O}_R, 0) = \deg_{H^\sigma}(u - P_\sigma^{-1}K(|u|^{1-\tau}u), \mathscr{O}_{R,\delta_1}, 0). \tag{4.66}$$

It follows from (4.64)–(4.66) that for $R > C$, (1.22) is proved. Theorem 1.3 follows from the above. □

Using the theory of linear algebra, we give the proof of Corallary 1.1.

Proof of the Corollary 1.1 If $\sharp\mathscr{K}^- = 1$, from the proof of Theorem 1.3, we can easily obtain the conclusion. If for any distinct $P, Q \in \mathscr{K}^-$, $\Delta_{g_0}K(P)\Delta_{g_0}K(Q) < 9K(P)K(Q)$, we claim that there is no integer $k \geq 2$ such that $q^{(1)}, \ldots, q^{(k)} \in \mathscr{K}^-$, $\mu(M(q^{(1)}, \ldots, q^{(k)})) > 0$.

In fact, for any distinct $q^{(1)}, \ldots, q^{(k)} \in \mathscr{K}^-$, $k \geq 2$, $\mu(M(q^1, \ldots, q^{(k)})) > 0$ if and only if $M(q^{(1)}, \ldots, q^{(k)})$ is a positive definite matrix. By (1.23), we have the fact that 2-order principle minor determinant strictly less than zero. Therefore, we proved the claim. Obviously, Corollary 1.1 follows from the claim. □

We next prove Theorem 1.4.

Proof of the Theorem 1.4 Since the Morse functions in $C^2(\mathbb{S}^3)^+\backslash\mathscr{A} = \partial\mathscr{A}$ are dense in $\partial\mathscr{A}$, without loss of generality we consider the case that $K \in \partial\mathscr{A}$ is a Morse function. First recall the definition of $\mathscr{K}$ and $\mathscr{K}^+$, we can assume here $\mathscr{K}\backslash\mathscr{K}^+ = \{q^{(1)}, \ldots, q^{(m)}\}$, $m \in \mathbb{N}_+$. From the definition of $\mathscr{A}$ and $K \in \partial\mathscr{A}$, we know that there exists $1 \leq i_1 < \cdots < i_k \leq m$, $k \geq 1$, such that

$$\mu(M(q^{(i_1)}, \ldots, q^{(i_k)})) = 0. \tag{4.67}$$

By perturbing the function K near its some critical points to change the Hessian matrix of K at these points, we obtain a sequence of K_ℓ satisfying: $K_\ell \to K$ in $C^2(\mathbb{S}^3)^+$ as $\ell \to \infty$; K_ℓ are identically the same as K except in some small balls and have the same critical points with the same Morse index; there is only one such $(i_1, \ldots, i_k)$ such that (4.67) is true for any ℓ. Refer to the perturbation method as in the proof of Li [27, Theorem 0.8] for more details. Using the same C^2 perturbation method for K_ℓ, we can obtain a smooth, one-parameter family of Morse functions $\{K_{\ell,t}\}$ $(-1 \leq t \leq 1)$ with the following properties:

(a) $K_{\ell,t}$ $(-1 \leq t \leq 1)$ are identically the same as K_ℓ except in some small balls around $q^{(i_1)}, \ldots, q^{(i_k)}$ and $K_0 = K_\ell$. $K_{\ell,t}$ have the same critical points with the same Morse index for any $-1 \leq t \leq 1$.
(b) $\mu(M_{\ell,t}(q^{(j_1)}, \ldots, q^{(j_s)}))$ have the same sign for $-1 < t < 1$ for any $1 \leq j_1 < \cdots < j_s \leq m$, $(j_1, \ldots, j_s) \neq (i_1, \ldots, i_k)$. $\mu(M_{\ell,t}(q^{(i_1)}, \ldots, q^{(i_k)})) < 0$ for $-1 < t < 0$, and $\mu(M_{\ell,t}(q^{(i_1)}, \ldots, q^{(i_k)})) > 0$ for $0 < t < 1$.

It is easily seen that $K_{\ell,t} \in \mathscr{A}$ when $t \neq 0$. From the definition of Index, we have

$$\mathrm{Index}(K_1) = \mathrm{Index}(K_{-1}) + (-1)^{k-1+\sum_{j=1}^{k} i(q^{(i_j)})},$$

thus, $\mathrm{Index}(K_1) \neq \mathrm{Index}(K_{-1})$. By the homotopy invariance of the Leray-Schauder degree, there exists $t_i \to 0$ and $v_{\ell,i} \in \mathscr{M}_{K_{\ell,t_i}}$, such that

$$\lim_{i\to\infty} \|v_{\ell,i}\|_{C^{2,\alpha}(\mathbb{S}^3)} = \infty \quad \text{or} \quad \lim_{i\to\infty} (\min_{\mathbb{S}^3} v_{\ell,i}) = 0.$$

Combining the Harnack inequality in [20, Lemma 4.3] and Schauder estimates in [20, Theorem 2.11], we deduce that (1.24) holds. It follows from Theorem 1.2, $K_{\ell,t} \in \mathscr{A}$

($t \neq 0$) and Theorem 1.1 that $K_{\ell,t_i} \to K_\ell$ and $\{v_{\ell,i}\}$ blows up exactly at k points $q^{(i_1)}, \dots, q^{(i_k)}$.

From the above, we know that there exists a sequence of $K_i \to K$ in $C^2(\mathbb{S}^3)$, $v_i \in \mathscr{M}_{K_i}$ such that $\{v_i\}$ blows up at precisely the k points $q^{(i_1)}, \dots, q^{(i_k)}$. We have, thus, proved Theorem 1.4. □

Proof of Theorem 1.5 By using Theorem 1.1 we can prove the Part (i) of Theorem 1.5. The Part (ii) of Theorem 1.5 is similar to the proof of Theorem 1.4, we omit it here. □

A Appendix

In this appendix, we provide some elementary calculations which have been used in the proof of Theorem 1.3.

Lemma A.1 *Let $\alpha \geq 2$, there exists a positive constant C depending only on α such that, for any $a \geq 0$, $b \in \mathbb{R}$,*

$$\left||a+b|^{\alpha-1}(a+b) - a^\alpha - \alpha a^{\alpha-1}b - \frac{\alpha(\alpha-1)}{2}a^{\alpha-2}b^2\right| \leq C\left(|b|^\alpha + a^\gamma|b|^{\alpha-\gamma}\right),$$

where $\gamma = \max\{0, \alpha-3\}$.

Lemma A.2 *Let $1 < \beta < 2$, there exists a universal positive constant C such that, for any $a > 0$, $b \in \mathbb{R}$,*

$$\left||a+b|^{\beta-1}(a+b) - a^\beta - \beta a^{\beta-1}b\right| \leq C|b|^\beta.$$

Lemma A.3 *Let $\beta > 1$ and $k \in \mathbb{N}_+$, there exists a constant C, such that for any $(a_1, \dots, a_k) \in \mathbb{R}^k$,*

$$\left|\Big(\sum_{i=1}^k a_i\Big)^\beta - \sum_{i=1}^k a_i^\beta\right| \leq C\sum_{i\neq j}|a_i|^{\beta-1}|a_j|.$$

Lemma A.4 *Let $\varepsilon_0, \tau > 0$ be suitably small and $A > 0$ be suitably large. Let $A^{-1}\tau^{-1/2} < t_1, t_2 < A\tau^{-1/2}$, $P_1, P_2 \in \mathbb{S}^3$, $|P_1 - P_2| \geq \varepsilon_0$, δ_{P_i,t_i} be as in (4.3) and $G_{P_1}(P_2)$ be as in (1.12) ($|P_1 - P_2|$ represents the distance between two points P_1*

and P_2 after through a stereographic projection). Then, we have,

$$\int_{\mathbb{S}^3} \delta_{P_1,t_1}^2 \delta_{P_2,t_2} = 4\pi|\mathbb{S}^2| \frac{G_{P_1}(P_2)}{t_1 t_2} + O(\tau^{3/2}), \tag{A.1}$$

$$\int_{\mathbb{S}^3} \delta_{P_1,t_1}^{2-\tau} \delta_{P_2,t_2} = O(\tau), \tag{A.2}$$

$$\frac{\partial}{\partial t_1} \int_{\mathbb{S}^3} \delta_{P_1,t_1}^2 \delta_{P_2,t_2} = -4\pi|\mathbb{S}^2| \frac{G_{P_1}(P_2)}{t_1^2 t_2} + O(\tau^2), \tag{A.3}$$

$$\frac{\partial}{\partial t_1} \int_{\mathbb{S}^3} \delta_{P_2,t_2} \delta_{P_1,t_1}^{2-\tau} = \int_{\mathbb{S}^3} \delta_{P_2,t_2} \delta_{P_1,t_1}^2 + O(\tau^{5/2}|\log\tau|), \tag{A.4}$$

$$\frac{\partial}{\partial t_1} \int_{\mathbb{S}^3} \delta_{P_2,t_2}^{2-\tau} \delta_{P_1,t_1} = \frac{\partial}{\partial t_1} \int_{\mathbb{S}^3} \delta_{P_2,t_2}^2 \delta_{P_1,t_1} + O(\tau^{5/2}|\log\tau|), \tag{A.5}$$

$$\int_{\mathbb{S}^3} |P - P_1|^2 \delta_{P_1,t_1}^{3-\tau} = \frac{1}{t_1^2} \frac{3\pi}{2} |\mathbb{S}^2| + O(\tau^2|\log\tau|), \tag{A.6}$$

$$\frac{\partial}{\partial t_1} \int_{\mathbb{S}^3} \delta_{P_1,t_1}^{3-\tau} = -\frac{\tau}{t_1} \frac{\pi}{2} |\mathbb{S}^2| + O(\tau^{5/2}|\log\tau|), \tag{A.7}$$

$$\frac{\partial}{\partial t_1} \int_{\mathbb{S}^3} |P - P_1|^2 \delta_{P_1,t_1}^{3-\tau} = -\frac{3\pi}{t_1^3} |\mathbb{S}^2| + O(\tau^{5/2}|\log\tau|). \tag{A.8}$$

Proof Because the computation is elementary and routine, we only take (A.2) as an example to prove.

By the stereographic projection, we have

$$\begin{aligned} \int_{\mathbb{S}^3} \delta_{P_1,t_1}^{2-\tau} \delta_{P_2,t_2} &= \int_{\mathbb{R}^3} \left(\frac{2t_1}{(1 + t_1^2|y - y_1|^2)} \right)^{2-\tau} \left(\frac{2t_2}{(1 + t_2^2|y - y_2|^2)} \right) \left(\frac{2}{1 + |y|^2} \right)^{\tau} dy \\ &\le C_1 \int_{\mathbb{R}^3} \frac{t_1^{2-\tau} t_2}{(1 + t_1^2|y - y_1|^2)^{2-\tau}(1 + t_2^2|y - y_2|^2)} dy =: C_1 \widetilde{\mathcal{I}}_{2-\tau}. \end{aligned}$$

Let

$$a_{12} = \frac{y_2 - y_1}{2}; \quad z = y - \frac{y_1 + y_2}{2}.$$

Then we have

$$\widetilde{\mathcal{I}}_{2-\tau} = \frac{1}{t_1^{2-\tau} t_2} \int_{\mathbb{R}^3} \frac{1}{(\frac{1}{t_1^2} + |z + a_{12}|^2)^{2-\tau} (\frac{1}{t_2^2} + |z - a_{12}|^2)} dy.$$

By symmetry arguments, we may assume

$$t_2 \le t_1.$$

Because we have

$$t_1 t_2 |a_{12}|^2 \geq C \frac{t_1}{t_2}, \quad \text{with} C \text{a large constant.}$$

Thus,

$$t_2^2 |a_{12}|^2 \geq C; \quad t_1^2 |a_{12}|^2 \geq C,$$

and it follows that if $|z + a_{12}| \leq \frac{1}{t_1} \leq |a_{12}|$, then $|z - a_{12}| \geq |a_{12}|$; and if $|z - a_{12}| \leq \frac{1}{t_2} \leq |a_{12}|$, then $|z + a_{12}| \geq |a_{12}|$.

Then

$$\begin{aligned} \widetilde{\mathcal{I}}_{2-\tau} \leq \frac{1}{t_1^{2-\tau} t_2} \Big\{ \int_{|z+a_{12}| \leq \frac{1}{t_1}} \frac{C_1 t_1^{2(2-\tau)}}{(\frac{1}{t_2^2} + |a_{12}|^2)} + \int_{|z-a_{12}| \leq \frac{1}{t_2}} \frac{C_1 t_2^2}{(\frac{1}{t_1^2} + |a_{12}|^2)^{2-\tau}} \Big\} \\ + \frac{2^3}{t_1^{2-\tau} t_2} \Big\{ \int_{\substack{|z+a_{12}| > \frac{1}{t_1} \\ |z-a_{12}| > \frac{1}{t_2}}} \frac{C_1}{|z + a_{12}|^{2(2-\tau)} |z - a_{12}|^2} \, dz \Big\}, \end{aligned} \tag{A.9}$$

where C_1 is a constant.

Since

$$\begin{aligned} &\int_{\substack{|z+a_{12}| > \frac{1}{t_1} \\ |z-a_{12}| > \frac{1}{t_2}}} \frac{1}{|z + a_{12}|^{2(2-\tau)} |z - a_{12}|^2} \, dz \\ &\leq \int_{\frac{1}{t_1} < |z+a_{12}| \leq |a_{12}|} \frac{dz}{|z + a_{12}|^{2(2-\tau)} |a_{12}|^2} + \int_{\frac{1}{t_2} < |z-a_{12}| \leq |a_{12}|} \frac{dz}{|a_{12}|^{2(2-\tau)} |z - a_{12}|^2} \\ &\quad + \int_{\substack{|z+a_{12}| > |a_{12}| \\ |z-a_{12}| > |a_{12}|}} \frac{dz}{|z + a_{12}|^{2(2-\tau)} |z - a_{12}|^2}. \end{aligned} \tag{A.10}$$

It is easy to see that if $|z \pm a_{12}| \leq |a_{12}|$, then $|z \mp a_{12}| \geq |a_{12}|$ again, and if $|z + a_{12}| \geq |a_{12}|$ and $|z - a_{12}| \geq |a_{12}|$, then $|z - a_{12}| \geq \frac{1}{C_2^{\frac{1}{3}}} |z + a_{12}|$ with C_2 large enough. if $|z - a_{12}| > |a_{12}|$, then $|z - a_{12}| > |a_{12}| > |z + a_{12}|$, we have

$$\begin{aligned} &\int_{\substack{|z+a_{12}| > |a_{12}| \\ |z-a_{12}| > |a_{12}|}} \frac{dz}{|z + a_{12}|^{2(2-\tau)} |z - a_{12}|^2} \\ &\leq C_2 \int_{|z+a_{12}| \geq |a_{12}|} \frac{dz}{|z + a_{12}|^{2(2-\tau)+2}} = \frac{C_2 |\mathbb{S}^2|}{(3 - 2\tau)} \frac{1}{|a_{12}|^{3-2\tau}}. \end{aligned}$$

On the other hand,

$$\frac{1}{|a_{12}|^2}\int_{\frac{1}{t_1}<|z+a_{12}|\le|a_{12}|}\frac{dz}{|z+a_{12}|^{2(2-\tau)}}$$
$$=\frac{|\mathbb{S}^2|}{|a_{12}|^2}\int_{\frac{1}{t_1}}^{|a_{12}|}\frac{1}{r^{2-2\tau}}dr=\frac{|\mathbb{S}^2|}{|a_{12}|^2}\frac{1}{2\tau-1}(|a_{12}|^{2\tau-1}-\frac{1}{t_1^{2\tau-1}}),\tag{A.11}$$

and

$$\frac{1}{|a_{12}|^{2(2-\tau)}}\int_{\frac{1}{t_2}<|z-a_{12}|\le|a_{12}|}\frac{dz}{|z-a_{12}|^2}$$
$$=\frac{|\mathbb{S}|^2}{|a_{12}|^{2(2-\tau)}}\int_{\frac{1}{t_2}<|z-a_{12}|\le|a_{12}|}\frac{1}{r}dr=\frac{|\mathbb{S}|^2}{|a_{12}|^{2(2-\tau)}}(|a_{12}|-\frac{1}{t_2}).\tag{A.12}$$

By (A.10), (A.11) and (A.12), we have

$$\frac{2^3}{t_1^{2-\tau}t_2}\int_{\substack{|z+a_{12}|>\frac{1}{t_1}\\|z-a_{12}|>\frac{1}{t_2}}}\frac{1}{|z+a_{12}|^{2(2-\tau)}|z-a_{12}|^2}dz$$
$$\le C\left(\frac{1}{\mu^{\frac{3}{2}-\tau}t_2^{\tau}}+\frac{1}{\mu t_1^{\tau}}+\frac{1}{\mu^{\frac{3}{2}-\tau}t_2^{\tau}}+\frac{1}{\mu^{2-\tau}t_2^{\tau}}+\frac{1}{\mu^{\frac{3}{2}-\tau}t_2^{\tau}}\right)$$
$$\le C(\tau^{\frac{3}{2}}+\tau+\tau^2)=O(\tau),\tag{A.13}$$

recalling that (A.9), one has

$$\frac{2^3}{t_1^{2-\tau}t_2}\int_{|z+a_{12}|\le\frac{1}{t_1}}\frac{C_1t_1^{2(2-\tau)}}{(\frac{1}{t_2^2}+|a_{12}|^2)}$$
$$\le C\frac{1}{t_1^{\tau}(\frac{t_1}{t_2}+\mu)}=O\left(\mu^{-1}\right)=O(\tau),\tag{A.14}$$

similarly,

$$\frac{2^3}{t_1^{2-\tau}t_2}\int_{|z-a_{12}|\le\frac{1}{t_2}}\frac{C_1t_2^2}{(\frac{1}{t_1^2}+|a_{12}|^2)^{2-\tau}}=O\left(\mu^{-(2-\tau)}\right)=O(\tau^2).\tag{A.15}$$

Thus, by (A.13), (A.14) and (A.15), we obtain (A.2). □

Lemma A.5 *Under the hypotheses of Lemma A.4, in addition that Γ_1, Γ_2 are positive constants independent of τ. Then, we have,*

$$\langle \delta_{P_1,t_1}, \delta_{P_1,t_1} \rangle = |\mathbb{S}^3|, \quad \left\langle \frac{\partial \delta_{P_1,t_1}}{\partial P_1^{(\ell)}}, \frac{\partial \delta_{P_1,t_1}}{\partial P_1^{(\ell)}} \right\rangle = \Gamma_1 t_1^2, \tag{A.16}$$

$$\left\langle \frac{\partial \delta_{P_1,t_1}}{\partial t_1}, \frac{\partial \delta_{P_1,t_1}}{\partial t_1} \right\rangle = \Gamma_2 t_1^{-2}, \tag{A.17}$$

$$\langle \delta_{P_1,t_1}, \delta_{P_2,t_2} \rangle = O(\tau), \tag{A.18}$$

$$\|\delta_{P_1,t_1}^{1-\tau} \delta_{P_2,t_2}\|_{L^{3/2}(\mathbb{S}^3)} = O(\tau |\log \tau|), \tag{A.19}$$

$$\left\| \delta_{P_2,t_2}^{1-\tau} \frac{\partial \delta_{P_1,t_1}}{\partial t_1} \right\|_{L^{3/2}(\mathbb{S}^3)} = O(\tau^{3/2} |\log \tau|), \tag{A.20}$$

$$\left\| \delta_{P_1,t_1}^{1-\tau} \delta_{P_2,t_2} \frac{\partial \delta_{P_1,t_1}}{\partial t_1} \right\|_{L^1(\mathbb{S}^3)} = O(\tau^{3/2} |\log \tau|), \tag{A.21}$$

$$\begin{aligned} &\|\delta_{P_1,t_1}^{2-\tau} - \delta_{P_1,t_1}^2\|_{L^{3/2}(\mathbb{S}^3)} = O(\tau |\log \tau|), \\ &\|\delta_{P_1,t_1}^{1-\tau} - \delta_{P_1,t_1}\|_{L^3(\mathbb{S}^3)} = O(\tau |\log \tau|), \end{aligned} \tag{A.22}$$

$$\|\delta_{P_1,t_1}^{3-\tau} - \delta_{P_1,t_1}^3\|_{L^1(\mathbb{S}^3)} = O(\tau |\log \tau|), \tag{A.23}$$

$$\begin{aligned} &\big\| |P - P_1| \delta_{P_1,t_1}^2 \big\|_{L^{3/2}(\mathbb{S}^3)} = O(\tau^{1/2}), \\ &\big\| |P - P_1|^2 \delta_{P_1,t_1}^2 \big\|_{L^{3/2}(\mathbb{S}^3)} = O(\tau), \end{aligned} \tag{A.24}$$

$$\int_{\mathbb{S}^3} |P - P_1|^3 \delta_{P_1,t_1} \frac{\partial \delta_{P_1,t_1}}{\partial t_1} = O(\tau^2). \tag{A.25}$$

Proof Because the computation is elementary and routine, we only take (A.16) and (A.17) as an example to prove.

Proof of:(A.16)

$$\begin{aligned} \langle \delta_{P_1,t_1}, \delta_{P_1,t_1} \rangle = \int_{\mathbb{S}^3} \delta_{P_1,t_1}^3 &= 2^3 \int_{\mathbb{R}^3} \frac{t_1^3}{(1 + t_1^2 |y|^2)^3} \, dy \\ &= 2^3 |\mathbb{S}^2| \int_0^\infty \frac{r^2}{(1 + r^2)^3} \, dr = |\mathbb{S}^3|. \end{aligned}$$

Since $P_\sigma \delta_{P_1,t_1} = \delta^2_{P_1,t_1}$, then $P_\sigma\big(\frac{\partial \delta_{P_1,t_1}}{\partial P_1}\big) = 2\delta_{P_1,t_1}\big(\frac{\partial \delta_{P_1,t_1}}{\partial P_1}\big)$. It follows that

$$
\begin{aligned}
\left\langle \frac{\partial \delta_{P_1,t_1}}{\partial P_1^{(\ell)}}, \frac{\partial \delta_{P_1,t_1}}{\partial P_1^{(\ell)}} \right\rangle &= \int_{\mathbb{S}^3} \left(P_\sigma \frac{\partial \delta_{P_1,t_1}}{\partial P_1^{(\ell)}}\right) \frac{\partial \delta_{P_1,t_1}}{\partial P_1^{(\ell)}} \\
&= \int_{\mathbb{S}^3} 2\delta_{P_1,t_1} \left(\frac{\partial \delta_{P_1,t_1}}{\partial P_1^{(\ell)}}\right)^2 \\
&= \int_{\mathbb{R}^3} \frac{2t_1}{1+t_1^2|y|^2}\left[\frac{4t_1^3(y)^{(\ell)}}{(1+t_1^2|y|^2)^2}\right]^2 dy \\
&= 2^5 \int_{\mathbb{R}^3} \frac{t_1^2(x^{(\ell)})^2}{(1+|x|^2)^5} = \Gamma_1 t_1^2.
\end{aligned}
$$

where Γ_1 is a constant.

Proof of:(A.17) Since $P_\sigma \delta_{P_1,t_1} = \delta^2_{P_1,t_1}$, then $P_\sigma\big(\frac{\partial \delta_{P_1,t_1}}{\partial t_1}\big) = 2\delta_{P_1,t_1}\big(\frac{\partial \delta_{P_1,t_1}}{\partial t_1}\big)$. It follows that

$$
\begin{aligned}
&\int_{\mathbb{S}^3} 2\delta_{P_1,t_1}\left(\frac{\partial \delta_{P_1,t_1}}{\partial t_1}\right)^2 \\
&= 2\int_{\mathbb{R}^3} \frac{2t_1}{1+t_1^2|y|^2}\left(\frac{2}{1+t_1^2|y|^2} - \frac{2^2 t_1^2|y|^2}{(1+t_1^2|y|^2)^2}\right)^2 dy \\
&= 2^4 t_1^{-2} \int_{\mathbb{R}^3} \frac{1}{(1+|x|^2)^3}\left(1 - \frac{2|x|^2}{1+|x|^2}\right)^2 dy := \Gamma_2 t_1^{-2}
\end{aligned}
$$

where Γ_2 is a constant. □

Lemma A.6 *Let ε_0, τ, A be as in Lemma A.4, $P_1, P_2, P_3 \in \mathbb{S}^3$ satisfy $|P_i - P_j| \geq \varepsilon_0$, $i \neq j$, and $A^{-1}\tau^{-1/2} < t_1, t_2, t_3 \leq A\tau^{-1/2}$. Then, we have,*

$$
\left|\frac{\partial}{\partial P_1}\int_{\mathbb{S}^3} \delta^{3-\tau}_{P_1,t_1}\right| = O(\tau^{1/2}), \quad \left\|\delta_{P_2,t_2}\frac{\partial \delta_{P_1,t_1}}{\partial P_1}\right\|_{L^{3/2}(\mathbb{S}^3)} = O(\tau^{1/2}|\log\tau|), \tag{A.26}
$$

$$
\left|\frac{\partial}{\partial P_1}\int_{\mathbb{S}^3} \delta^{2-\tau}_{P_2,t_2}\delta_{P_1,t_1}\right| = O(\tau^{1/2}), \quad \left|\frac{\partial}{\partial P_1}\int_{\mathbb{S}^3} \delta^{2}_{P_2,t_2}\delta_{P_1,t_1}\right| = O(\tau^{1/2}), \tag{A.27}
$$

$$
\left\|\delta^{1-\tau}_{P_2,t_2}\delta_{P_3,t_3}\frac{\partial \delta_{P_1,t_1}}{\partial P_1}\right\|_{L^1(\mathbb{S}^3)} = O(\tau^{1/2}|\log\tau|). \tag{A.28}
$$

Lemma A.7 *In addition to the hypotheses of Lemma A.4, we assume that $K \in C^1(\mathbb{S}^3)$. Then*

$$
\frac{\partial}{\partial t_1}\int_{\mathbb{S}^3}(K(P) - K(P_2))\delta^{2-\tau}_{P_2,t_2}\delta_{P_1,t_1} = O(\tau^2). \tag{A.29}
$$

Lemma A.8 *In addition to the hypotheses of Lemma A.7, we assume that $v \in E_{P_1,t_1}$. Then*

$$\int_{\mathbb{S}^3} (K(P) - K(P_1))\delta_{P_1,t_1}^{1-\tau} \frac{\partial \delta_{P_1,t_1}}{\partial P_1} v = O(\tau^{1/2}|\log\tau|\|v\|_\sigma). \tag{A.30}$$

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On the Existence of Solutions for Prescribing Fractional Q-curvature Problem on $\mathbb{S}^n$

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Abstract The aim of this paper is to investigate the existence of solutions to the prescribing fractional Q-curvature problem on $\mathbb{S}^n$ under some reasonable assumption of the Laplacian sign at the critical point of prescribing curvature function K. Due to the lack of compactness, we choose to return to the basic elements of variational theory and study the deformation along the flow lines. The novelty of the paper is that we obtain the existence without assuming any symmetry and periodicity on K. In addition, to overcome the loss of compactness for high-order operator problem, we need more delicate estimates with the second order cases.

Keywords Prescribing curvature problem, existence, critical exponent

MR(2020) Subject Classification 35G20, 35A15, 35B38

1 Introduction

Conformal geometry is an in-depth and complex research field aimed at revealing the geometric properties of manifolds, which has received extensive research in recent years. In this paper, we mainly study the prescribing fractional Q-curvature problem on n-dimensional standard sphere $(\mathbb{S}^n, g_0)$ in conformal geometry. This problem can be described as: which function K on $(\mathbb{S}^n, g_0)$ is the fractional Q-curvature of a metric g that is conformal to g_0? This problem involves a class of nonlinear partial differential equations derived from conformal deformations of Riemannian metrics. To be more precise, if we denote $g = v^{\frac{4}{(n-2\sigma)}} g_0$, this problem can be represented as finding the solution of the following nonlinear equation with critical exponent:

$$P_\sigma^{g_0}(v) = c(n,\sigma) K v^{\frac{n+2\sigma}{n-2\sigma}} \quad \text{on } \mathbb{S}^n, \tag{1.1}$$

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where $n \geq 2$, $0 < \sigma < \frac{n}{2}$, $c(n,\sigma) = \Gamma(\frac{n}{2}+\sigma)/\Gamma(\frac{n}{2}-\sigma)$, Γ is the Gamma function, K is a function defined on $\mathbb{S}^n$, and $P_\sigma^{g_0}$ is an intertwining operator of 2σ-order:

$$P_\sigma^{g_0} = \frac{\Gamma(B+\frac{1}{2}+\sigma)}{\Gamma(B+\frac{1}{2}-\sigma)}, \quad B = \sqrt{-\Delta_{g_0} + \Big(\frac{n-1}{2}\Big)^2}.$$

It can be viewed as the pull back operator of the σ power of the Laplacian $(-\Delta)^\sigma$ on $\mathbb{R}^n$ via the stereographic projection:

$$(P_\sigma^{g_0}(v)) \circ F = |J_F|^{-\frac{n+2\sigma}{2n}} (-\Delta)^\sigma (|J_F|^{\frac{n-2\sigma}{2n}} (v \circ F)) \quad \text{for } v \in C^{2\sigma}(\mathbb{S}^n),$$

where F is the inverse of the stereographic projection and $|J_F|$ is the determinant of the Jacobian of F. The Green's function of $P_\sigma^{g_0}$ is the spherical Riesz potential, i.e.,

$$(P_\sigma^{g_0})^{-1} f(\xi) = c_{n,\sigma} \int_{\mathbb{S}^n} \frac{f(\zeta)}{|\xi-\zeta|^{n-2\sigma}} dv_{g_0}(\zeta) \quad \text{for } f \in L^p(\mathbb{S}^n),$$

where $c_{n,\sigma} = \frac{\Gamma(\frac{n-2\sigma}{2})}{2^{2\sigma}\pi^{\frac{n}{2}}\Gamma(\sigma)}$, $p > 1$, and $|\cdot|$ is the Euclidean distance in $\mathbb{R}^{n+1}$.

The prescribing fractional Q-curvature problem can be viewed as extensions and generalizations of several problems in conformal geometry. For $\sigma = 1$, it is the scalar curvature problem or the classical Nirenberg problem: which function K on $(\mathbb{S}^n, g_0)$ is the scalar curvature (Gauss curvature in dimension $n = 2$) of a metric g that is conformal to g_0? If we denote $g = \mathrm{e}^{2v} g_0$ in the two dimensional case and $g = v^{\frac{4}{n-2}} g_0$ in the n ($n \geq 3$) dimensional case, this problem is equivalent to solving the following nonlinear elliptic equations:

$$-\Delta_{g_0} v + 1 = K\mathrm{e}^{2v} \quad \text{on } \mathbb{S}^2,$$

and

$$-\Delta_{g_0} v + c(n) R_0 v = c(n) K v^{\frac{n+2}{n-2}} \quad \text{on } \mathbb{S}^n, \quad n \geq 3,$$

where Δ_{g_0} is the Laplace–Beltrami operator, $c(n) = \frac{n-2}{4(n-1)}$, $R_0 = n(n-1)$ is the scalar curvature associated to g_0. There have been many papers on the problem and related ones, see e.g., [4, 10, 12, 14, 20, 21, 28, 29, 32, 33, 39, 44, 45, 47]. For $\sigma = 2$, it is the Paneitz–Branson curvature problem, the fourth order conformally invariant Paneitz operator and Branson's Q-curvature on Riemannian manifolds (M, g) are given by

$$P_2^g = \Delta_g^2 - \mathrm{div}_g(a_n R_g g + b_n \mathrm{Ric}_g) d + \frac{n-4}{2} Q_g,$$
$$Q_g = -\frac{1}{2(n-1)} \Delta_g R_g + c_n R_g^2 - \frac{2}{(n-2)^2} |\mathrm{Ric}_g|^2,$$

where R_g and Ric_g denote the scalar curvature and Ricci tensor of g respectively, and $a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}$, $b_n = -\frac{4}{n-2}$, $c_n = \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}$. For this topic, we refer for example to [11, 13, 38, 43, 46, 48] and references therein. For $\sigma = k$ if n is odd, and for $k \in \{1, \ldots, \frac{n}{2}\}$ if n is even, it is the higher order Nirenberg problems which are associated to Graham, Jenne, Mason, and Sparling operators, see [17]. For the case where σ is a non-integer, it is the prescribing fractional Q-curvature problem (1.1), see [9, 18, 25, 26, 42] for properties of those fractional and high order operators, as well as for various existence results on the higher order operators and related problems. One can see from these papers that the higher order operator presents new and challenging features compared with the Laplace operator.

The problem of determining which K admits a solution to prescribing fractional Q-curvature problem (1.1) has been studied extensively. First of all, (1.1) is not always solvable. A first necessary condition for the existence is that $\max_{\mathbb{S}^n} K > 0$, but there are also some obstructions. For example, a necessary condition is the following Kazdan–Warner type obstruction: for any conformal Killing vector field X on $\mathbb{S}^n$, there holds

$$\int_{\mathbb{S}^n} (\nabla_X K) v^{\frac{2n}{n-2\sigma}} \mathrm{d}v_{g_0} = 0,$$

for any solution v of (1.1), see [7, 49].

Moreover, the flatness of the prescribing curvature function K plays a crucial role in the study of this problem. Roughly, the function K is said to satisfy the β-flatness condition if there exists $\beta \in (1, n)$ such that

$$K(x) = K(x_0) + \sum_{j=1}^{n} a_j |x_j - x_{0,j}|^\beta + \mathcal{R}(x),$$

where $a_j \neq 0, \sum_{j=1}^n a_j \neq 0$, and $\mathcal{R}(x)$ satisfies certain degenerating conditions. For $\sigma \in (0,1)$ and $\beta \in (n-2\sigma, n)$, Jin–Li–Xiong [22, 23] proved the existence of the solutions to (1.1) and derived some compactness properties when K satisfies the β-flatness condition by using the approach based on approximation of the solutions to (1.1) by a blow up subcritical method. For $\sigma \in (0, \frac{n}{2})$ and $\beta \in (n-2\sigma, n)$, Jin–Li–Xiong [24] developed a unified approach to establish blow up profiles, compactness and existence of positive solutions to (1.1) when K satisfies the β-flatness condition by making use of integral representations. For $\sigma \in (0, \frac{n}{2})$ and $\beta \in [n-2\sigma, n)$, Li–Tang–Zhou obtained the unified results of existence for prescribing fractional Q-curvature problem, see our work [29–31]. Existence results of the solutions to (1.1) were given when $\beta \in (1, n-2\sigma]$ by Abdelhedi–Chtioui–Hajaiej [2], and when $\beta \in [n-2\sigma, n)$ by Chtioui and Abdelhedi [16]. Under a so-called "non-degenerate condition", Khadijah and Chtioui [27] studied the lack of compactness and provided the existence results for (1.1) when $\beta = n - 2\sigma = 2,\ \sigma \in (0, \frac{n}{2})$.

On the other hand, the periodicity and symmetry of the curvature function K also have a certain impact on the existence of solutions. For $\sigma = 1$, when K is positive and periodic, Li–Wei–Xu [34] proved the existence of multi-bump solutions where the centers of bumps can be placed in some lattices in $\mathbb{R}^n$, including infinite lattices. When K is positive and rotationally symmetric, Wei and Yan [47] proved that (1.1) possesses infinitely many non-radial solutions by using singular perturbation method. Subsequently, Liu and Ren [36] extended this result to the case of $\sigma \in (0,1)$. For more research work in this area, one may refer to [19, 40] and the references therein.

From the above, it can be seen that all existence results of this problem are based on a certain flatness, symmetry or periodicity assumption of the prescribing curvature function K, for instance radial symmetry, symmetry under a subgroup of the group of rotations, or periodicity in one variable.

We want to refer the readers to work by Bianchi [6], where the author studied the existence for the prescribing scalar curvature problem with $\sigma = 1$, when K is a function bounded from above and below by positive constants and no symmetry assumption on K is made. Motivated the work of Bianchi [6], we study the existence for the prescribing fractional Q-curvature problem (1.1) with $\sigma \in (0, \frac{n}{2})$ without assuming any symmetry and periodicity on K. In addition, due to its deep geometry and physics roots, the higher order equations (1.1) for any $\sigma > 1$ have been found considerable interest.

The aim of this paper is two-fold. Firstly, we will discuss the existence of solutions to (1.1) under some reasonable conditions near the global maximum point of K. Secondly, when the condition weakens to the local maximum point of K, we further investigate the existence of the solution. Obviously, due to the existence of critical Sobolev exponent, the lack of compactness leads to that the problem cannot be directly solved by variational methods. Instead, we can return to the basic elements of variational theory and study the deformation along the flow lines, and further restrict the set of critical points of K which may cause loss of compactness. To be more precise, the research of Bahri and Coron in [3, 4] enables us to regain compactness along gradient flow, at certain values of variational functional which are related to the value of the coefficient K in some of its critical points. By doing so, similar as in [6], we use the min-max method and look for a min-max over the class of continuous paths connecting two "critical points at infinity". We assume that the prescribing curvature function K satisfies the following conditions:

For $K \in C^2(\mathbb{S}^n)$ and $d \geq 0$, we define $K_{\max} := \max_{\mathbb{S}^n} K$,

$$\mathscr{K}_d := \{x \in \mathbb{S}^n : K(x) \geq d\}, \tag{1.2}$$

and

$$\mathscr{A}_d := \{K \in C^2(\mathbb{S}^n) : K > 0 \text{ on } \mathbb{S}^n,\ K \text{ has only finitely many critical points in } \mathscr{K}_d\}. \tag{1.3}$$

Our first result is about the existence of solutions to (1.1) under some reasonable conditions near the global maximum point of K.

Theorem 1.1 *Let $\sigma \in (0, \frac{n}{2})$, $\mathscr{A}_d$ be as in (1.3), and $K \in \mathscr{A}_d$ with $d = 2^{-\frac{2\sigma}{n-2\sigma}} K_{\max}$. Let x_0, $x_1 \in \mathbb{S}^n$ be two distinct points belonging two different connected components of $\{x \in \mathbb{S}^n : K(x) = K_{\max}\}$. Suppose also that there exist some positive constant c_0 and some $\beta > n - 2\sigma$, such that in some geodesic normal coordinate system centered at x_i, $i = 0, 1$,*

$$K(x) \geq K(x_i) - c_0|x - x_i|^\beta.$$

Furthermore, if $\tilde{x}$ is a critical point of K and $K(\tilde{x}) \in (2^{-\frac{2\sigma}{n-2\sigma}} K_{\max}, K_{\max})$, then either $\Delta_{g_0} K(\tilde{x}) < 0$ and $\tilde{x}$ is a strict local maximum or $\Delta_{g_0} K(\tilde{x}) > 0$. Then there exists a solution to (1.1).

Remark 1.2 The assumption of Theorem 1.1 indicates that there are neither critical points with $\Delta_{g_0} K(\tilde{x}) = 0$ nor "saddle" points with $\Delta_{g_0} K(\tilde{x}) < 0$ near the global maximum point of K.

Our second result is about the existence of solutions to (1.1) when the condition weakens to the local maximum point of K.

Theorem 1.3 *Let $\sigma \in (0, \frac{n}{2})$ and $\mathscr{A}_d$ be as in (1.3). Let $K \in C^2(\mathbb{S}^n)$ be a positive function and x_1 be a strict local maximum of K with $\Delta_{g_0} K(x_1) \neq 0$. Let $\gamma(t)$ be a continuous path that connecting x_1 to some point x_0 different from x_1, where $K(x_0) \geq K(x_1)$. Suppose also that $K \in \mathscr{A}_d$ with $d < \min_t K(\gamma(t))$ and*

$$\min_t K(\gamma(t)) > 2^{\frac{2}{n-2\sigma}} K_{\max}.$$

Furthermore, if a point $\tilde{x}$ *satisfies* $\nabla_{g_0} K(\tilde{x}) = 0$ *and* $K(\tilde{x}) \in [\min_t K(\gamma(t)), K(x_1))$*, then either* $\Delta_{g_0} K(\tilde{x}) < 0$ *and* $\tilde{x}$ *is a strict local maximum or* $\Delta_{g_0} K(\tilde{x}) > 0$*. Then there exists a solution to* (1.1)*.*

The paper is organized as follows:

In Section 2, we introduce some definitions and notations, as well as some properties of gradient flow. In Section 3, we give the set of admissible paths to run the inf-sup scheme and provide estimates of the upper and lower bounds on the inf-sup value of the functional on the admissible paths. Section 4 is devoted to the proof of Theorems 1.1 and 1.3, by using the estimates obtained in Section 3 and constructing a path that satisfies a specific condition. In Appendix, we provide some useful technical results and elementary estimates.

2 Notations and Preliminaries

In this section, we introduce some definitions and notations, as well as some properties of gradient flow. In order to further restrict the set of critical points of prescribing curvature function K that may cause compactness loss, we first investigate the behavior of the flow lines for the negative gradient of functional associated to (1.1) near the critical points at infinity.

In what follows, $P_\sigma^{g_0}$ is simply written as P_σ. Denote $H^\sigma(\mathbb{S}^n)$ as the σ order fractional Sobolev space with the norm

$$\|v\|_\sigma = \left(\int_{\mathbb{S}^n} v\, P_\sigma v\, dv_{g_0} \right)^{\frac{1}{2}},$$

where dv_{g_0} is the volume element of $\mathbb{S}^n$ with respect to the standard metric g_0. Let

$$\langle u,\, v\rangle_\sigma = \int_{\mathbb{S}^n} u P_\sigma v\, dv_{g_0}$$

be the corresponding scalar product. The sharp Sobolev inequality on $\mathbb{S}^n$ (see Beckner [5]) asserts that

$$\left(\int_{\mathbb{S}^n} |v|^{\frac{2n}{n-2\sigma}} dv_{g_0} \right)^{\frac{n-2\sigma}{n}} \le \frac{\Gamma(\frac{n}{2}-\sigma)}{|\mathbb{S}^n|^{2\sigma/n}\Gamma(\frac{n}{2}+\sigma)} \int_{\mathbb{S}^n} v P_\sigma(v) dv_{g_0}, \quad \forall v \in H^\sigma(\mathbb{S}^n). \tag{2.1}$$

Let

$$\Sigma^+ := \{ v \in H^\sigma(\mathbb{S}^n) :\ \|v\|_\sigma = 1,\, v \ge 0 \}. \tag{2.2}$$

For $K \in C^2(\mathbb{S}^n)$ and $K > 0$, we define $J_K : \Sigma^+ \to \mathbb{R}$ as follows:

$$J_K(v) = \frac{1}{\int_{\mathbb{S}^n} K v^{2_\sigma^*}\, dv_{g_0}}, \quad v \in \Sigma^+, \tag{2.3}$$

where $2_\sigma^* = \frac{2n}{n-2\sigma}$.

The exponent 2_σ^* is critical for the Sobolev embedding $H^\sigma(\mathbb{S}^n) \to L^q(\mathbb{S}^n)$. This embedding is continuous and not compact. The functional J_K does not satisfy the Palais–Smale condition on Σ^+, but the sequences which violate the Palais-Smale condition are known. In order to describe them, let us introduce some notation. For $a \in \mathbb{S}^n$ and $\lambda > 0$, define

$$\delta_{a,\lambda}(x) = c_0 \left(\frac{\lambda}{1 + \frac{\lambda^2 - 1}{2}(1 - \cos\, d(x,a))} \right)^{\frac{n-2\sigma}{2}}, \quad x \in \mathbb{S}^n, \tag{2.4}$$

where c_0 is a constant, depending only on the dimension n, such that $\|\delta_{a,\lambda}\|_\sigma = 1$ and $d(\cdot,\cdot)$ is the geodesic distance (with respect to the standard metric g_0). After multiplication by a suitable constant, they are the only positive solutions of

$$P_\sigma v = c(n,\sigma)v^{2^*_\sigma - 1}, \quad v > 0 \ \text{ on } \mathbb{S}^n,$$

where $c(n,\sigma)$ is as in (1.1).

Define

$$\mathcal{S} := |\mathbb{S}^n|^{\frac{\sigma}{n}} c(n,\sigma)^{\frac{1}{2}}, \tag{2.5}$$

then by (2.1) and $\|\delta_{a,\lambda}\|_\sigma = 1$, we have

$$\int_{\mathbb{S}^n} \delta_{a,\lambda}^{2^*_\sigma} = (|\mathbb{S}^n|^{\frac{\sigma}{n}} c(n,\sigma)^{\frac{1}{2}})^{-2^*_\sigma} = \mathcal{S}^{-2^*_\sigma}. \tag{2.6}$$

For a positive integer p and $\varepsilon > 0$, we define the set $\Omega(p,\varepsilon)$ of potential critical points at infinity as follows (see for instance [3]):

$$\begin{aligned}\Omega(p,\varepsilon) = \Big\{ & u \in \Sigma^+ : \exists\, a_1, \ldots, a_p \in \mathbb{S}^n,\ \lambda_1, \ldots, \lambda_p > 0, \\ & \text{with } \Big\| u - c\sum_{i=1}^p K(a_i)^{\frac{2\sigma-n}{4}} \delta_{a_i,\lambda_i} \Big\|_\sigma < \varepsilon, \text{ where } c = \Big(\sum_{i=1}^p K(a_i)^{\frac{n-2\sigma}{2}} \Big)^{-\frac{1}{2}}, \\ & \lambda_i > \frac{1}{\varepsilon},\ \varepsilon_{i,j}^{\frac{2}{2\sigma-n}} = \frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j d(a_i,\, a_j)^2 > \frac{1}{\varepsilon}. \Big\}\end{aligned} \tag{2.7}$$

Following [8] and [35], the failure of the Palais–Smale condition can be described as follows.

Lemma 2.1 *Let $K \in C^2(\mathbb{S}^n)$ be a positive function, Σ^+ be as in (2.2), and J_K be as in (2.3). Assume J_K has no critical point on Σ^+. Let $\{u_m\}$ be a sequence in Σ^+ such that $J_K(u_m)$ is bounded and $\nabla J_K(u_m)$ goes to zero. Then there exist a subsequence of $\{u_m\}$ (still denote the subsequence by $\{u_m\}$), a sequence $\{\varepsilon_m\}$ tending to zero, and a positive integer p such that $u_m \in \Omega(p,\varepsilon_m)$.*

If u is a function in $\Omega(p,\varepsilon)$, one can find an optimal representation, following the ideas introduced in [3]. Namely, we have:

Lemma 2.2 *For any $p \in \mathbb{N}^+$, there is $\varepsilon_p > 0$ such that if $\varepsilon \le \varepsilon_p$ and $u \in \Omega(p,\varepsilon)$, then the minimization problem*

$$(\mathrm{M}): \qquad \min_{\alpha_i > 0,\, \lambda_i > 0,\, a_i \in \mathbb{S}^n} \Big\| u - \sum_{i=1}^p \alpha_i \delta_{a_i,\lambda_i} \Big\|_\sigma$$

has unique solution up to a permutations.

If we denote

$$v = u - \sum_{i=1}^p \alpha_i \delta_{a_i,\lambda_i},$$

then v belongs to $H^\sigma(\mathbb{S}^n)$ and satisfies the following condition:

$$(E_{a,\lambda}): \ \|v\|_\sigma \le \varepsilon, \ \text{and } \langle v, \varphi_i \rangle_\sigma = 0, \ \text{for } i = 1, \ldots, p, \ \text{and } \varphi_i = \delta_{a_i,\lambda_i}, \frac{\partial \delta_{a_i,\lambda_i}}{\partial \lambda_i}, \frac{\partial \delta_{a_i,\lambda_i}}{\partial a_i}.$$

Here and subsequently, we denote $v \in (E_{a,\lambda})$ to say that v satisfies $(E_{a,\lambda})$.

Lemma 2.3 *Under the assumptions of Lemma* 2.1, *and let* $(a_i)_m$ *be the points associated to* u_m *via the minimization problem* (M) *and* $a_i := \lim_{m\to\infty}(a_i)_m$, *then we have*

$$J_K(u_m) \to \mathcal{S}^{2^*_\sigma}\Big(\sum_{i=1}^{p}\frac{1}{K(a_i)^{\frac{n-2\sigma}{2}}}\Big)^{\frac{2}{n-2\sigma}},$$

as $m\to\infty$, *where* $\mathcal{S}$ *is as in* (2.5).

Proof By using Lemma 2.1 and the optimal representation of u_m, after a simple calculation, we can obtain Lemma 2.3.

Lemma 2.4 *Under the assumptions of Lemma* 2.1, *in addition that* $\mathcal{S}$ *is as in* (2.5), *and*

$$\lim_{m\to\infty} J_K(u_m) < 2^{\frac{2}{n-2\sigma}}\mathcal{S}^{2^*_\sigma}K_{\max}^{-1}.$$

Then there exists a subsequence of $\{u_m\}$ (*still denote the subsequence by* $\{u_m\}$) *and a sequence* $\{\varepsilon_m\}$ *tending to zero such that* $u_m \in \Omega(1,\varepsilon_m)$. *Moreover, there exists* $a_1 \in \mathscr{K}_d$ *for some* $d < \mathcal{S}^{2^*_\sigma}(\lim_{m\to\infty} J_K(u_m))^{-1}$, *such that, as* $m\to\infty$,

$$(a_1)_m \to a_1 \quad \text{and} \quad J_K(u_m) \to \frac{\mathcal{S}^{2^*_\sigma}}{K(a_1)},$$

where $\mathscr{K}_d$ *is as in* (1.2) *and* $(a_1)_m$ *is the point associated to* u_m *via the minimization problem* (M).

We now state the definition of critical point at infinity.

Definition 2.5 *Let* $K \in C^2(\mathbb{S}^n)$ *be a positive function,* Σ^+ *be as in* (2.2), *and* J_K *be as in* (2.3). *Given* $u_0 \in H^\sigma(\mathbb{S}^n)$, *a critical point at infinity of* J_K *on* Σ^+ *is a limit of a flow-line* $u(t)$ *of the following equation:*

$$\begin{cases}\dfrac{\partial u}{\partial t} = -\nabla J_K(u(t)),\\ u(0) = u_0,\end{cases} \tag{2.8}$$

such that $u(t)$ *remains entirely within* $\Omega(p,\varepsilon)$ *for any* ε *small enough and some* $p \geq 1$.

Lemma 2.6 *Under the assumptions of Lemma* 2.4, *in addition that* $K \in \mathscr{A}_d$ *for some* $d < \mathcal{S}^{2^*_\sigma}(\lim_{m\to\infty} J_K(u_m))^{-1}$. *Let* $u(t)$ *be as in* (2.8), *and*

$$\lim_{t\to\infty} J_K(u(t)) < 2^{\frac{2}{n-2\sigma}}\mathcal{S}^{2^*_\sigma}K_{\max}^{-1}.$$

Then $p=1$ *and if* $a(t)$ *denotes the point associated to* $u(t)$ *via the minimization problem* (M), *then* $a(t)$ *converges to some point* $a \in \mathbb{S}^n$ *with* $\nabla_{g_0}K(a) = 0$ *and* $\Delta_{g_0}K(a) \leq 0$ *as* $t\to\infty$.

The proofs of Lemmas 2.4 and 2.6 are similar to Lemmas 1.3 and 1.4 in [6], respectively. We omit it here.

Lemma 2.7 *Let* $u(t)$ *be a solution to* (2.8) *then* $\lim_{t\to\infty}\|\nabla J_K(u(t))\| = 0$.

The proof of this lemma can refer to Appendix A in [4].

3 The Inf-sup Scheme Argument for Existence of Solutions

In this section, we first give the set of admissible paths to run the Inf-sup scheme, which we are going to use in order to prove existence result. Then we provide estimates of the upper and lower bounds on the inf-sup value of the functional on the admissible paths. These estimates play a crucial role in our proof of existence results.

Now we introduce the set of admissible paths. In the notation below, the point a_s is the one given by the minimization problem (M). Let x_0, x_1 be two distinct points on $\mathbb{S}^n$. Let $\mathcal{P}_{x_0,x_1}$ be the set of all continuous paths $\gamma_s : (0,1) \to \Sigma^+$ defined by

$$\begin{aligned}\mathcal{P}_{x_0,x_1} = \Big\{ \gamma_s : (0,1) \longrightarrow \Sigma^+ \text{ such that } \gamma_s \in \Omega(1,\varepsilon) \text{ for any } \varepsilon > 0, \\ \text{and } \lim_{s\to 0} a_s = x_0,\ \lim_{s\to 1} a_s = x_1 \Big\}.\end{aligned} \tag{3.1}$$

Since $K(x_0), K(x_1) > 0$, it is easy to see that the set of paths $\mathcal{P}_{x_0,x_1}$ is nonempty (see for instance the proof of Proposition 3.1 below).

Let J_K be as in (2.3). Define

$$C_{\mathcal{P}_{x_0,x_1}} := \inf_{\gamma_s \in \mathcal{P}_{x_0,x_1}} \sup_{s\in(0,1)} J_K(\gamma_s). \tag{3.2}$$

Proposition 3.1 *Let $K \in C^2(\mathbb{S}^n)$, $K > 0$, and x_0, $x_1 \in \mathbb{S}^n$ be two distinct points. Let J_K be as in* (2.3) *and $\mathcal{S}$ be as in* (2.5)*. Let $\mathcal{P}_{x_0,x_1}$ be as in* (3.1) *and $C_{\mathcal{P}_{x_0,x_1}}$ be as in* (3.2)*. Suppose also that there exist some positive constant c_0 and some $\beta > n - 2\sigma$, such that in some geodesic normal coordinate system centered at x_i, $i = 1, 2$,*

$$K(x) \geq K(x_i) - c_0|x - x_i|^\beta. \tag{3.3}$$

Then, we have

$$C_{\mathcal{P}_{x_0,x_1}} < (K(x_0)^{-\frac{n-2\sigma}{2\sigma}} + K(x_1)^{-\frac{n-2\sigma}{2\sigma}})^{\frac{2\sigma}{n-2\sigma}} \mathcal{S}^{2^*_\sigma}.$$

Proof For $s, t \geq 0$, $s^2 + t^2 = 1$, and λ sufficiently large, let

$$u_s = s\delta_{x_0,\lambda} + t\delta_{x_1,\lambda}, \tag{3.4}$$

where $\delta_{x_0,\lambda}$ and $\delta_{x_1,\lambda}$ are as in (2.4).

The path u_s, when properly normalized belongs to $\mathcal{P}_{x_0,x_1}$. Actually the behavior at the endpoints $\delta_{x_0,\lambda}$ and $\delta_{x_1,\lambda}$ is not completely correct, but we can continue the path with the desired property.

Here and subsequently, it is convenient to transform the problem on $\mathbb{R}^n$ by the stereographic projection when calculating estimates. We use the symbol y_i to represent a point x_i on the sphere after passing through the stereographic projection.

By (3.4) and Lemma 4.2, we have

$$\begin{aligned}\int_{\mathbb{S}^n} K(x) u_s^{2^*_\sigma} = &\int_{\mathbb{S}^n} K(x)(s^{2^*_\sigma}\delta_{x_0,\lambda}^{2^*_\sigma} + t^{2^*_\sigma}\delta_{x_1,\lambda}^{2^*_\sigma}) \\ &+ 2^*_\sigma\Big(s^{2^*_\sigma-1}t\int_{\mathbb{S}^n} K(x)\delta_{x_0,\lambda}^{2^*_\sigma-1}\delta_{x_1,\lambda} + st^{2^*_\sigma-1}\int_{\mathbb{S}^n} K(x)\delta_{x_0,\lambda}\delta_{x_1,\lambda}^{2^*_\sigma-1}\Big) \\ &+ R(\lambda),\end{aligned} \tag{3.5}$$

where

$$R(\lambda) = \begin{cases} O\Big(\displaystyle\int_{\mathbb{S}^n} \delta_{x_0,\lambda}^{2^*_\sigma/2}\delta_{x_1,\lambda}^{2^*_\sigma/2}\Big), & \text{for } n \geq 6, \\ O\Big(\displaystyle\int_{\mathbb{S}^n} \delta_{x_0,\lambda}^{2^*_\sigma-2}\delta_{x_1,\lambda}^{2}\Big), & \text{for } 3 \leq n < 6. \end{cases}$$

Now, we further provide an accurate estimate of $R(\lambda)$.

Case 1 when $n \geq 6$, we have $2 < 2_\sigma^* \leq 3$, then

$$\begin{aligned}
&(st)^{\frac{2_\sigma^*}{2}} \int_{\mathbb{S}^n} \delta_{x_0,\lambda}^{\frac{2_\sigma^*}{2}} \delta_{x_1,\lambda}^{\frac{2_\sigma^*}{2}} \\
&\quad \leq C \int_{\mathbb{R}^n} \left(\frac{\lambda}{1+\lambda^2|y-y_0|^2}\right)^{\frac{n}{2}} \left(\frac{\lambda}{1+\lambda^2|y-y_1|^2}\right)^{\frac{n}{2}} \\
&\quad \leq C\left(\int_{|y-y_0|<r} + \int_{|y-y_1|<r} + \int_{|y-y_0|\geq r,\, |y-y_1|\geq r}\right) \left(\frac{\lambda}{1+\lambda^2|y-y_0|^2}\right)^{\frac{n}{2}} \left(\frac{\lambda}{1+\lambda^2|y-y_1|^2}\right)^{\frac{n}{2}} \\
&\quad =: C(\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3),
\end{aligned}$$

where $|y_0 - y_1| > 2r$. Firstly, it is easy to see that

$$|\mathcal{A}_3| \leq O(\lambda^{-n}).$$

For $\mathcal{A}_1$, let $z = \lambda(y - y_0)$, we have

$$\begin{aligned}
|\mathcal{A}_1| &\leq \int_{|y-y_0|<r} \left(\frac{\lambda}{1+\lambda^2|y-y_0|^2}\right)^{\frac{n}{2}} \left(\frac{\lambda}{1+\lambda^2|y-y_0+y_0-y_1|^2}\right)^{\frac{n}{2}} \\
&\leq C\lambda^{-n} \int_{|z|<\lambda r} \left(\frac{1}{1+|z|^2}\right)^{\frac{n}{2}} \left(\frac{1}{\lambda^{-2}+|y_0-y_1+\lambda^{-1}z|^2}\right)^{\frac{n}{2}} \\
&\leq C\lambda^{n} \left(\frac{1}{\lambda^{-2}+\varepsilon}\right)^{\frac{n}{2}} \int_1^{\lambda r} \frac{\xi^{n-1}}{(1+\xi^2)^{\frac{n}{2}}} = O(\lambda^{-n}\ln(\lambda)).
\end{aligned}$$

By the same reason, we can obtain

$$\mathcal{A}_2 = O(\lambda^{-n}\ln(\lambda)).$$

Case 2 when $3 \leq n \leq 5$, we have $3 < \frac{10}{3} < 2_\sigma^* < 6$, then

$$\begin{aligned}
&\int_{\mathbb{S}^n} \delta_{x_0,\lambda}^{2_\sigma^*-2} \delta_{x_1,\lambda}^2 \\
&\quad \leq \int_{\mathbb{R}^n} \left(\frac{\lambda}{1+\lambda^2|y-y_0|^2}\right)^{2\sigma} \left(\frac{\lambda}{1+\lambda^2|y-y_1|^2}\right)^{n-2\sigma} \\
&\quad = C\left(\int_{|y-y_0|<r} + \int_{|y-y_1|<r} + \int_{|y-y_0|\geq r,\, |y-y_1|\geq r}\right) \left(\frac{\lambda}{1+\lambda^2|y-y_0|^2}\right)^{2\sigma} \left(\frac{\lambda}{1+\lambda^2|y-y_1|^2}\right)^{n-2\sigma} \\
&\quad = \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3.
\end{aligned}$$

It is clear that as $\lambda \to \infty$,

$$|\mathcal{B}_3| \leq \lambda^{-n} \int_{|y-y_0|\geq r,\, |y-y_1|\geq r} \frac{1}{|y-y_0|^{2\sigma}|y-y_1|^{n-2\sigma}} = O(\lambda^{-n}).$$

By direct calculation, we can obtain

$$\begin{aligned}
|\mathcal{B}_1| &\leq \int_{|y-y_0|<r} \left(\frac{\lambda}{1+\lambda^2|y-y_0|^2}\right)^{2\sigma} \left(\frac{\lambda}{1+\lambda^2|y-y_0+y_0-y_1|^2}\right)^{n-2\sigma} \\
&\leq \lambda^{2(2\sigma-n)} \int_{|z|<\lambda r} \left(\frac{1}{1+|z|^2}\right)^{2\sigma} \left(\frac{1}{\lambda^{-2}+|y_1-y_0+\lambda^{-1}z|^2}\right)^{n-2\sigma} \\
&\leq C\lambda^{2(2\sigma-n)} \int_0^{\lambda r} r^{n-1-4\sigma} = o(\lambda^{2\sigma-n}).
\end{aligned}$$

Using a similar argument, we have

$$\mathcal{B}_2 = o(\lambda^{2\sigma-n}).$$

From what has already been proved, we conclude that

$$R(y) = o(\lambda^{2\sigma-n}). \tag{3.6}$$

Next, we estimate the second term at the right-hand side of (3.5). Let F be the stereographic projection with x_i being the south pole and $\widetilde{K}(y) := K(F(y))$, $x_i = F(y_i)$. We consider a ball B_i centered at y_i, it follows from (2.6) that

$$\begin{aligned}
&\int_{\mathbb{S}^n} K(x)\delta_{x_0,\lambda}^{2^*_\sigma-1}\delta_{x_1,\lambda}\\
&= K(x_0)\int_{\mathbb{S}^n}\delta_{x_0,\lambda}^{2^*_\sigma-1}\delta_{x_1,\lambda} + \int_{\mathbb{S}^n}(K(x)-K(x_0))\delta_{x_0,\lambda}^{2^*_\sigma-1}\delta_{x_1,\lambda}\\
&= \left(\int_{B_0} + \int_{B_0^c}\right)(\widetilde{K}(y)-\widetilde{K}(y_0))\left(\frac{\lambda}{1+\lambda^2|y-y_0|^2}\right)^{\frac{n+2\sigma}{2}}\left(\frac{\lambda}{1+\lambda^2|y-y_1|^2}\right)^{\frac{n-2\sigma}{2}}\\
&= \mathcal{C}K(x_0)\lambda^{2\sigma-n}|x_0-x_1|^{2\sigma-n} + o(\lambda^{2\sigma-n}),
\end{aligned} \tag{3.7}$$

where $\mathcal{C}$ is a universal constant independent of λ, and $|x_0 - x_1|$ represents the distance between two points x_0 and x_1 after through a stereographic projection.

By (3.5), (3.6), and (3.7), we have

$$\begin{aligned}
\int_{\mathbb{S}^n} K(x)u_s^{2^*_\sigma} &= \mathcal{S}^{-2^*_\sigma}(s^{2^*_\sigma}K(x_0) + t^{2^*_\sigma}K(x_1))\\
&\quad + s^{2^*_\sigma}\int_{\mathbb{S}^n}(K(x)-K(x_0))\delta_{x_0,\lambda}^{2^*_\sigma} + t^{2^*_\sigma}\int_{\mathbb{S}^n}(K(x)-K(x_1))\delta_{x_1,\lambda}^{2^*_\sigma}\\
&\quad + 2^*_\sigma\mathcal{C}\mathcal{S}^{2^*_\sigma}\lambda^{2\sigma-n}|x_0-x_1|^{2\sigma-n}(s^{2^*_\sigma-1}tK(x_0) + st^{2^*_\sigma}K(x_1)) + o(\lambda^{2\sigma-n}).
\end{aligned}$$

From the definition of u_s, a direct calculation gives

$$\begin{aligned}
\|u_s\|_\sigma^{2^*_\sigma} &= \left(\int_{\mathbb{S}^n}(s\delta_{x_0,\lambda}+t\delta_{x_1,\lambda})P_\sigma(s\delta_{x_0,\lambda}+t\delta_{x_1,\lambda})\right)^{\frac{2^*_\sigma}{2}}\\
&= \left(s^2\|\delta_{x_0,\lambda}\|_\sigma^2 + t^2\|\delta_{x_0,\lambda}\|_\sigma^2 + 2st\int_{\mathbb{S}^n}\delta_{x_0,\lambda}P_\sigma\delta_{x_1,\lambda}\right)^{\frac{2^*_\sigma}{2}}\\
&= \left(1 + 2st\mathcal{S}^{2^*_\sigma}\int_{\mathbb{S}^n}\delta_{x_0,\lambda}\delta_{x_1,\lambda}^{2^*_\sigma-1}\right)^{\frac{2^*_\sigma}{2}}\\
&= 1 + st2^*_\sigma\mathcal{S}^{2\sigma^*}\mathcal{C}\lambda^{2\sigma-n}|x_0-x_1|^{2\sigma-n} + o(\lambda^{2\sigma-n}).
\end{aligned} \tag{3.8}$$

By (3.3) and (3.8), it is easy to see that

$$\begin{aligned}
\lim_{\lambda\to\infty} J_K(u_s) = \lim_{\lambda\to\infty}\frac{\|u_s\|_\sigma^{2^*_\sigma}}{\int_{\mathbb{S}^n}K(x)u_s^{2^*_\sigma}} &= \mathcal{S}^{2^*_\sigma}(s^{2^*_\sigma}K(x_0) + t^{2^*_\sigma}K(x_1))^{-1}\\
&< \mathcal{S}^{2^*_\sigma}\left(K(x_0)^{-\frac{n-2\sigma}{2\sigma}} + K(x_0)^{-\frac{n-2\sigma}{2\sigma}}\right)^{\frac{2\sigma}{n-2\sigma}},
\end{aligned} \tag{3.9}$$

with equality if and only if

$$s = \hat{s} = \frac{K(x_1)^{\frac{n-2\sigma}{4\sigma}}}{\sqrt{K(x_0)^{\frac{n-2\sigma}{2\sigma}} + K(x_1)^{\frac{n-2\sigma}{2\sigma}}}},$$

$$t=\hat{t}=\frac{K(x_0)^{\frac{n-2\sigma}{4\sigma}}}{\sqrt{K(x_0)^{\frac{n-2\sigma}{2\sigma}}+K(x_1)^{\frac{n-2\sigma}{2\sigma}}}}.$$

Then, we just need to prove that (3.9) is true when $s=\hat{s}$ and $t=\hat{t}$. In this case, we have

$$\|u_{\hat{s}}\|_{\sigma}^{2_\sigma^*}=1+2_\sigma^*\mathcal{S}^{2_\sigma^*}\mathcal{C}\frac{(K(x_0)K(x_1))^{\frac{n-2\sigma}{4\sigma}}}{K(x_0)^{\frac{n-2\sigma}{2\sigma}}+K(x_1)^{\frac{n-2\sigma}{2\sigma}}}\lambda^{2\sigma-n}|x_0-x_1|^{2\sigma-n}+o(\lambda^{2\sigma-n}),$$

and

$$\begin{aligned}&\frac{1}{\int_{\mathbb{S}^n}K(x)u_{\hat{s}}^{2_\sigma^*}}\\&\quad=\mathcal{S}^{2_\sigma^*}(K(x_0)^{-\frac{n-2\sigma}{2\sigma}}+K(x_1)^{-\frac{n-2\sigma}{2\sigma}})^{\frac{2}{n-2\sigma}}\\&\qquad\cdot\left(1-2\mathcal{C}(\mathcal{S}^{2_\sigma^*})^2\frac{(K(x_0)K(x_1))^{\frac{n-2\sigma}{4\sigma}}}{K(x_0)^{\frac{n-2\sigma}{2\sigma}}+K(x_1)^{\frac{n-2\sigma}{2\sigma}}}|x_0-x_1|^{2\sigma-n}\lambda^{2\sigma-n}-\mathcal{E}(\lambda)\right)+o(\lambda^{2\sigma-n}),\end{aligned}$$

with

$$\mathcal{E}(\lambda)=c_2\int_{\mathbb{S}^n}(K(x)-K(x_0))\delta_{x_0,\lambda}^{2_\sigma^*}+c_3\int_{\mathbb{S}^n}(K(x)-K(x_1))\delta_{x_1,\lambda}^{2_\sigma^*},$$

where c_2 and c_3 are positive constants. We might as well assume $d(x_0,x_1)>2\rho$ for some $\rho>0$. It is easy to see that

$$\begin{aligned}&\int_{\mathbb{S}^n}(K(x)-K(x_0))\delta_{x_0,\lambda}^{2_\sigma^*}\\&\quad=\int_{B(x_0,\rho)}(K(x)-K(x_0))\delta_{x_0,\lambda}^{2_\sigma^*}+\int_{B^c(x_0,\rho)}(K(x)-K(x_0))\delta_{x_0,\lambda}^{2_\sigma^*}\\&\quad=\int_{B(x_0,\rho)}(K(x)-K(x_0))\delta_{x_0,\lambda}^{2_\sigma^*}+O(\lambda^{-n}),\end{aligned}$$

where $B(x_0,\rho)$ is the geodesic ball centered in x_0 with radius ρ. It follows from (3.3) that

$$\int_{B(x_0,\rho)}(K(x)-K(x_0))\delta_{x_0,\lambda}^{2_\sigma^*}=o(\lambda^{2\sigma-n}).$$

By the same reason, we have

$$\int_{\mathbb{S}^n}(K(x)-K(x_1))\delta_{x_1,\lambda}^{2_\sigma^*}=o(\lambda^{2\sigma-n}).$$

It follows that $\mathcal{E}(\lambda)=o(\lambda^{2\sigma-n})$. Therefore, we can obtain

$$\begin{aligned}J_K(u_{\hat{s}})&=\frac{\|u\|_\sigma^{2_\sigma^*}}{\int_{\mathbb{S}^n}Ku_{\hat{s}}^{2_\sigma^*}}\\&\le(K(x_0)^{-\frac{n-2\sigma}{2}}-K(x_1)^{-\frac{n-2\sigma}{2}})^{\frac{2}{n-2\sigma}}\mathcal{S}^{2_\sigma^*}\\&\quad\cdot\left(1-2\mathcal{C}(\mathcal{S}^{2_\sigma^*})^2\frac{(K(x_0)K(x_1))^{\frac{n-2\sigma}{4\sigma}}}{K(x_0)^{\frac{n-2\sigma}{2\sigma}}+K(x_1)^{\frac{n-2\sigma}{2\sigma}}}|x_0-x_1|^{2\sigma-n}\lambda^{2\sigma-n}+\mathcal{E}(\lambda)\right)+o(\lambda^{2\sigma-n})\\&<(K(x_0)^{-\frac{n-2\sigma}{2}}-K(x_1)^{-\frac{n-2\sigma}{2}})^{\frac{2}{n-2\sigma}}\mathcal{S}^{2_\sigma^*}.\end{aligned}\tag{3.10}$$

From (3.9) and (3.10), we prove that Proposition 3.1 holds. □

Proposition 3.2 *Let $K \in C^2(\mathbb{S}^n)$ and $K > 0$. Let x_0, x_1 belong to two different connected components of $\{x \in \mathbb{S}^n : K(x) = K_{\max}\}$. Let J_K be as in (2.3) and $\mathcal{S}$ be as in (2.5). Let $\mathcal{P}_{x_0,x_1}$ be as in (3.1) and $C_{\mathcal{P}_{x_0,x_1}}$ be as in (3.2). Then, we have*

$$C_{\mathcal{P}_{x_0,x_1}} > \frac{\mathcal{S}^{2^*_\sigma}}{K_{\max}}. \tag{3.11}$$

The proof of this proposition is based on the concept of concentration-compactness in [41] for fractional Sobolev space. Analysis similar to that in the proof of Lemma 2.2 in [6] shows that (3.11) holds.

Proposition 3.3 *Let $K \in C^2(\mathbb{S}^n)$ and $K > 0$. Let $x_1 \in \mathbb{S}^n$ be a strict local maximum of K and $\Delta_{g_0} K(x_1) < 0$. Let $x_0 \in \mathbb{S}^n$ be different from x_1 such that $K(x_0) \geq K(x_1)$. Let J_K be as in (2.3) and $\mathcal{S}$ be as in (2.5). Let $\mathcal{P}_{x_0,x_1}$ be as in (3.1) and $C_{\mathcal{P}_{x_0,x_1}}$ be as in (3.2). Then there exists $c_0 > 0$ such that*

$$C_{\mathcal{P}_{x_0,x_1}} > K(x_1)^{-1}\mathcal{S}^{2^*_\sigma} + c_0.$$

Proof Let $\gamma_s \in \mathcal{P}_{x_0,x_1}$. By the definition of $\mathcal{P}_{x_0,x_1}$, for any $\varepsilon > 0$, there exists $s_\varepsilon > 0$ such that when $|s| > s_\varepsilon$, γ_s stays definitively in $\Omega(1,\varepsilon)$. Let $\alpha_s > 0$, $a_s \in \mathbb{S}^n$, $\lambda_s > 0$ be the parameters associated to γ_s via the minimization problem (M), by Lemma 2.2 we can write

$$\gamma_s(x) = \alpha_s \delta_{a_s,\lambda_s}(x) + v_s(x), \tag{3.12}$$

where v_s satisfies $(E_{a,\lambda})$. The definition of γ_s implies that $\|v_s\|_\sigma \to 0$, $a_s \to x_1$, $\lambda_s \to +\infty$ as $s \to 1$.

Next, we consider the expansion of J_K as $s \to 1$. Firstly, by Lemma 4.1 in Appendix, we have

$$\begin{aligned}\int_{\mathbb{S}^n} K(x)\gamma_s^{2^*_\sigma} &= \alpha_s^{2^*}\int_{\mathbb{S}^n} K(x)\delta_{a_s,\lambda_s}^{2^*_\sigma} + 2^*_\sigma \alpha_s^{2^*_\sigma-1}\int_{\mathbb{S}^n} K(x)\delta_{a_s,\lambda_s}^{2^*_\sigma-1}v_s \\ &\quad + \frac{2^*_\sigma(2^*_\sigma-1)}{2}\alpha_s^{2^*_\sigma-2}\int_{\mathbb{S}^n} K(x)\delta_{a_s,\lambda_s}^{2^*_\sigma-2}v_s^2 + \mathcal{R}(s) \\ &=: \alpha_s^{2^*}\mathcal{M}_1 + 2^*_\sigma\alpha_s^{2^*_\sigma-1}\mathcal{M}_2 + \frac{2^*_\sigma(2^*_\sigma-1)}{2}\alpha_s^{2^*_\sigma-2}\mathcal{M}_3 + \mathcal{R}(s),\end{aligned} \tag{3.13}$$

where a direct calculation yields

$$\begin{aligned}|\mathcal{R}(s)| &\leq C\int_{\mathbb{S}^n}(|v_s|^{2^*_\sigma} + |v_s|^3\delta_{a_s,\lambda_s}^{2^*_\sigma-3}) \\ &\leq C\|v_s\|_\sigma^{2^*_\sigma} + C\bigg(\int_{\mathbb{S}^n}|v_s|^{2^*_\sigma}\bigg)^{\frac{3}{2^*_\sigma}}\bigg(\int_{\mathbb{S}^n}\delta_{a_s,\lambda_s}^{2^*_\sigma}\bigg)^{\frac{2^*_\sigma-3}{2^*_\sigma}} \\ &\leq C(\|v_s\|_\sigma^{2^*_\sigma} + \|v_s\|_\sigma^3) \\ &= O(\|v_s\|_\sigma^3).\end{aligned}$$

For the term $\mathcal{M}_1$ in (3.13), let $\rho > 0$ be sufficiently small. By using the same computing method as in [3, 4], we have

$$\begin{aligned}\mathcal{M}_1 = \int_{\mathbb{S}^n} K(x)\delta_{a_s,\lambda_s}^{2^*_\sigma} = K(a_s)\int_{\mathbb{S}^n}\delta_{a_s,\lambda_s}^{2^*_\sigma} &+ \int_{B(a_s,\rho)}(K(x)-K(a_s))\delta_{a_s,\lambda_s}^{2^*_\sigma} \\ &+ \int_{B(a_s,\rho)^c}(K(x)-K(a_s))\delta_{a_s,\lambda_s}^{2^*_\sigma}\end{aligned}$$

$$=: K(a_s)\mathcal{S}^{-2^*_\sigma} + \mathcal{R}_1 + \mathcal{R}_2, \tag{3.14}$$

where $B(a_s, \rho)$ is the geodesic ball centered in a_s with radius ρ. In order to estimate $\mathcal{R}_1$ and $\mathcal{R}_2$ in (3.14), we still transform the problem on $\mathbb{R}^n$ for calculation as before. Let F be the stereographic projection with a_s being the south pole and $\widetilde{K}(y) := K(F(y))$, then we have

$$\begin{aligned}
\mathcal{R}_1 &= \int_{|y|\le r} (\widetilde{K}(y) - \widetilde{K}(0))\Big(\frac{2\lambda_s}{1+\lambda_s^2|y|^2}\Big)^n \\
&= \int_{|y|\le r} x\cdot(\nabla \widetilde{K}(0))\Big(\frac{2\lambda_s}{1+\lambda_s^2|y|^2}\Big)^n + \frac{1}{2n}\int_{|y|\le r} \Delta\widetilde{K}(0)|y|^2\Big(\frac{2\lambda_s}{1+\lambda_s^2|y|^2}\Big)^n + o\Big(\frac{1}{\lambda_s^2}\Big) \\
&= \frac{2^{n-1}}{n}\frac{\Delta\widetilde{K}(0)}{\lambda_s^2}\int_{\mathbb{R}^n}\frac{|y|^2}{(1+|y|^2)^n} + o\Big(\frac{1}{\lambda_s^2}\Big) \\
&= c_1\frac{\Delta_{g_0}K(a_s)}{\lambda_s^2} + o\Big(\frac{1}{\lambda_s^2}\Big),
\end{aligned} \tag{3.15}$$

where c_1 is a positive constant independent of s. It is easily seen that

$$\mathcal{R}_2 = \int_{|y|>r} (\widetilde{K}(y) - \widetilde{K}(0))\Big(\frac{2\lambda_s}{1+\lambda_s^2|y|^2}\Big)^n \le C\frac{1}{\lambda_s^n} = O\Big(\frac{1}{\lambda_s^n}\Big). \tag{3.16}$$

Thus, by (3.14), (3.15), and (3.16), we can obtain

$$\begin{aligned}
\mathcal{M}_1 &= \int_{\mathbb{S}^n} K(x)\delta_{a_s,\lambda_s}^{2^*_\sigma} \\
&= K(a_s)\int_{\mathbb{S}^n}\delta_{a_s,\lambda_s}^{2^*_\sigma} + \int_{B(a_s,\rho)}(K(x)-K(a_s))\delta_{a_s,\lambda_s}^{2^*_\sigma} \\
&\quad + \int_{B(a_s,\rho)^c}(K(x)-K(a_s))\delta_{a_s,\lambda_s}^{2^*_\sigma} \\
&= K(a_s)\mathcal{S}^{-2^*_\sigma} + c_1\frac{\Delta_{g_0}K(a_s)}{\lambda_s^2} + o\Big(\frac{1}{\lambda_s^2}\Big).
\end{aligned} \tag{3.17}$$

For the term $\mathcal{M}_2$ in (3.13), since $a_s \to x_1$ as $s \to 1$ and $\nabla_{g_0}K(x_1) = 0$, it follows from Hölder inequality and $(E_{a,\lambda})$ condition that

$$\begin{aligned}
|\mathcal{M}_2| &= \Big|\int_{\mathbb{S}^n} K(x)\delta_{a_s,\lambda_s}^{2^*_\sigma-1}v_s\Big| \\
&= \Big|\int_{\mathbb{S}^n}(K(x)-K(a_s))\delta_{a_s,\lambda_s}^{2^*_\sigma-1}v_s\Big| \\
&\le c_2\frac{|\nabla_{g_0}K(a_s)|}{\lambda_s}\|v_s\|_\sigma + o\Big(\frac{1}{\lambda_s}\Big)\|v_s\|_\sigma \\
&\le o\Big(\frac{1}{\lambda_s}\Big)\|v_s\|_\sigma.
\end{aligned} \tag{3.18}$$

For the term $\mathcal{M}_3$ in (3.13), by using Hölder inequality and (2.1), we have

$$\mathcal{M}_3 = \int_{\mathbb{S}^n} K(x)\delta_{a_s,\lambda_s}^{2^*_\sigma-2}v_s^2 = K(a_s)\int_{\mathbb{S}^n}\delta_{a_s,\lambda_s}^{2^*_\sigma-2}v_s^2 + o(\|v_s\|_\sigma^2). \tag{3.19}$$

By (3.12) and $\|\gamma_s\|_\sigma = 1$, we have

$$\alpha_s^2 + \|v_s\|_\sigma^2 = 1.$$

Substituting (3.17), (3.18), (3.19), and $\alpha_s = (1 - \|v_s\|_\sigma^2)^{\frac{1}{2}}$ into (3.13), we can obtain

$$\begin{aligned}\int_{\mathbb{S}^n} K(x)\gamma_s^{2_\sigma^*} = {} & K(a_s)\mathcal{S}^{-2_\sigma^*} + c_1 \frac{\Delta_{g_0} K(a_s)}{\lambda_s^2} \\ & - \frac{2_\sigma^*}{2} K(a_s)\mathcal{S}^{-2_\sigma^*}\left(\|v_s\|_\sigma^2 - \mathcal{S}^{2_\sigma^*}(2_\sigma^* - 1)\int_{\mathbb{S}^n} \delta_{a_s,\lambda_s}^{2_\sigma^*-2} v_s^2\right) \\ & + o(\|v_s\|_\sigma^2) + o\left(\frac{1}{\lambda_s^2}\right) + o\left(\|v_s\|_\sigma \frac{1}{\lambda_s}\right). \end{aligned} \tag{3.20}$$

To be more precise, by (3.17) and $\alpha_s = (1 - \|v_s\|_\sigma^2)^{\frac{1}{2}}$, we have

$$\begin{aligned}\alpha_s^{2^*}\mathcal{M}_1 &= (1 - \|v_s\|_\sigma^2)^{2_\sigma^*/2}\int_{\mathbb{S}^n} K(x)\delta_{a_s,\lambda_s}^{2_\sigma^*} \\ &= \left(1 - \frac{2_\sigma^*}{2}\|v_s\|_\sigma^2 + o(\|v_s\|_\sigma^2)\right)\left(K(a_s)\mathcal{S}^{-2_\sigma^*} + c_1\frac{\Delta_{g_0}K(a_s)}{\lambda_s^2} + o\left(\frac{1}{\lambda_s^2}\right)\right) \\ &= K(a_s)\mathcal{S}^{-2_\sigma^*} + c_1\frac{\Delta_{g_0}K(a_s)}{\lambda_s^2} + o\left(\frac{1}{\lambda_s^2}\right) \\ &\quad - \frac{2_\sigma^*}{2}\|v_s\|_\sigma^2\left(K(a_s)\mathcal{S}^{-2_\sigma^*} + c_1\frac{\Delta_{g_0}K(a_s)}{\lambda_s^2}\right) + o\left(\|v_s\|_\sigma^2\frac{1}{\lambda_s^2}\right) \\ &\quad + o(\|v_s\|_\sigma^2) + o\left(\|v_s\|_\sigma^2\frac{1}{\lambda_s^2}\right) \\ &= K(a_s)\mathcal{S}^{-2_\sigma^*} + c_1\frac{\Delta_{g_0}K(a_s)}{\lambda_s^2} - \frac{2_\sigma^*}{2}\|v_s\|_\sigma^2 K(a_s)\mathcal{S}^{-2_\sigma^*} \\ &\quad + o(\|v_s\|_\sigma^2) + o\left(\frac{1}{\lambda_s^2}\right) + o\left(\|v_s\|_\sigma^2\frac{1}{\lambda_s^2}\right). \end{aligned}$$

From (3.18) and $\alpha_s = (1 - \|v_s\|_\sigma^2)^{\frac{1}{2}}$, we can obtain

$$\begin{aligned}|2_\sigma^*\alpha_s^{2_\sigma^*-1}\mathcal{M}_2| &\le 2_\sigma^*(1 - \|v_s\|_\sigma^2)^{\frac{2_\sigma^*-1}{2}} o\left(\frac{1}{\lambda_s}\right)\|v_s\|_\sigma \\ &= 2_\sigma^*\left(1 - \frac{2_\sigma^*-1}{2}\|v_s\|_\sigma^2 + o(\|v_s\|_\sigma^2)\right) o\left(\frac{1}{\lambda_s}\right)\|v_s\|_\sigma \\ &= 2_\sigma^* o\left(\frac{1}{\lambda_s}\right)\|v_s\|_\sigma + o(\|v_s\|_\sigma^2). \end{aligned}$$

It follows from (3.19) and $\alpha_s = (1 - \|v_s\|_\sigma^2)^{\frac{1}{2}}$ that

$$\begin{aligned}&\frac{2_\sigma^*(2_\sigma^*-1)}{2}\alpha_s^{2_\sigma^*-2}\mathcal{M}_3 \\ &= \frac{2_\sigma^*(2_\sigma^*-1)}{2}\left(1 - \frac{2_\sigma^*-2}{2}\|v_s\|_\sigma^2 + o(\|v_s\|_\sigma^2)\right)\left(K(a_s)\int_{\mathbb{S}^n}\delta_{a_s,\lambda_s}^{2_\sigma^*-2}v_s^2 + o(\|v_s\|_\sigma^2)\right) \\ &= \frac{2_\sigma^*(2_\sigma^*-1)}{2}K(a_s)\int_{\mathbb{S}^n}\delta_{a_s,\lambda_s}^{2_\sigma^*-2}v_s^2 + o(\|v_s\|_\sigma^2). \end{aligned}$$

Therefore, we get (3.20).

Let $\langle Q_s v_s, v_s\rangle_\sigma$ be the quadratic form on (E_{a_s,λ_s}) given by

$$\langle Q_s v_s, v_s\rangle_\sigma = \|v_s\|_\sigma^2 - \frac{(2_\sigma^*-1)}{\mathcal{S}^{2_\sigma^*}}\int_{\mathbb{S}^n}\delta_{a_s,\lambda_s}^{2_\sigma^*-2}v_s^2.$$

By the same method of proving the coercivity of the quadratic form Q_s in [1, 15, 37], we obtain that there exists a positive constant $c_2 = c_2(s)$ such that

$$\langle Q_s v_s, v_s\rangle_\sigma \geq c_2 \|v\|_\sigma^2, \quad \text{on } (E_{a_s,\lambda_s}). \tag{3.21}$$

By (3.20) and (3.21), we have

$$\begin{aligned}\int_{\mathbb{S}^n} K(x)\gamma_s^{2^*_\sigma} \leq K(x_1)\mathcal{S}^{-2^*_\sigma}\biggl(1 + \frac{K(a_s)-K(x_1)}{K(x_1)} + c_1\mathcal{S}^{2^*_\sigma}\frac{\Delta_{g_0}K(a_s)}{K(x_1)\lambda_s^2} - c_2\frac{2^*_\sigma K(a_s)}{2K(x_1)}\|v_s\|_\sigma^2 \\ + o(\|v_s\|_\sigma^2) + o\biggl(\frac{1}{\lambda_s^2}\biggr) + o\biggl(\|v_s\|_\sigma \frac{1}{\lambda_s}\biggr)\biggr).\end{aligned}$$

Since x_1 is a global maximum for K with $\Delta_{g_0}K(x_1) < 0$ and $\Delta_{g_0}K(a_s) = \Delta_{g_0}K(x_1) + o(1) < 0$, as $s \to 1$, there exists a constant $c_0 > 0$ such that

$$\sup_{s\in(0,1)} J_K(u_s) < K(x_1)^{-1}\mathcal{S}^{2^*_\sigma}(1-c_0).$$

It follows that Proposition 3.3 holds.

Proposition 3.4 *Let $K \in C^2(\mathbb{S}^n)$ and $K > 0$. Let x_0, x_1 be two distinct points on $\mathbb{S}^n$ and $\zeta(s)$ be a continuous path on $\mathbb{S}^n$ with $\lim_{s\to 0}\zeta(s) = x_0$ and $\lim_{s\to 1}\zeta(s) = x_1$. Let J_K be as in* (2.3) *and $\mathcal{S}$ be as in* (2.5). *Let $\mathcal{P}_{x_0,x_1}$ be as in* (3.1) *and $C_{\mathcal{P}_{x_0,x_1}}$ be as in* (3.2). *Then we have*

$$C_{\mathcal{P}_{x_0,x_1}} \leq \frac{\mathcal{S}^{2^*_\sigma}}{\min_{s\in(0,1)} K(\zeta(x))}.$$

Proof Let $\lambda > 0$, $\lambda \to +\infty$ and $\gamma(s) := \delta_{\zeta(s),\lambda}$. This path is in $\mathcal{P}_{x_0,x_1}$. Actually the behavior at the endpoints is not exactly correct, but we can continue the path with the desired property. Let $\rho > 0$ be small enough. For each fixed s, we have

$$\begin{aligned}\int_{\mathbb{S}^n} K(x)\delta_{\zeta(s),\lambda}^{2^*_\sigma} &= K(\zeta(s))\int_{\mathbb{S}^n}\delta_{\zeta(s),\lambda}^{2^*_\sigma} + \biggl(\int_{B(\zeta(s),\rho)} + \int_{B(\zeta(s),\rho)^c}\biggr)(K(x) - K(\zeta(s)))\delta_{\zeta(s),\lambda}^{2^*_\sigma} \\ &= K(\zeta(s))\mathcal{S}^{-2^*_\sigma} + \widetilde{\mathcal{R}}_1 + \widetilde{\mathcal{R}}_2.\end{aligned}$$

It follows from the calculation technique in Proposition 3.3 that there exist positive constants c_4, c_5 independent of $\zeta(s)$ such that

$$\widetilde{\mathcal{R}}_1 \leq \|K\|_{C^2(\mathbb{S}^n)}\frac{1}{\lambda}\int_{\mathbb{R}^n}|y|\biggl(\frac{2}{1+|y|^2}\biggr)^n \leq \frac{c_4}{\lambda},$$

and

$$|\widetilde{\mathcal{R}}_2| \leq \frac{c_5}{\lambda^n}.$$

Thus, for sufficiently large λ, we know that

$$\int_{\mathbb{S}^n} K(x)\delta_{\zeta(s),\lambda}^{2^*_\sigma} \geq K(\zeta(s))\mathcal{S}^{-2^*_\sigma} - \frac{c_4}{\lambda}.$$

We obtain that Proposition 3.4 holds.

4 Proof of the Existence to Solutions

This section is devoted to proving the Theorems 1.1 and 1.3. We first use the estimates in Section 3 to obtain the upper and lower bounds of the inf-sup value of functional J_K, which is related to K. Then, assuming that (1.1) has no solution, we construct a path and estimate the value of the functional on that path, ultimately leading to the contradiction, and prove Theorem 1.1. Similar arguments apply to the case that the condition weakens to the local maximum point of K.

Proof of Theorem 1.1 Let $\mathcal{P}_{x_0,x_1}$ be as in (3.1) and $C_{\mathcal{P}_{x_0,x_1}}$ be as in (3.2). It follows from Propositions 3.1 and 3.3 that

$$\frac{\mathcal{S}^{2^*_\sigma}}{K_{\max}} < C_{\mathcal{P}_{x_0,x_1}} < 2^{\frac{2\sigma}{n-2\sigma}} \frac{\mathcal{S}^{2^*_\sigma}}{K_{\max}}.$$

Since $2^{-\frac{2\sigma}{n-2\sigma}} K_{\max} < \mathcal{S}^{2^*_\sigma} C^{-1}_{\mathcal{P}_{x_0,x_1}} < K_{\max}$, we define

$$\mathcal{Q} := \{\tilde{x} \in \mathbb{S}^n \,:\, K(\tilde{x}) = \mathcal{S}^{2^*_\sigma} C^{-1}_{\mathcal{P}_{x_0,x_1}},\, \nabla_{g_0} K(\tilde{x}) = 0,\, \Delta_{g_0} K(\tilde{x}) \le 0\}.$$

It follows from the assumption of theorem that each point $\tilde{x} \in \mathcal{Q}$ is a strict local maximum for K. By Proposition 3.3, we can obtain that there exists a constant $c_{\tilde{x}} > 0$ such that

$$\max_{u\in\gamma} J_K(u) > \mathcal{S}^{2^*_\sigma} K(\tilde{x})^{-1} + c_{\tilde{x}} \tag{4.1}$$

for each $\gamma \in \mathcal{P}_{x_0,x_1}$. Let $c_0 = \min_{\tilde{x}\in\mathcal{Q}} c_{\tilde{x}}$. Choose

$$\theta < \min(c_0, 2^{\frac{2\sigma}{n-2\sigma}} \mathcal{S}^{2^*_\sigma} K^{-1}_{\max} - C_{\mathcal{P}_{x_0,x_1}}), \tag{4.2}$$

and also assume that there is no critical point $\tilde{x}$ of K with

$$K(\tilde{x}) \in (\mathcal{S}^{2^*_\sigma} C^{-1}_{\mathcal{P}_{x_0,x_1}}, \mathcal{S}^{2^*_\sigma} C^{-1}_{\mathcal{P}_{x_0,\,x_1}} + \theta).$$

If there is no solution to (1.1), let $\gamma(s) \in \mathcal{P}_{x_0,x_1}$ be a path such that $J_K(\gamma(s)) < C_{\mathcal{P}_{x_0,x_1}} + \theta$. Then, there exists a constant $\bar{s}$ such that denoting by $u(t)$ is the solution of the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\nabla J_K(u(t)), \\ u(0) = \gamma(\bar{s}), \end{cases}$$

and

$$J_K(u(t)) \in (C_{\mathcal{P}_{x_0,x_1}}, C_{\mathcal{P}_{x_0,x_1}} + \theta). \tag{4.3}$$

This conclusion follows by the same method as in Lemma 3.1 in [6]. By using Lemma 2.7, we know that each of subsequence of $u(t)$ is a Palais–Smale sequence. It follows from (4.2) and (4.3) that

$$\lim_{t\to\infty} J_K(u(t)) \le C_{\mathcal{P}_{x_0,x_1}} + \theta < 2^{\frac{2\sigma}{n-2\sigma}} \mathcal{S}^{2^*_\sigma} K^{-1}_{\max}.$$

Thus, by $d = 2^{\frac{2\sigma}{n-2\sigma}} K_{\max}$ and Lemma 2.4, we conclude that $u(t)$ stay definitively in $\Omega(1, \varepsilon_t)$ with $\varepsilon_t \to 0$. Let $a(t)$ be the point associated to $u(t)$ via problem (M). Lemma 2.6 shows that the point $a(t)$ converges to some point $a \in \mathbb{S}^n$ with $\nabla_{g_0} K(a) = 0$ and $\Delta_{g_0} K(a) \le 0$. A trivial verification shows that

$$\lim_{t\to\infty} J_K(u(t)) = \mathcal{S}^{2^*_\sigma} K(a)^{-1}.$$

Since there is no critical point $\tilde{x}$ for K with $K(\tilde{x}) \in (\mathcal{S}^{2^*_\sigma} C^{-1}_{\mathcal{P}_{x_0,x_1}}, \mathcal{S}^{2^*_\sigma} C^{-1}_{\mathcal{P}_{x_0,\,x_1}} + \theta)$, then we have

$$K(a) = \mathcal{S}^{2^*_\sigma} C^{-1}_{\mathcal{P}_{x_0,x_1}}.$$

It follows from the definition of $\mathcal{Q}$ that $a \in \mathcal{Q}$. Consider the path $\tilde{\gamma}$ that is the union of $u(t)$, $t \geq 0$ and $\gamma(s)$ with $0 < s \leq \bar{s}$. We know that this path belongs to $\mathcal{P}_{x_0,a}$ and the value of $\max_{u\in\tilde{\gamma}} J_K(u) \leq C_{\mathcal{P}_{x_0,x_1}} + \theta$. This contradicts (4.1). □

By using Propositions 3.3 and 3.4, and similar arguments as the above proof, we next prove Theorem 1.3.

Proof of Theorem 1.3 Let $\mathcal{P}_{x_0,x_1}$ be as in (3.1) and $C_{\mathcal{P}_{x_0,x_1}}$ be as in (3.2). In what follows, we choose $d > 2^{-\frac{2\sigma}{n-2\sigma}} K_{\max}$. By Propositions 3.3 and 3.4, and $d < \min_t K(\gamma(t))$, we can obtain

$$\frac{\mathcal{S}^{2^*_\sigma}}{K(x_1)} < C_{\mathcal{P}_{x_0,x_1}} < \frac{\mathcal{S}^{2^*_\sigma}}{d}.$$

Moreover, from $d < \min_t K(\gamma(t))$ and $K(x_1) \leq K_{\max}$, it is easily seen that

$$\frac{\mathcal{S}^{2^*_\sigma}}{K_{\max}} \leq \frac{\mathcal{S}^{2^*_\sigma}}{K(x_1)} < C_{\mathcal{P}_{x_0,x_1}} < \frac{\mathcal{S}^{2^*_\sigma}}{d} < 2^{\frac{2\sigma}{n-2\sigma}} \frac{\mathcal{S}^{2^*_\sigma}}{K_{\max}}.$$

Then the rest of the proof is similar to Theorem 1.1. □

Appendix

In this appendix, we provide two inequalities that were respectively used in [33] and [37], both of which have already been applied in Section 3.

Lemma 4.1 ([33]) *Let $2 \leq \alpha \leq \beta$. There exists a positive constant C depending only on β such that, for any $a \geq 0$, $b \in \mathbb{R}$, we have*

$$\left| |a+b|^{\alpha-1}(a+b) - a^\alpha - \alpha a^{\alpha-1}b - \frac{\alpha(\alpha-1)}{2}a^{\alpha-2}b^2 \right| \leq C(|b|^\alpha + a^\gamma |b|^{\alpha-\gamma}),$$

where $\gamma = \max\{0, \alpha-3\}$.

Lemma 4.2 ([37]) *Let $a, b \geq 0$. There exists a positive constant C such that*

$$||a+b|^p - a^p - b^p - pa^{p-1}b - pab^{p-1}| \leq \begin{cases} Ca^{p/2}b^{p/2}, & 2 < p \leq 3, \\ Ca^2b^{p-2} + Ca^{p-2}b^2, & p > 3. \end{cases}$$

Conflict of Interest The authors declare no conflict of interest.

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Unified results for existence and compactness in the prescribed fractional Q-curvature problem

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Abstract. In this paper we study the problem of prescribing fractional Q-curvature of order 2σ for a conformal metric on the standard sphere $\mathbb{S}^n$ with $\sigma \in (0, n/2)$ and $n \geq 3$. Compactness and existence results are obtained in terms of the flatness order β of the prescribed curvature function K. Making use of integral representations and perturbation result, we develop a unified approach to obtain these results when $\beta \in [n - 2\sigma, n)$ for all $\sigma \in (0, n/2)$. This work generalizes the corresponding results of Jin-Li-Xiong (Math Ann 369:109–151, 2017) for $\beta \in (n - 2\sigma, n)$.

Mathematics Subject Classification. 35R09, 35B44, 35J35.

Keywords. Prescribing fractional Q-curvatures problem, Blow-up analysis, Existence and compactness.

1. Introduction

Let $(\mathbb{S}^n, g_0)$ be the standard sphere in $\mathbb{R}^{n+1}$. The prescribing fractional Q-curvature problem of order 2σ on $\mathbb{S}^n$ can be described as: which function K on $\mathbb{S}^n$ is the fractional Q-curvature of a metric g on $\mathbb{S}^n$ conformally equivalent to g_0? If we denote $g = v^{4/(n-2\sigma)} g_0$, this problem can be represented as finding the solution of the following nonlinear equation with critical exponent:

$$P_\sigma(v) = c(n, \sigma) K v^{\frac{n+2\sigma}{n-2\sigma}} \quad \text{on } \mathbb{S}^n, \tag{1.1}$$

where $n \geq 3$, $0 < \sigma < n/2$, $c(n, \sigma) = \Gamma(\frac{n}{2} + \sigma)/\Gamma(\frac{n}{2} - \sigma)$, Γ is the Gamma function, K is a function defined on $\mathbb{S}^n$, and P_σ is an intertwining operator of 2σ-order:

$$P_\sigma = \frac{\Gamma(B + \frac{1}{2} + \sigma)}{\Gamma(B + \frac{1}{2} - \sigma)}, \quad B = \sqrt{-\Delta_{g_0} + \left(\frac{n-1}{2}\right)^2},$$

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where Δ_{g_0} is the Laplace-Beltrami operator on $(\mathbb{S}^n, g_0)$. The operator P_σ can be viewed as the pull back operator of the fractional Laplacian $(-\Delta)^\sigma$ on $\mathbb{R}^n$ via the stereographic projection:

$$(P_\sigma(v)) \circ F = |J_F|^{-\frac{n+2\sigma}{2n}}(-\Delta)^\sigma(|J_F|^{\frac{n-2\sigma}{2n}}(v \circ F)) \quad \text{for } v \in C^{2\sigma}(\mathbb{S}^n),$$

where F is the inverse of the stereographic projection and $|J_F| = (\frac{2}{1+|x|^2})^n$ is the determinant of the Jacobian of F. In addition, the Green's function of P_σ is the spherical Riesz potential, i.e.,

$$P_\sigma^{-1} f(\xi) = c_{n,\sigma} \int_{\mathbb{S}^n} \frac{f(\zeta)}{|\xi - \zeta|^{n-2\sigma}} \, \mathrm{dvol}_{g_0}(\zeta) \quad \text{for } f \in L^p(\mathbb{S}^n), \tag{1.2}$$

where $c_{n,\sigma} = \frac{\Gamma(\frac{n-2\sigma}{2})}{2^{2\sigma}\pi^{n/2}\Gamma(\sigma)}$, $p > 1$, and $|\cdot|$ is the Euclidean distance in $\mathbb{R}^{n+1}$.

Equation (1.1) has a variational structure and involves critical exponent because of the Sobolev embeddings. A natural function space for finding solutions is $H^\sigma(\mathbb{S}^n)$, the σ-order fractional Sobolev space that consists of all functions $v \in L^2(\mathbb{S}^n)$ such that $(1 - \Delta_{g_0})^{\sigma/2} v \in L^2(\mathbb{S}^n)$, with the norm $\|v\|_{H^\sigma(\mathbb{S}^n)} := \|(1 - \Delta_{g_0})^{\sigma/2} v\|_{L^2(\mathbb{S}^n)}$. The sharp Sobolev inequality on $\mathbb{S}^n$ (see Beckner [7]) asserts that

$$\left(⨍_{\mathbb{S}^n} |v|^{\frac{2n}{n-2\sigma}} \, \mathrm{dvol}_{g_0}\right)^{\frac{n-2\sigma}{n}} \le \frac{\Gamma(\frac{n}{2} - \sigma)}{\Gamma(\frac{n}{2} + \sigma)} ⨍_{\mathbb{S}^n} v P_\sigma(v) \, \mathrm{dvol}_{g_0} \quad \text{for } v \in H^\sigma(\mathbb{S}^n). \tag{1.3}$$

Due to the non-compactness of the embedding of $H^\sigma(\mathbb{S}^n)$ into $L^{2n/(n-2\sigma)}(\mathbb{S}^n)$, the Euler functional associated to (1.1) does not satisfy the Palais-Smale condition, which leads to the failure of the standard critical point theory. Moreover, beside the obvious necessary condition that K be positive somewhere, there are topological obstructions of Kazdan-Warner type to solve (1.1) (see [27,38]).

Problem (1.1) can be seen as the generalization of the classical Nirenberg problem: which function K on $\mathbb{S}^n$ is the scalar curvature of a metric conformal to the standard one? This is equivalent to solving

$$P_1 w + 1 = -\Delta_{g_0} w + 1 = K e^w \quad \text{on } \mathbb{S}^2,$$

and

$$P_1 v = -\Delta_{g_0} v + \frac{n(n-2)}{4} v = \frac{n-2}{4(n-1)} K v^{\frac{n+2}{n-2}} \quad \text{on } \mathbb{S}^n, \ n \ge 3, \tag{1.4}$$

where $g = e^{2w} g_0$ and $v = e^{\frac{n-2}{4} w}$. There has been vast literature on the Nirenberg problem and related ones and it would be impossible to mention here all works in this area. One significant aspect most directly related to this paper is the fine analysis of blow-up solutions or the compactness of the solution set. These were studied in [5,10,11,24,25,27,33,34,39]. For more recent and further studies, see our work [31,32] and related references therein.

Another stimulating situation is the study of higher orders and fractional order conformally invariant pseudo-differential operators $P_k^{g_0}$ on $(\mathbb{S}^n, g_0)$, which exist for all positive integers k if n is odd and for $k \in \{1, \ldots, n/2\}$ if n is even. These operators defined on Riemannian manifolds have also been studied. For any Riemannian manifold (M, g), let R_g be the scalar curvature of (M, g), and the conformal Laplacian be defined as $P_1^g = -\Delta_g + \frac{n-2}{4(n-1)} R_g$. The Paneitz operator P_2^g is another conformal invariant operator, which was discovered by Paneitz [36]. Graham et al. [22] constructed a sequence of conformally covariant elliptic operators $\{P_k\}$ on Riemannian manifolds for all positive integers k if n is odd, and for $k \in \{1, \ldots, n/2\}$ if n is even, which are called GJMS operators. Juhl [28,29] found an explicit formula and a recursive formula for GJMS operators and Q-curvatures (see also Fefferman and Graham [21]). Graham and Zworski [23] presented a family of fractional order conformally invariant operators P_σ^g of non-integer order $\sigma \in (0, n/2)$ on the conformal infinity of asymptotically hyperbolic manifolds. In addition, Chang and González [9] showed that the operator P_σ^g with $\sigma \in (0, n/2)$ can be defined as a Dirichlet-to-Neumann operator of a conformally compact Einstein manifold by using localization method in [8], they also provided some new interpretations and properties of those fractional operators and their associated fractional Q-curvatures. There are many research conducted on the fractional operators P_σ^g and their associated fractional Q-curvature, for instance, see [1,2,4,14–16,18–20,25–27,31,32].

Directly related to our current work are some work on blow up analysis, a priori estimates, and existence and compactness of solutions to (1.1). For $\sigma \in (0, 1)$, Jin-Li-Xiong [25,26] proved the existence of the solutions to (1.1) and derived some compactness properties. More precisely, thanks to a very subtle approach based on approximation of the solutions to (1.1) by a blow-up subcritical method, they proved the existence of solutions to (1.1). For $\sigma \in (0, n/2)$, Jin-Li-Xiong [27] developed a unified approach to establish blow up profiles, compactness and existence of positive solutions to (1.1) by making use of integral representations. Their main hypothesis is the so-called flatness condition. The definition is as follows:

Definition 1.1. Let $K \in C^1(\mathbb{S}^n)$ ($K \in C^{1,1}(\mathbb{S}^n)$ if $0 < \sigma \leq 1/2$) be a positive function. We say that K satisfies the flatness condition $(K)_\beta$ for some $\beta > 0$ if for each critical point q_0 of K, in some geodesic normal coordinates $\{y_1, \ldots, y_n\}$ centered at q_0, there exists a small neighborhood $\mathscr{O}$ of 0 such that

$$K(y) = K(0) + Q_{(q_0)}^{(\beta)}(y) + R_{(q_0)}(y) \quad \text{for } |y| \ll 1, \tag{1.5}$$

where $Q_{(q_0)}^{(\beta)}$ satisfies

$$Q_{(q_0)}^{(\beta)}(\lambda y) = \lambda^\beta Q_{(q_0)}^{(\beta)}(y), \quad \text{for all } \lambda > 0,\, y \in \mathbb{R}^n,\, Q_{(q_0)}^{(\beta)} \in C^{[\beta]-1,1}(\mathbb{S}^{n-1}),$$

$R_{(q_0)}(y)$ is $C^{[\beta]-1,1}$ near 0 with $\lim_{y\to 0} \sum_{0\leq|\alpha|\leq[\beta]} |\partial^\alpha R_{(q_0)}(y)||y|^{-\beta+|\alpha|} = 0$. Here $C^{[\beta]-1,1}$ means that up to $[\beta]-1$ derivatives are Lipschitz functions, and $[\beta]$ is the integer part β. We call β the flatness order.

However, they were only able to handle the case $\beta \in (n - 2\sigma, n)$ in the flatness hypothesis. When the flatness order β of K is $n - 2\sigma$, the $L^\infty(\mathbb{S}^n)$ estimates of the solutions to (1.1) fail, see [27] for more details.

The flatness condition excludes some very interesting functions K. In fact, note that an important class of functions, which is worth including in any results of existence for (1.1), are the Morse functions (C^2 having only nondegenerate critical points). Such functions can be written in the form $(K)_\beta$ with $\beta = 2$. This special flatness type condition $\beta = 2$ has been applied to obtain existence and compactness results, see Li [34] for $\sigma = 1$ with $n = 4$ in the classical Nirenberg problem (1.4); Djadli-Malchiodi-Ahmedou [19] in Paneitz operator $\sigma = 2$ with $n = 6$; Li-Tang-Zhou [31] in the half Laplacian $\sigma = 1/2$ with $n = 3$ and [32] in Q-curvatures problems.

Recently, there have been some works devoted to the existence results via studying the flatness condition effect, and those mainly use the critical points at infinity techniques introduced by Bahri-Coron [5,6]. For $\sigma \in (0, 1)$, see Abdelhedi-Chtioui-Hajaiej [2] with $\beta \in (1, n-2\sigma]$; Abdelhedi-Chtioui [1] for a non-degeneracy condition $n = 2$ and $\sigma = 1/2$. For $\sigma = 2$, see Chtioui-Bensouf-Al-Ghamdi1 [17] with $\beta = n$; Al-Ghamdi-Chtioui-Rigane [3] with $\beta \in [1, n-4)$; Chtioui-Rigan [16] with $\beta \in [n-4, n)$. However, for higher order case including $\sigma \in (0, n/2)$, there are still plenty of technical difficulties which demand new ideas.

Convinced that the nondegeneracy assumption would exclude some interesting class of functions K, we adopt the flatness hypothesis used in [25–27]. But again, in order to include all plausible cases $\beta \in [n - 2\sigma, n)$ with $\sigma \in (0, n/2)$, we need to develop a new line of attack with new ideas. This is essentially due to the structure of the multiple blow-up points, which is much more complicated than in the classical setting. Many new phenomena emerge. More precisely, it turns out that the strong interaction between the bubbles, in the case $\beta \in (n-2\sigma, n)$, forces all blow-up points to be single, and $\beta = n-2\sigma$ can present multiple blow-up points and there is a phenomenon of balance that is the interaction of two bubbles of the same order with respect to the self interaction.

Our goal in this paper is to include a larger class of functions K in the existence and compactness results for (1.1). For this aim, we develop a self-contained approach which enables us to include the case $\beta \in [n-2\sigma, n)$ for all $\sigma \in (0, n/2)$. In order to state our results, we need the following notations and assumptions.

Suppose that $K(x)$ satisfies $(K)_\beta$ condition with $\beta \in [n-2\sigma, n)$, assume also

$$|\nabla Q^{(\beta)}_{(q_0)}(y)| \sim |y|^{\beta-1} \quad \text{for all } |y| \ll 1, \tag{1.6}$$

$$\begin{pmatrix} \int_{\mathbb{R}^n} \nabla Q^{(\beta)}_{(q_0)}(y+\xi)(1+|y|^2)^{-n}\,\mathrm{d}y \\ \\ \int_{\mathbb{R}^n} Q^{(\beta)}_{(q_0)}(y+\xi)\frac{1-|y|^2}{1+|y|^2}(1+|y|^2)^{-n}\,\mathrm{d}y \end{pmatrix} \neq 0, \quad \text{for all } \xi \in \mathbb{R}^n, \tag{1.7}$$

$$\begin{pmatrix} \int_{\mathbb{R}^n} \nabla Q^{(\beta)}_{(q_0)}(y+\xi)(1+|y|^2)^{-n}\,\mathrm{d}y \\ \\ \int_{\mathbb{R}^n} Q^{(\beta)}_{(q_0)}(y+\xi)(1+|y|^2)^{-n}\,\mathrm{d}y \end{pmatrix} \neq 0, \quad \text{for all } \xi \in \mathbb{R}^n. \tag{1.8}$$

For any $\alpha \in [n-2\sigma, n)$, define

$$\mathscr{K}_\alpha = \{q_0 \in \mathbb{S}^n : \nabla_{g_0} K(q_0) = 0,\ \beta(q_0) = \alpha\},$$

where $\beta(q_0)$ represents the flatness order of K at the point q_0. For $q_0 \in \mathscr{K}_{n-2\sigma}$, we assume

$$\int_{\mathbb{R}^n} \nabla Q_{(q_0)}^{(n-2\sigma)}(y+\xi)(1+|y|^2)^{-n}\,\mathrm{d}y = 0 \quad \text{if and only if } \xi = 0. \tag{1.9}$$

Let

$$\mathscr{K}_{n-2\sigma}^{-} = \Big\{q_0 \in \mathscr{K}_{n-2\sigma} : \int_{\mathbb{R}^n} z \cdot \nabla Q_{(q_0)}^{(n-2\sigma)}(z)(1+|z|^2)^{-n}\,\mathrm{d}z < 0\Big\}, \tag{1.10}$$

and for any distinct $q^{(1)}, q^{(2)} \in \mathscr{K}_{n-2\sigma}^{-}$, $M = M(q^{(1)}, q^{(2)})$ is a symmetric 2×2 matrix given by

$$M_{ij} = \begin{cases} -K(q^{(j)})^{-\frac{1+\sigma}{\sigma}} \displaystyle\int_{\mathbb{R}^n} y \cdot \nabla Q_{q^{(j)}}^{(n-2\sigma)}(y)(1+|y|^2)^{-n}\,\mathrm{d}y, & i = j, \\ -\dfrac{2^{\frac{n-2\sigma}{2}}(n-2\sigma)^2}{4n} \dfrac{\pi^{n/2}}{\Gamma\left(\sigma+\frac{n}{2}\right)} \dfrac{G_{q^{(i)}}(q^{(j)})}{\sqrt{K(q^{(i)})K(q^{(j)})}}, & i \neq j, \end{cases} \tag{1.11}$$

where

$$G_{q^{(i)}}(q^{(j)}) = \left(\frac{1}{1-\cos d(q^{(i)}, q^{(j)})}\right)^{\frac{n-2\sigma}{2}} \tag{1.12}$$

is the Green's function of P_σ on $\mathbb{S}^n$, and $d(\cdot,\cdot)$ denotes the geodesic distance.

Let $\gamma \in (0,1)$, $C^\gamma(\Omega)$ denotes the standard Hölder space over the domain Ω. For simplicity, we use $C^\gamma(\Omega)$ to denote $C^{[\gamma],\gamma-[\gamma]}(\Omega)$ when $1 < \gamma \notin \mathbb{N}_+$. For $R > 0$, $\alpha \in (0,1)$ and $\sigma \in (0, n/2)$, we define

$$\mathscr{O}_R := \{v \in C^{2\sigma+\alpha}(\mathbb{S}^n) : 1/R < v < R,\ \|v\|_{C^{2\sigma+\alpha}} < R\}.$$

For $P \in \mathbb{S}^n$, $1 \le t < \infty$, let $\varphi_{P,t}$ be the Möbius transformation on $\mathbb{S}^n$ which, under stereographic projection with respect to the north pole P, sends y to ty (see [33]). The totality of such a set of conformal transforms is diffeomorphic to the unit ball B^{n+1} in $\mathbb{R}^{n+1}$, with the identity transformation identified with the origin in B^{n+1} and

$$\varphi_{P,t} \leftrightarrow \left(\frac{t-1}{t}\right) P =: p \in B^{n+1}. \tag{1.13}$$

Our main result is:

Theorem 1.1. *Let $K \in C^1(\mathbb{S}^n)$ ($K \in C^{1,1}(\mathbb{S}^n)$ if $\sigma \le 1/2$) be a positive function satisfying that for any critical point $q_0 \in \mathbb{S}^n$ of K, there exists some real numbers $\beta = \beta(q_0) \in [n-2\sigma, n)$ such that $K \in C^{[\beta],\beta-[\beta]}(\mathbb{S}^n)$ and (1.5)–(1.8) hold in some geodesic normal coordinate system centered at q_0. Suppose also that if either $\sharp\mathscr{K}_{n-2\sigma}^{-} \le 1$ or for any two distinct points $q^{(1)}, q^{(2)} \in \mathscr{K}_{n-2\sigma}^{-}$, we have $M_{11}M_{22} < M_{12}^2$, where $M = M(q^{(1)}, q^{(2)})$ is given by (1.11).*

Then for all $0 < \alpha$, $\varepsilon < 1$, there exists some constant $C = C(n, \delta, \varepsilon, \alpha)$ such that, for all $\varepsilon \le \mu \le 1$, any positive solution v to (1.1) with K replaced by $K_\mu = \mu K + (1-\mu)$, we have

$$1/C < v < C, \quad \|v\|_{C^{2\sigma+\alpha}(\mathbb{S}^n)} < C. \tag{1.14}$$

For all $P \in \mathbb{S}^n$, $t \geq C$, *we have*

$$\int_{\mathbb{S}^n} K \circ \varphi_{P,t}(x) x \, \mathrm{dvol}_{g_0} \neq 0, \tag{1.15}$$

and for all $R \geq C$, $t \geq C$,

$$\deg(v - (P_\sigma)^{-1} K v^{\frac{n+2\sigma}{n-2\sigma}}, \mathscr{O}_R, 0) = (-1)^n \deg\Big(\int_{\mathbb{S}^n} K \circ \varphi_{P,t}(x) x \, \mathrm{dvol}_{g_0}, B^{n+1}, 0 \Big). \tag{1.16}$$

If we further assume that

$$\deg\Big(\int_{\mathbb{S}^n} K \circ \varphi_{P,t}(x) x \, \mathrm{dvol}_{g_0}, B^{n+1}, 0 \Big) \neq 0 \quad \textit{for large } t,$$

then (1.1) *has at least one solution.*

Remark 1.1. For $n \geq 3$ and $\sigma = 1$, the above result was established by Li [34]. Theorem 1.1 gives the compactness and existence results of the solution to (1.1) when K satisfies the flatness condition $(K)_\beta$ with $\beta \in [n - 2\sigma, n)$. Moreover, Theorem 1.1 establishes a unified result on the compactness and existence of solutions corresponding to prescribing fractional Q-curvature problem.

If we consider a more specific expression of $K(y)$ with $Q(y) = \sum_{j=1}^n a_j |y_j|^\beta$, additional degree-counting formula of the solutions to (1.1) will be obtained.

Corollary 1.1. *Let* $K \in C^1(\mathbb{S}^n)$ *(*$K \in C^{1,1}(\mathbb{S}^n)$ *if* $0 < \sigma \leq 1/2$*) be a positive function satisfying that for any critical point* q_0 *of* K, *under the stereographic projection coordinate system* $\{y_1, \ldots, y_n\}$ *with* q_0 *as south pole, there exist some small neighbourhood* $\mathscr{O}$ *of* 0 *and some real number* $\beta \in [n - 2\sigma, n)$, *such that* $K \in C^{[\beta], \beta - [\beta]}(\mathbb{S}^n)$ *and*

$$K(y) = K(0) + \sum_{j=1}^n a_j |y_j|^{n-2\sigma} + R_{(q_0)}(y) \quad \textit{in } \mathscr{O},$$

where $a_j = a_j(q_0) \neq 0$, $\sum_{j=1}^n a_j \neq 0$, *and* $R_{(q_0)}(y) \in C^{[\beta]-1,1}(\mathscr{O})$ *with*

$$\sum_{|\alpha|=0}^{[\beta]} |\partial^\alpha R_{(q_0)}(y)| |y|^{-\beta + |\alpha|} \to 0 \quad \textit{as } |y| \to 0.$$

Suppose also that if either $\sharp \mathscr{K}^-_{n-2\sigma} \leq 1$ *or* $M_{11} M_{22} < M_{12}^2$ *for all distinct* $q^{(1)}$, $q^{(2)} \in \mathscr{K}^-_{n-2\sigma}$, *where* $\mathscr{K}^-_{n-2\sigma}$ *is as in* (1.10) *and* $M = M(q^{(1)}, q^{(2)})$ *is as in* (1.11)*. Then for all* $0 < \alpha < 1$, *there exists some constant* C *such that, for all solutions* v *of* (1.1), *we have*

$$1/C < v < C, \quad \|v\|_{C^{2\sigma+\alpha}(\mathbb{S}^n)} < C,$$

and for all $R \geq C$,

$$\deg(v - (P_\sigma)^{-1} K v^{\frac{n+2\sigma}{n-2\sigma}}, \mathscr{O}_R, 0) = -1 + (-1)^n \sum_{\substack{\nabla_{q_0} K(q_0) = 0 \\ \sum_{j=1}^n a_j(q_0) < 0}} (-1)^{i(q_0)},$$

where

$$i(q_0) = \sharp\{a_j(q_0) : a_j(q_0) < 0,\, 1 \le j \le n\}.$$

If we further assume that

$$\sum_{\substack{\nabla_{g_0} K(q_0)=0 \\ \sum_{j=1}^{n} a_j(q_0)<0}} (-1)^{i(q_0)} \neq (-1)^n,$$

then (1.1) *has at least one solution.*

Remark 1.2. When $\beta \in (n-2\sigma, n)$, Jin-Li-Xiong [27] obtained existence results for (1.1), but their assumptions on K differ from those in Corollary 1.1, and they did not provide a degree-counting formula of the solution. The result of Corollary 1.1 holds for all cases of $\beta \in [n-2\sigma, n)$. Furthermore, for the special case of $\beta = n-2\sigma = 2$, Li-Tang-Zhou [31] established an optimal compactness and existence result when $n = 3$ and $\sigma = 1$. For other integers with $\beta = n-2\sigma$, a similar result was demonstrated by Li-Tang-Zhou in [32].

Our methods rely on a readapted characterization of blow up behavior introduced by Schoen-Zhang [37,39] and used in the above mentioned papers [25–27,33,34]. However, there is a serious problem of divergence of the integrals when $\beta = n - 2\sigma$. To overcome this challenging problem, we perform a local analysis to give precise estimates to further characterize the blow up behavior. In detail, we obtain the necessary conditions for the solution to (1.1) blow up at more than one point by using the blow up analysis, the Pohozaev type identity (see Proposition A.1), and the assumptions of of K. This approach is different from the proof in [27] with the case $n - 2\sigma < \beta < n$.

The present paper is organized as the following. In Sect. 2, we characterize the blow up points for solutions to (1.1), which plays a key role in proving the compactness result of Theorem 1.1 (see Theorem 2.1). The proof of Theorem 2.1 is mainly based on Pohozaev type identity (see Proposition A.1) and the results of the blow up analysis established by Jin-Li-Xiong [27]. In Sect. 3, we follow the arguments of Jin-Li-Xiong [26] and establish a perturbation result for all $\sigma \in (0, n/2)$ (see Theorem 3.1), which is necessary to prove the existence result of Theorem 1.1. In Sect. 4, we complete the proof of Theorem 1.1 and Corollary 1.1 by using Theorem 2.1, Theorem 3.1 and some results in [33, Section 6]. In "Appendix A", we provide several technical results obtained in Jin-Li-Xiong [27], which is necessary in our proof.

2. Characterization of blow up behavior

In this section, we further characterizes the behavior of blow up points for solutions to (1.1) by using integral representation, some blow up estimates in "Appendix A" and the properties of matrix M, which plays a key role in proving main result concerning compactness and existence. Theorem 2.1 below also gives a necessary condition when the solution to (1.1) has more than one isolate simple blow up point.

We first review the definitions of blow up point. Let Ω be a domain in $\mathbb{R}^n$ and K_i are nonnegative bounded functions in $\mathbb{R}^n$. Let $\{\tau_i\}_{i=1}^\infty$ be a sequence of nonnegative constants satisfying $\lim_{i\to\infty}\tau_i=0$, and set

$$p_i=\frac{n+2\sigma}{n-2\sigma}-\tau_i.$$

Suppose that $0\le u_i\in L^\infty_{loc}(\mathbb{R}^n)$ satisfies the nonlinear integral equation

$$u_i(x)=\int_{\mathbb{R}^n}\frac{K_i(y)u_i(y)^{p_i}}{|x-y|^{n-2\sigma}}\,\mathrm{d}y \quad \text{in } \Omega. \tag{2.1}$$

We assume that $K_i\in C^1(\Omega)$ ($K_i\in C^{1,1}(\Omega)$ if $\sigma\le 1/2$) and, for some positive constants A_1 and A_2,

$$1/A_1\le K_i, \quad \text{and} \quad \|K_i\|_{C^1(\Omega)}\le A_2, \left(\|K_i\|_{C^{1,1}(\Omega)}\le A_2 \text{ if } \sigma\le\frac{1}{2}\right). \tag{2.2}$$

Definition 2.1. Suppose that $\{K_i\}$ satisfies (2.2) and $\{u_i\}$ satisfies (2.1). A point $\overline{y}\in\Omega$ is called a blow up point of $\{u_i\}$ if there exists a sequence y_i tending to $\overline{y}$ such that $u_i(y_i)\to\infty$.

Definition 2.2. A blow up point $\overline{y}\in\Omega$ is called an isolated blow up point of $\{u_i\}$ if there exists $0<\overline{r}<\operatorname{dist}(\overline{y},\Omega)$, $\overline{C}>0$, and *a* sequence y_i tending to $\overline{y}$, such that y_i is a local maximum point of u_i, $u_i(y_i)\to\infty$ and

$$u_i(y)\le\overline{C}|y-y_i|^{-2\sigma/(p_i-1)} \quad \text{for all } y\in B_{\overline{r}}(y_i). \tag{2.3}$$

Let $y_i\to\overline{y}$ be an isolated blow up point of $\{u_i\}$, and define, for $r>0$,

$$\overline{u}_i(r):=\frac{1}{|\partial B_r(y_i)|}\int_{\partial B_r(y_i)}u_i \quad \text{and} \quad \overline{w}_i(r):=r^{2\sigma/(p_i-1)}\overline{u}_i(r).$$

Definition 2.3. A point $y_i\to\overline{y}\in\Omega$ is called an isolated simple blow up point if $y_i\to\overline{y}$ is an isolated blow up point such that for some $\rho>0$ (independent of i), $\overline{w}_i$ has precisely one critical point in $(0,\rho)$ for large i.

In what follows, we consider a situation more general than the properties of set $\mathcal{K}^-_{n-2\sigma}$ given in (1.9) and (1.10). Let $K\in C^1(\mathbb{S}^n)$ ($K\in C^{1,1}(\mathbb{S}^n)$ if $\sigma\le 1/2$) be some positive function satisfying that for any critical point $q_0\in\mathbb{S}^n$ of K, there exists some real number $\beta=\beta(q_0)\in[n-2,n)$ such that (1.5)–(1.8) hold in some geodesic normal coordinate system centered at q_0. Let $\widehat{\mathscr{K}}^-_{n-2\sigma}$ denote the set of critical points q_0 of K with $\beta(q_0)=n-2\sigma$ and simultaneously for some $\eta_0\in\mathbb{R}^n$ satisfying

$$\begin{cases} \int_{\mathbb{R}^n}\nabla Q^{(n-2\sigma)}_{(q_0)}(y+\eta_0)(1+|y|^2)^{-n}\,\mathrm{d}y=0, \\ \\ \int_{\mathbb{R}^n}y\cdot\nabla Q^{(n-2\sigma)}_{(q_0)}(y+\eta_0)(1+|y|^2)^{-n}\,\mathrm{d}y<0. \end{cases} \tag{2.4}$$

When $\#\widehat{\mathscr{K}}^-_{n-2\sigma}\ge 2$, for distinct $q^{(1)},\dots,q^{(k)}\in\widehat{\mathscr{K}}^-_{n-2\sigma}$, $\eta^{(j)}\in\mathbb{R}^n$ ($1\le j\le k$), satisfying (2.4) with $q_0=q^{(j)}$, $\eta_0=\eta^{(j)}$, we define a $k\times k$ symmetric matrix $M = M\left(q^{(1)}, \ldots, q^{(k)}, \eta^{(1)}, \ldots, \eta^{(k)}\right)$ by

$$M_{ij} = \begin{cases} -K(q^{(j)})^{-\frac{1+\sigma}{\sigma}} \displaystyle\int_{\mathbb{R}^n} y \cdot \nabla Q^{(n-2\sigma)}_{q^{(j)}}(y+\eta^{(j)})(1+|y|^2)^{-n}\, \mathrm{d}y, & i=j, \\ -\dfrac{2^{\frac{n-2\sigma}{2}}(n-2\sigma)^2}{4n} \dfrac{\pi^{n/2}}{\Gamma(\sigma+\frac{n}{2})} \dfrac{G_{q^{(i)}}(q^{(j)})}{\sqrt{K(q^{(i)})K(q^{(j)})}}, & i \neq j. \end{cases} \tag{2.5}$$

The result about characterization of blow up behavior of the solutions to (1.1) is:

Theorem 2.1. *Let $K \in C^1(\mathbb{S}^n)$ ($K \in C^{1,1}(\mathbb{S}^n)$ if $\sigma \le 1/2$) be a positive function satisfying that for any critical point q_0 of K, there exists some real number $\beta = \beta(q_0) \in [n-2\sigma, n)$, such that (1.5)–(1.8) hold in some geodesic normal coordinate system centered at q_0. Let $\{v_i\}$ be a sequence of solutions to (1.1) that blows up at $\{q^{(1)}, \ldots, q^{(k)}\}$ with $k \ge 2$. Then we have $q^{(1)}, \ldots, q^{(k)} \in \widehat{\mathscr{K}^-_{n-2\sigma}}$, and for some $\eta^{(j)} \in \mathbb{R}^n$ satisfying (2.4) with $q_0 = q^{(j)}$, $\eta_0 = \eta^{(j)} (1 \le j \le k)$, the equation*

$$\sum_{\ell=1}^{k} M_{j\ell}\lambda_\ell = 0$$

has at least one solution $\lambda_\ell > 0$, $\ell = 1, \ldots, k$, where $\widehat{\mathscr{K}^-_{n-2\sigma}}$ is as in (2.4) and $M_{j\ell}$ is as in (2.5).

Proof. It follows from [27, Theorem 3.3] that, after passing to a subsequence, $\{v_i\}$ has only isolated simple blow up points. Moreover, if $\{v_i\}$ blows up at $\{q^{(1)}, \ldots, q^{(k)}\}$ with $k \ge 2$, we know from [27, Theorem 3.4] that $\beta(q^{(j)}) = n - 2\sigma$ for each $j \in \{1, \ldots, k\}$.

Using (1.2), we write (1.1) as the form

$$v_i(\xi) = \frac{\Gamma(\frac{n+2\sigma}{2})}{2^{2\sigma}\pi^{n/2}\Gamma(\sigma)} \int_{\mathbb{S}^n} \frac{K(\eta)v_i(\eta)^{\frac{n+2\sigma}{n-2\sigma}}}{|\xi-\eta|^{n-2\sigma}}\, \mathrm{d}\eta \quad \text{on } \mathbb{S}^n. \tag{2.6}$$

Let F be the stereographic projection with $q^{(j)}$ being the south pole:

$$F: \mathbb{R}^n \longrightarrow \mathbb{S}^n \backslash \{-q^{(j)}\},$$
$$x \longmapsto \Big(\frac{2x}{1+|x|^2}, \frac{|x|^2-1}{|x|^2+1}\Big),$$

and its Jacobi determinant takes $|J_F| = (\frac{2}{1+|x|^2})^n$. Via the stereographic projection, the Eq. (2.6) is translated to

$$u_i(x) = \frac{\Gamma(\frac{n+2\sigma}{2})}{2^{2\sigma}\pi^{n/2}\Gamma(\sigma)} \int_{\mathbb{R}^n} \frac{\widetilde{K}(y)u_i(y)^{\frac{n+2\sigma}{n-2\sigma}}}{|x-y|^{n-2\sigma}}\, \mathrm{d}y \quad \text{on } \mathbb{R}^n,$$

where

$$u_i(x) = H(x)v_i(F(x)), \quad \widetilde{K}(x) = K(F(x)), \quad H(x) = |J_F(x)|^{\frac{n-2\sigma}{2n}} = \Big(\frac{2}{1+|x|^2}\Big)^{\frac{n-2\sigma}{2}}. \tag{2.7}$$

Let $x_i^{(j)}$ be the local maximum of u_i and $x_i^{(j)} \to x^{(j)} = 0$. It follows from Propositions A.5 and A.6 that

$$\begin{aligned} &u_i(x_i^{(j)})u_i(x) \to h^{(j)}(x) := aK(q^{(j)})^{\frac{2\sigma-n}{2\sigma}}|x|^{2\sigma-n} + b^{(j)}(x) \\ &\quad \text{in } C^2_{loc}(\mathbb{R}^n \setminus \cup_{\ell=1}^k \{x^{(\ell)}\}), \end{aligned} \tag{2.8}$$

where

$$a = 2^n c(n,\sigma)c_{n,\sigma}\int_{\mathbb{R}^n}\Big(\frac{1}{1+|y|^2}\Big)^{\frac{n+2\sigma}{2}}\,\mathrm{d}y = 2^{n-1}c(n,\sigma)c_{n,\sigma}B(\delta, n/2) = 2^{n-2\sigma}, \tag{2.9}$$

and $b^{(j)}(y) > 0$ is some regular harmonic function in $\mathbb{R}^n \setminus \cup_{\ell\neq j}\{x^{(\ell)}\}$. Coming back to v_i on $\mathbb{S}^n$, by (2.7), we have

$$\lim_{i\to\infty} v_i(q_i^{(j)})v_i(q) = \lim_{i\to\infty}\Big(\frac{1+|x|^2}{2^2}\Big)^{\frac{n-2\sigma}{2}} u_i(x_i^{(j)})u_i(x).$$

Thus, for $q \neq q^{(j)}$ and close to $q^{(j)}$,

$$\begin{aligned} \lim_{i\to\infty} v_i(q_i^{(j)})v_i(q) &= a2^{\frac{2\sigma-n}{2}}K(q^{(j)})^{\frac{2\sigma-n}{2\sigma}}G_{q^{(j)}}(q) + \widetilde{b}^{(j)}(q) \\ &\quad \text{in } C^2_{loc}(\mathbb{S}^n\setminus\{q^{(1)},\ldots,q^{(k)}\}), \end{aligned}$$

where $G_{q^{(i)}}(q^{(j)})$ is as in (1.12), and $\widetilde{b}^{(j)}(q)$ is some regular function near $q^{(j)}$ satisfying $P_\sigma \widetilde{b}^{(j)} = 0$.

Then, taking into account the contribution of all the poles, we deduce that

$$\lim_{i\to\infty} v_i(q_i^{(j)})v_i(q) = a2^{\frac{2\sigma-n}{2}}\Big\{\frac{G_{q^{(j)}}(q)}{K(q^{(j)})^{\frac{n-2\sigma}{2\sigma}}} + \sum_{\ell\neq j}\lim_{i\to\infty}\frac{v_i(q_i^{(j)})}{v_i(q_i^{(\ell)})}\frac{G_{q^{(\ell)}}(q)}{K(q^{(\ell)})^{\frac{n-2\sigma}{2\sigma}}}\Big\}. \tag{2.10}$$

It follows that for $|x| > 0$ small,

$$\begin{aligned} &\lim_{i\to\infty} u_i(x_i^{(j)})u_i(x) \\ &\quad = aK(q^{(j)})^{\frac{2\sigma-n}{2\sigma}}|x|^{2\sigma-n} + a2^{\frac{n-2\sigma}{2}}\sum_{\ell\neq j}\lim_{i\to\infty}\frac{v_i(q_i^{(j)})}{v_i(q_i^{(\ell)})}\frac{G_{q^{(\ell)}}(q^{(j)})}{K(q^{(\ell)})^{\frac{n-2\sigma}{2\sigma}}} + O(|x|) \\ &\quad =: h^{(j)}(x). \end{aligned} \tag{2.11}$$

For sufficiently small $\delta > 0$, u_i satisfy

$$u_i(x) = c_{n,\sigma}c(n,\sigma)\int_{B_\delta(x_i^{(j)})}\frac{\widetilde{K}(y)u_i(y)^{\frac{n+2\sigma}{n-2\sigma}}}{|x-y|^{n-2\sigma}}\,\mathrm{d}y + h_\delta(x), \tag{2.12}$$

where

$$h_\delta(x) := c_{n,\sigma}c(n,\sigma)\int_{\mathbb{R}^n\setminus B_\delta(x_i^{(j)})}\frac{\widetilde{K}(y)u_i(y)^{\frac{n+2\sigma}{n-2\sigma}}}{|x-y|^{n-2\sigma}}\,\mathrm{d}y. \tag{2.13}$$

By Proposition A.1, we have

$$
\begin{aligned}
&-\frac{2n}{n-2\sigma}\int_{B_\delta(x_i^{(j)})}(x-x_i^{(j)})\cdot\nabla\widetilde{K}(x)u_i(x)^{\frac{2n}{n-2\sigma}}\,\mathrm{d}x\\
&\quad=\frac{n-2\sigma}{2}\int_{B_\delta(x_i^{(j)})}\widetilde{K}(x)u_i(x)^{\frac{n+2\sigma}{n-2\sigma}}h_\delta(x)\,\mathrm{d}x\\
&\qquad+\int_{B_\delta(x_i^{(j)})}(x-x_i^{(j)})\cdot\nabla h_\delta(x)\widetilde{K}(x)u_i(x)^{\frac{n+2\sigma}{n-2\sigma}}\,\mathrm{d}x\\
&\qquad-\frac{2n}{n-2\sigma}\delta\int_{\partial B_\delta(x_i^{(j)})}\widetilde{K}(x)u_i(x)^{\frac{2n}{n-2\sigma}}\,\mathrm{d}s.
\end{aligned}\tag{2.14}
$$

Let $m_{ij}:=u_i(x_i^{(j)})$, by [27, Lemma 2.18], we have

$$
|\nabla\widetilde{K}(x_i^{(j)})|\le Cm_{ij}^{-\frac{2(n-2\sigma-1)}{n-2\sigma}}.
$$

On the other hand, it follows from (1.5) and (1.6) that

$$
\nabla\widetilde{K}(x_i^{(j)})=\nabla Q^{(n-2\sigma)}_{(q^{(j)})}(x_i^{(j)})+o_\delta(1)|x_i^{(j)}|^{n-2\sigma-1},
$$

then, we obtain

$$
|x_i^{(j)}|^{n-2\sigma-1}\le Cm_{ij}^{-\frac{2(n-2\sigma-1)}{n-2\sigma}}.\tag{2.15}
$$

It follows from the above that

$$
\begin{aligned}
&m_{ij}^2\Big|\int_{B_\delta}y\cdot\nabla R_{(q^{(j)})}(y+x_i^{(j)})u_i(y+x_i^{(j)})^{\frac{2n}{n-2\sigma}}\,\mathrm{d}y\Big|\\
&\quad\le m_{ij}^2o_\delta(1)\int_{B_\delta}|y||y+x_i^{(j)}|^{n-2\sigma-1}u_i(y+x_i^{(j)})^{\frac{2n}{n-2\sigma}}\,\mathrm{d}y\\
&\quad\le o_\delta(1)\int_{B_\delta}(|y|^{n-2\sigma}+|y||x_i^{(j)}|^{n-2\sigma-1})u_i(y+x_i^{(j)})^{\frac{2n}{n-2\sigma}}\,\mathrm{d}y\\
&\quad=o_\delta(1).
\end{aligned}\tag{2.16}
$$

For the left hand side of (2.14), by Proposition A.3, Proposition A.7, (2.16), and letting $i\to\infty$, we have

$$
\begin{aligned}
&-\frac{2n}{n-2\sigma}m_{ij}^2\int_{B_\delta(x_i^{(j)})}(x-x_i^{(j)})\cdot\nabla\widetilde{K}(x)u_i(x)^{\frac{2n}{n-2\sigma}}\,\mathrm{d}x\\
&\quad=-\frac{2n}{n-2\sigma}m_{ij}^2\int_{B_\delta}y\cdot\nabla\widetilde{K}(y+x_i^{(j)})u_i(y+x_i^{(j)})^{\frac{2n}{n-2\sigma}}\,\mathrm{d}y\\
&\quad=-\frac{2n}{n-2\sigma}m_{ij}^2\int_{B_\delta}y\cdot\nabla Q^{(n-2\sigma)}_{(q^{(j)})}(y+x_i^{(j)})\,\mathrm{d}y+o_\delta(1)\\
&\quad=-\frac{2n}{n-2\sigma}\int_{\mathbb{R}^n}\frac{z\cdot\nabla Q^{(n-2\sigma)}_{(q^{(j)})}(z+\xi^{(j)})}{(1+k^{(j)}|z|^2)^n}\,\mathrm{d}z+o_\delta(1),
\end{aligned}\tag{2.17}
$$

where $\xi^{(j)}=\lim_{i\to\infty}m_{ij}^{\frac{2}{n-2\sigma}}x_i^{(j)}$ and $k^{(j)}=K(q^{(j)})^{1/\sigma}/4$.

For the first term on the right hand side of (2.14), by Proposition A.3, Proposition A.6, (2.11), and letting $i \to \infty$, we have

$$\begin{aligned} & m_{ij}^2 \frac{n-2\sigma}{2} \int_{B_\delta(x_i^{(j)})} \widetilde{K}(x) u_i(x)^{\frac{n+2\sigma}{n-2\sigma}} h_\delta(x)\,\mathrm{d}x \\ &= m_{ij}^2 \frac{n-2\sigma}{2} \int_{B_\delta(x_i^{(j)})} (\widetilde{K}(x_i^{(j)}) + (x - x_i^{(j)}) \cdot \nabla \widetilde{K}(x_i^{(j)}) \\ &\quad + O(|x - x_i^{(j)}|^2)) h_\delta(x) u_i(x)^{\frac{n+2\sigma}{n-2\sigma}}\,\mathrm{d}x \\ &= \frac{n-2\sigma}{2} \widetilde{K}(x_i^{(j)}) \int_{|y| \le R_i} (m_{ij}^{-1} u_i(m_{ij}^{\frac{-2}{n-2\sigma}} y + x_i^{(j)}))^{\frac{n+2\sigma}{n-2\sigma}} h_\delta(m_{ij}^{-1} y + x_i^{(j)})\,\mathrm{d}y \\ &\quad + o(1) \\ &= \frac{n-2\sigma}{2} K(q^{(j)}) \int_{\mathbb{R}^n} \frac{b^{(j)}(0)}{(1 + k^{(j)}|y|^2)^{\frac{n+2\sigma}{2}}}\,\mathrm{d}y. \end{aligned} \tag{2.18}$$

A direct calculation gives that when $|x - x_i^{(j)}| < \delta$,

$$|\nabla h_\delta(x)| \le \begin{cases} C \dfrac{|\delta^{2\sigma-1} - (\delta - |x - x_i^{(j)}|)^{2\sigma-1}|}{2\sigma - 1} m_{ij}^{-1} & \text{if } \sigma \ne 1/2, \\ C|\log\delta - \log(\delta - |x - x_i^{(j)}|)| m_{ij}^{-1} & \text{if } \sigma = 1/2. \end{cases} \tag{2.19}$$

For the second term on the right hand side of (2.14), from (2.19) and Proposition A.3, we have

$$\begin{aligned} & m_{ij}^2 \Big| \int_{B_\delta(x_i^{(j)})} (x - x_i^{(j)}) \cdot \nabla h_\delta(x) \widetilde{K}(x) u_i(x)^{\frac{n+2\sigma}{n-2\sigma}}\,\mathrm{d}x \Big| \\ &\le C m_{ij} \int_{B_\delta(x_i^{(j)})} |x - x_i^{(j)}| u_i(x)^{\frac{n+2\sigma}{n-2\sigma}}\,\mathrm{d}x \\ &= C m_{ij}^{1 - \frac{2}{n-2\sigma} - \frac{2n}{n-2\sigma} + \frac{n+2\sigma}{n-2\sigma}} \int_{|y| < R_i} |y| (m_{ij}^{-1} u_i(m_{ij}^{-\frac{2}{n-2\sigma}} y + x_i^{(j)})^{\frac{n+2\sigma}{n-2\sigma}}\,\mathrm{d}y \\ &= m_{ij}^{-\frac{2}{n-2\sigma}} \int_{\mathbb{R}^n} \frac{|y|}{(1 + k^{(j)}|y|^2)^{\frac{n+2\sigma}{2}}}\,\mathrm{d}y = o(1). \end{aligned} \tag{2.20}$$

For the third term on the right side of (2.14), Proposition A.7 led to

$$\begin{aligned} & \lim_{i\to\infty} \Big| - m_{ij}^2 \frac{2n}{n-2\sigma} \delta \int_{\partial B_\delta(x_i^{(j)})} \widetilde{K}(x) u_i(x)^{\frac{2n}{n-2\sigma}}\,\mathrm{d}s \Big| \\ &\le C(\delta) \lim_{i\to\infty} m_{ij}^2 m_{ij}^{-\frac{2n}{n-2\sigma}} = 0. \end{aligned} \tag{2.21}$$

Let

$$\lambda_j := K(q^{(j)})^{\frac{1-n+2\sigma}{2\sigma}} \lim_{i\to\infty} v_i(q_i^{(1)}) v_i(q_1^{(j)}). \tag{2.22}$$

It follows from Propositions A.4 and A.5 that $0 < \lambda_j < \infty$. Therefore, by (2.14), (2.17), (2.18), (2.21), (2.20) and letting $\delta \to 0$, we have

$$-\frac{2n}{n-2\sigma}\int_{\mathbb{R}^n}\frac{z\cdot\nabla Q^{(n-2\sigma)}_{(q^{(j)})}(z+\xi^{(j)})}{(1+k^{(j)}|z|^2)^n}\,\mathrm{d}z=\frac{n-2\sigma}{2}K(q^{(j)})\int_{\mathbb{R}^n}\frac{b^{(j)}(0)}{(1+k^{(j)}|y|^2)^{\frac{n+2\sigma}{2}}}\,\mathrm{d}y. \tag{2.23}$$

By (2.9) and (2.22), we obtain

$$b^{(j)}(0)=2^{\frac{3(n-2\sigma)}{2}}\sum_{\ell\neq j}\frac{\lambda_\ell}{\lambda_j}\frac{K(q^{(j)})^{\frac{1-n+2\sigma}{2\sigma}}}{K(q^{(\ell)})^{\frac{1}{2\sigma}}}G_{q^{(\ell)}}(q^{(j)}). \tag{2.24}$$

Substituting (2.24) into (2.23) and making a change of variable, it holds

$$\begin{aligned}&-\frac{2n}{n-2\sigma}2^{2(n-\sigma)}K(q^{(j)})^{\frac{\sigma-n}{\sigma}}\int_{\mathbb{R}^n}\frac{y\cdot\nabla Q^{(n-2\sigma)}_{q^{(j)}}(y+\sqrt{k^j}\xi^{(j)})}{(1+|y|^2)^n}\,\mathrm{d}y\\&=(n-2\sigma)2^{n-2+\frac{3(n-2\sigma)}{2}}\frac{2\pi^{n/2}}{\Gamma(\sigma+n/2)}\sum_{\ell\neq j}\frac{\lambda_\ell}{\lambda_j}\frac{K(q^{(j)})^{\frac{1-2n+4\sigma}{2\sigma}}}{K(q^{(\ell)})^{\frac{1}{2\sigma}}}G_{q^{(\ell)}}(q^{(j)}),\end{aligned}$$

where $\eta^{(j)}:=\sqrt{k^{(j)}}\xi^{(j)}$. It follows that

$$\begin{aligned}&-K(q^{(j)})^{-\frac{1+\sigma}{\sigma}}\lambda_j\int_{\mathbb{R}^n}\frac{y\cdot\nabla Q^{(n-2\sigma)}_{q^{(j)}}(y+\eta^{(j)})}{(1+|y|^2)^n}\,\mathrm{d}y\\&=\frac{2^{\frac{n-2\sigma}{2}}(n-2\sigma)^2}{4n}\frac{\pi^{n/2}}{\Gamma(\sigma+\frac{n}{2})}\sum_{\ell\neq j}\frac{G_{q^{(\ell)}}(q^{(j)})}{K(q^{(\ell)})^{\frac{1}{2\sigma}}K(q^{(j)})^{\frac{1}{2\sigma}}}\lambda_\ell.\end{aligned} \tag{2.25}$$

We next prove that $q^{(j)}$, $\eta^{(j)}$ $(j=1,\ldots,k)$, satisfy (2.4) with $q_0=q^{(j)}$, $\eta_0=\eta^{(j)}$. In fact, due to (2.25), we only need to prove

$$\int_{\mathbb{R}^n}\nabla Q^{(n-2\sigma)}_{(q^{(j)})}(y+\eta^{(j)})(1+|y|^2)^{-n}\,\mathrm{d}y=0. \tag{2.26}$$

We first claim that

$$\int_{B_\delta}\nabla\widetilde{K}(x+x_i^{(j)})u_i(x+x_i^{(j)})^{\frac{2n}{n-2\sigma}}\,\mathrm{d}x=O(m_{ij}^{-2}). \tag{2.27}$$

Indeed, by using symmetry, we have

$$\begin{aligned}&\frac{n-2\sigma}{2n}\int_{B_\delta(x_i^{(j)})}\widetilde{K}(x)\nabla u_i(x)^{\frac{2n}{n-2\sigma}}\,\mathrm{d}x\\&=\int_{B_\delta(x_i^{(j)})}\widetilde{K}(x)u_i(x)^{\frac{n+2\sigma}{n-2\sigma}}\nabla u_i(x)\,\mathrm{d}x\\&=(2\sigma-n)\int_{B_\delta(x_i^{(j)})}\widetilde{K}(x)u_i(x)^{\frac{n+2\sigma}{n-2\sigma}}\int_{B_\delta(x_i^{(j)})}\frac{(x-y)\widetilde{K}(y)u_i(y)^{\frac{n+2\sigma}{n-2\sigma}}}{|x-y|^{n-2\sigma+2}}\,\mathrm{d}y\,\mathrm{d}x\\&\quad+\int_{B_\delta(x_i^{(j)})}\widetilde{K}(x)u_i(x)^{\frac{n+2\sigma}{n-2\sigma}}\nabla h_\delta(x)\,\mathrm{d}x\\&=\int_{B_\delta(x_i^{(j)})}\widetilde{K}(x)u_i(x)^{\frac{n+2\sigma}{n-2\sigma}}\nabla h_\delta(x)\,\mathrm{d}x,\end{aligned}$$

and by the divergence theorem,

$$\begin{aligned}&\int_{B_\delta(x_i^{(j)})}\widetilde{K}(x)\nabla u_i(x)^{\frac{2n}{n-2\sigma}}\,\mathrm{d}x\\&\quad=-\int_{B_\delta(x_i^{(j)})}\nabla\widetilde{K}(x)u_i(x)^{\frac{2n}{n-2\sigma}}\,\mathrm{d}x+\delta\int_{\partial B_\delta(x_i^{(j)})}\widetilde{K}(x)u_i(x)^{\frac{2n}{n-2\sigma}}(x-x_i^{(j)})\,\mathrm{d}s.\end{aligned}$$

It follows that

$$\begin{aligned}&-\frac{n-2\sigma}{2n}\int_{B_\delta(x_i^{(j)})}\nabla\widetilde{K}(x)u_i(x)^{\frac{2n}{n-2\sigma}}\,\mathrm{d}x\\&\qquad+\frac{n-2\sigma}{2n}\delta\int_{\partial B_\delta(x_i^{(j)})}\widetilde{K}(x)u_i(x)^{\frac{2n}{n-2\sigma}}(x-x_i^{(j)})\,\mathrm{d}s\\&\quad=\int_{B_\delta(x_i^{(j)})}\widetilde{K}(x)u_i(x)^{\frac{n+2\sigma}{n-2\sigma}}\nabla h_\delta(x)\,\mathrm{d}x.\end{aligned}$$

Then by using Proposition A.5 and (2.19), we have

$$\begin{aligned}&\int_{B_\delta(x_i^{(j)})}\nabla\widetilde{K}(x)u_i(x)^{\frac{2n}{n-2\sigma}}\,\mathrm{d}x\\&\quad=\delta\int_{\partial B_\delta(x_i^{(j)})}\widetilde{K}(x)u_i(x)^{\frac{2n}{n-2\sigma}}(x-x_i^{(j)})\,\mathrm{d}s\\&\qquad-\frac{2n}{n-2\sigma}\int_{B_\delta(x_i^{(j)})}\widetilde{K}(x)u_i(x)^{\frac{n+2\sigma}{n-2\sigma}}\nabla h_\delta(x)\,\mathrm{d}x\\&\quad=O(m_{ij}^{-2}).\end{aligned}$$

Therefore, (2.27) can be obtained from the above.

Multiplying (2.27) by $m_{ij}^{\frac{2}{n-2\sigma}(n-2\sigma-1)}$, we obtain

$$\begin{aligned}&\int_{B_\delta}\nabla Q_{(q^{(j)})}^{(n-2\sigma)}(m_{ij}^{\frac{2}{n-2\sigma}}x+m_{ij}^{\frac{2}{n-2\sigma}}x_i^{(j)})u_i(x+x_i^{(j)})^{\frac{2n}{n-2\sigma}}\,\mathrm{d}x\\&\quad=o_\delta(1)\int_{B_\delta}|m_{ij}^{\frac{2}{n-2\sigma}}x+m_{ij}^{\frac{2}{n-2\sigma}}x_i^{(j)}|^{n-2\sigma-1}u_i(x+x_i^{(j)})^{\frac{2n}{n-2\sigma}}\,\mathrm{d}x+o(1).\end{aligned}\tag{2.28}$$

It follows from (2.15) and Proposition A.7 that

$$\begin{aligned}&m_{ij}^{\frac{2}{n-2\sigma}(n-2\sigma-1)}\Big|\int_{B_\delta}\nabla R_{(q^{(j)})}(x+x_i^{(j)})u_i(x+x_i^{(j)})^{\frac{2n}{n-2\sigma}}\,\mathrm{d}x\Big|\\&\quad\le m_{ij}^{\frac{2}{n-2\sigma}(n-2\sigma-1)}\int_{B_\delta}|x+x_i^{(j)}|^{n-2\sigma-1}u_i(x+x_i^{(j)})^{\frac{2n}{n-2\sigma}}\,\mathrm{d}x\\&\quad\le m_{ij}^{\frac{2}{n-2\sigma}(n-2\sigma-1)}\int_{B_\delta}|x|^{n-2\sigma-1}u_i(x+x_i^{(j)})^{\frac{2n}{n-2\sigma}}\,\mathrm{d}x\\&\qquad+m_{ij}^{\frac{2}{n-2\sigma}(n-2\sigma-1)}\int_{B_\delta}|x_i^{(j)}|^{n-2\sigma-1}u_i(x+x_i^{(j)})^{\frac{2n}{n-2\sigma}}\,\mathrm{d}x\\&\quad=O(1),\end{aligned}\tag{2.29}$$

and using Proposition A.3, we conclude

$$
\begin{aligned}
&\int_{B_\delta} \nabla Q^{(n-2\sigma)}_{(q^{(j)})}(m_{ij}^{\frac{2}{n-2\sigma}} x + m_{ij}^{\frac{2}{n-2\sigma}} x_i^{(j)}) u_i(x+x_i^{(j)})^{\frac{2n}{n-2\sigma}}\,\mathrm{d}x \\
&\quad = \int_{|x|\le r_i} \nabla Q^{(n-2\sigma)}_{(q^{(j)})}(m_{ij}^{\frac{2}{n-2\sigma}} x + m_{ij}^{\frac{2}{n-2\sigma}} x_i^{(j)}) u_i(x+x_i^{(j)})^{\frac{2n}{n-2\sigma}}\,\mathrm{d}x + o(1) \\
&\quad = \int_{|y|\le R_i} Q^{(n-2\sigma)}_{(q^{(j)})}(y + m_{ij}^{\frac{2}{n-2\sigma}} x_i^{(j)})(m_{ij}^{-1} u_i(m_{ij}^{\frac{2}{n-2\sigma}} y + x_i^{(j)}))^{\frac{2n}{n-2\sigma}}\,\mathrm{d}y + o(1).
\end{aligned}
\tag{2.30}
$$

By (2.28)–(2.30), and letting $i \to \infty$ and $\delta \to 0$, we have

$$
\int_{\mathbb{R}^n} \nabla Q^{(n-2\sigma)}_{(q^{(j)})}(z+\xi^{(j)})(1+k^{(j)}|z|^2)^{-n}\,\mathrm{d}z = 0.
$$

Making a change of variable, we establish (2.26).

Theorem 2.1 follows from (2.25) and (2.26). □

3. Perturbation method for existence results

In this section, we use the method in [26] to establish a perturbation result Theorem 3.1. Similar results in the classical Nirenberg problem were obtained in [11,13,33]. In this section, we only concern with the case $1 \le \sigma < n/2$, since the proof of $0<\sigma<1$ can be found in [26, Section 3].

For a conformal transformation $\varphi_{P,t}$ (see (1.13)), we let

$$
T_{\varphi_{P,t}} v = v \circ \varphi_{P,t} |\det \mathrm{d}\varphi_{P,t}|^{\frac{n-2\sigma}{2n}},
$$

where $\mathrm{d}\varphi_{P,t}$ denotes the Jacobian of $\varphi_{P,t}$ satisfying

$$
\varphi_{P,t}^* g_0 = |\det \mathrm{d}\varphi_{P,t}|^{2/n} g_0.
$$

Let

$$
\mathscr{S} := \Big\{ v \in H^\sigma(\mathbb{S}^n) : ⨍_{\mathbb{S}^n} |v|^{\frac{2n}{n-2\sigma}}\,\mathrm{dvol}_{g_0} = 1 \Big\},
$$

$$
\mathscr{S}_0 := \Big\{ v \in \mathscr{S} : ⨍_{\mathbb{S}^n} x|v|^{\frac{2n}{n-2\sigma}}\,\mathrm{dvol}_{g_0} = 0 \Big\}.
$$

For $w \in \mathscr{S}_0$ and $p \in B^{n+1}$, let $\pi(w,p)$ be defined by $\pi(w,0) = w$ and $\pi(w,p) = T^{-1}_{\varphi_{P,t}} w$. It can be checked that the map $\pi : \mathscr{S}_0 \times B^{n+1} \to \mathscr{S}$ is a C^2 diffeomorphism, see [33].

It is easy to see that $1 \in \mathscr{S} \cap \mathscr{S}_0$. By some direct computations and elementary properties of spherical harmonics, we have

$$
T_1\mathscr{S} = \Big\{ \phi : \int_{\mathbb{S}^n} \phi = 0 \Big\} = \mathrm{span}\{\text{spherical harmonics of degree} \ge 1\},
$$

and

$$
T_1\mathscr{S}_0 = \mathrm{span}\{\text{spherical harmonics of degree} \ge 2\},
$$

where $T_1\mathscr{S}$ and $T_1\mathscr{S}_0$ denote the tangent space of at 1, respectively.

Let us use $\widetilde{w} \in T_1\mathscr{S}_0$ as local coordinates of $w \in \mathscr{S}_0$ near $w = 1$, and $\widetilde{w} = 0$ corresponds to $w = 1$. Using implicit function theorem, it is easy to check that $\mathscr{S}_0$ is represented locally near 1 as a graph over $T_1\mathscr{S}_0$: For all $\widetilde{w} \in T_1\mathscr{S}_0$, $\widetilde{w}$ sufficiently close to 0, there is a twice differentiable map $\mu(\widetilde{w}) \in \mathbb{R}$, $\eta(\widetilde{w}) \in \mathbb{R}^{n+1}$ defined in a neighborhood of 0 in $T_1\mathscr{S}_0$ with $\mu(0) = 0$, $\eta(0) = 0$, $D\mu(0) = 0$, and $D\eta(0) = 0$ such that

$$\fint_{\mathbb{S}^n} |1 + \widetilde{w} + \mu + \eta \cdot x|^{2n/(n-2\sigma)} = 1$$

and

$$\fint_{\mathbb{S}^n} |1 + \widetilde{w} + \mu + \eta \cdot x|^{2n/(n-2\sigma)} x = 0.$$

Now we consider a functional on $\mathscr{S}$:

$$E_K(v) = \frac{\fint_{\mathbb{S}^n} vP_\sigma(v)\,\mathrm{dvol}_{g_0}}{(\fint_{\mathbb{S}^n} K|v|^{2n/(n-2\sigma)}\,\mathrm{dvol}_{g_0})^{(n-2\sigma)/n}}. \tag{3.1}$$

We have

Proposition 3.1. *Let $n \geq 3$, $1 \leq \sigma < n/2$, and $K \in C^1(\mathbb{S}^n)$ be a positive function. There exist some constants $\varepsilon_1 = \varepsilon_1(n,\sigma) > 0$ and $\varepsilon_2 = \varepsilon_2(n,\sigma) > 0$, such that, if $\|K - 1\|_{L^\infty(\mathbb{S}^n)} \leq \varepsilon_1$, then*

$$\min_{w\in\mathscr{S}_0,\, \|w-1\|_{H^\sigma(\mathbb{S}^n)}\leq\varepsilon_2} E_K(w)$$

has a unique minimizer $w_K > 0$. Furthermore, $D^2E_K|_{\mathscr{S}_0}(w_K)$ is positive definite, and there exists a constant $C = C(n,\sigma)$ such that

$$\|w_K - 1\|_{H^\sigma(\mathbb{S}^n)} \leq C \inf_{c\in\mathbb{R}} \|K - c\|_{L^{2n/(n+2\sigma)}(\mathbb{S}^n)}. \tag{3.2}$$

Proof. Using the conformal invariance of P_σ and (3.1), for $\widetilde{w} \in T_1\mathscr{S}_0$ and $\widetilde{w}$ close to 0, we have

$$\widetilde{E}(\widetilde{w}) := E_1(w) = \fint_{\mathbb{S}^n} wP_\sigma(w)\,\mathrm{dvol}_{g_0},$$

where $w = 1 + \widetilde{w} + \mu(\widetilde{w}) + \eta(\widetilde{w}) \cdot x$.

It is well known (see [35]) that P_σ has eigenfunctions the spherical harmonics and eigenvalues

$$\lambda_k = \frac{\Gamma(k + \frac{n}{2} + \sigma)}{\Gamma(k + \frac{n}{2} - \sigma)}, \quad k \geq 0, \tag{3.3}$$

with multiplicity $(2k + n - 1)(k + n - 2)!/(n - 1)!k!$. It follows that

$$\widetilde{E}(\widetilde{w}) = P_\sigma(1)(1 + 2\mu(\widetilde{w})) + \int_{\mathbb{S}^n} \widetilde{w}P_\sigma(\widetilde{w})\,\mathrm{dvol}_{g_0} + O(\|\widetilde{w}\|^2_{H^\sigma(\mathbb{S}^n)}). \tag{3.4}$$

Thus,

$$\mu(\widetilde{w}) = -\frac{1}{2} \cdot \frac{n + 2\sigma}{n - 2\sigma} \int_{\mathbb{S}^n} \widetilde{w}^2\,\mathrm{dvol}_{g_0} + o(\|\widetilde{w}\|^2_{H^\sigma(\mathbb{S}^n)}). \tag{3.5}$$

Note that $\lambda_1 = \frac{n+2\sigma}{n-2\sigma} P_\sigma(1)$ (see (3.3)), then, by (3.4) and (3.5),

$$\widetilde{E}(\widetilde{w}) = P_\sigma(1) + \fint_{\mathbb{S}^n} (\widetilde{w} P_\sigma(\widetilde{w}) - \lambda_1 \widetilde{w}^2) \, \mathrm{dvol}_{g_0} + o(\|\widetilde{w}\|^2_{H^\sigma(\mathbb{S}^n)}). \tag{3.6}$$

Set $Q(\widetilde{w}) := \fint_{\mathbb{S}^n} (\widetilde{w} P_\sigma(\widetilde{w}) - \lambda_1 \widetilde{w}^2)$. It is clear that for any $\widetilde{w}$, $\widetilde{v} \in T_1 \mathscr{S}_0$,

$$D^2 Q(\widetilde{w})(\widetilde{v}, \widetilde{v}) = 2 \fint_{\mathbb{S}^n} (\widetilde{v} P_\sigma(\widetilde{v}) - \lambda_1 \widetilde{v}^2) \, \mathrm{dvol}_{g_0} \geq 2\Big(1 - \frac{\lambda_1}{\lambda_2}\Big) \|\widetilde{v}\|^2_{H^\sigma(\mathbb{S}^n)}, \tag{3.7}$$

which means the quadratic form $Q(\widetilde{w})$ is positive definite in $T_1 \mathscr{S}_0$. Furthermore, we derive from Sobolev inequality (1.3) that for some $\varepsilon_0 = \varepsilon_0(n, \sigma) > 0$,

$$\|E_K|_{\mathscr{S}_0} - E_1|_{\mathscr{S}_0}\|_{C^2(B_{\varepsilon_0}(1))} \leq O(\varepsilon), \tag{3.8}$$

provided $\|K - 1\|_{L^\infty(\mathbb{S}^n)} \leq \varepsilon$. Here $B_{\varepsilon_0}(1)$ denotes the ball in $\mathscr{S}_0$ of radius ε_1 centered at 1. Then we verify by direct computations that, for any $\widetilde{w} \in T_1 \mathscr{S}_0$ and any constant c,

$$\langle DE_K|_{\mathscr{S}_0}(1), \widetilde{w} \rangle = -2 P_\sigma(1) \Big(\fint_{\mathbb{S}^n} K \, \mathrm{dvol}_{g_0} \Big)^{(2\sigma - 2n)/n} \fint_{\mathbb{S}^n} (K - c) \widetilde{w} \, \mathrm{dvol}_{g_0}.$$

Therefore,

$$\|DE_K|_{\mathscr{S}_0}(1)\| \leq C \|K - c\|_{L^{2n/(n+2\sigma)}(\mathbb{S}^n)}. \tag{3.9}$$

As a consequence, we see from (3.6)–(3.8) that the minimizing problem has a unique minimizer w_K and $D^2 E_K|_{\mathscr{S}_0}(w_K)$ is positive definite. Estimate (3.2) follows from (3.7)–(3.9) with some standard functional analysis arguments.

We are left to prove the positivity of w_K.

Since w_K is a constrained local minimum, w_K satisfies the Euler–Lagrange equation for some Lagrange multiplier $\Lambda_K \in \mathbb{R}^{n+1}$:

$$P_\sigma(w_K) = (\lambda_K K - \Lambda_K \cdot x) |w_K|^{4\sigma/(n-2\sigma)} w_K \quad \text{on } \mathbb{S}^n, \tag{3.10}$$

where

$$\lambda_K = \frac{\fint_{\mathbb{S}^n} w_K P_\sigma(w_K) \, \mathrm{dvol}_{g_0}}{\fint_{\mathbb{S}^n} K |w_K|^{\frac{2n}{n-2\sigma}} \, \mathrm{dvol}_{g_0}}.$$

Then, using the same argument in [26, Lemma 3.6], we obtain $w_K \geq 0$. Note that Eq. (3.10) can be equivalently rewritten as

$$w_K(\xi) = c_{n,\sigma} \int_{\mathbb{S}^n} \frac{(\lambda_K K(\eta) - \Lambda_K \cdot \eta) w_K(\eta)^{(n+2\sigma)/(n-2\sigma)}}{|\xi - \eta|^{n-2\sigma}} \, \mathrm{d}\eta,$$

by using (1.2). If there exists some $\xi_0 \in \mathbb{S}^n$ such that $w_K(\xi_0) = 0$, then using the facts $\|K - 1\|_{L^\infty(\mathbb{S}^n)} \leq \varepsilon$, $|\lambda_k - c(n, \sigma)| = O(\varepsilon)$, and $|\Lambda_K| = O(\varepsilon)$, we immediately obtain $(\lambda_K K - \Lambda_K \cdot x) > 0$ for sufficient small ε. It follows that $w_K \equiv 0$. However, it is a contradiction because $w_K \in \mathscr{S}_0$, which in turn implies that $w_K > 0$. The proof is finished. $\square$

As illustrated before, we write $v = \pi(w, p) = T^{-1}_{\varphi_{P,t}} w$ for any $v \in \mathscr{S}$ with $w \in \mathscr{S}_0$, $p = sP \in B^{n+1}$, $t \geq 1$ and $s = (t-1)/t$. It is easy to check that $E_K(v) = E_{K \circ \varphi_{P,t}}(w)$. Let us rewrite $E_K(v)$ in the (w, p) variables:

$$I(w, p) := E_K(v) = E_{K \circ \varphi_{P,t}}(w). \tag{3.11}$$

By Proposition 3.1, for some $\varepsilon_2 > 0$, we have

$$\min_{w\in\mathscr{S}_0,\|w-1\|_{H^\sigma(\mathbb{S}^n)}\le\varepsilon_2} I(w,p) = \min_{w\in\mathscr{S}_0,\|w-1\|_{H^\sigma(\mathbb{S}^n)}\le\varepsilon_2} E_{K\circ\varphi_{P,t}}(w). \tag{3.12}$$

Furthermore, if $\|K-1\|_{L^\infty(\mathbb{S}^n)} \le \varepsilon_1$, the minimizer exists and we denote it as w_p.

Let ε_2 be as in Proposition 3.1, we define

$$\mathcal{N}_1 := \{w \in \mathscr{S}_0 : \|w-1\|_{H^\sigma(\mathbb{S}^n)} \le \varepsilon_2\}. \tag{3.13}$$

For $1 < t \le \infty$, define

$$\begin{aligned}\mathcal{N}_2(t) := \{v \in \mathscr{S} : v = \pi(w,p) \text{ for some } w \in \mathcal{N}_1 \text{ and } p = sP,\\ P\in\mathbb{S}^n, s = \frac{\zeta-1}{\zeta}, 1\le\zeta<t\},\end{aligned}$$

and

$$\mathcal{N}_3(t) := \{v \in H^\sigma(\mathbb{S}^n)\backslash\{0\} : cv \in \mathcal{N}_2(t) \text{ for some constant } c > 0\}.$$

Using the Proposition 3.1 and the properties of the integral equation, as well as a natural fibration of H^σ, we can obtain the following perturbation result:

Theorem 3.1. *Let $n \ge 3$, $0 < \sigma < n/2$, and $K \in C^1(\mathbb{S}^n)$ ($K \in C^{1,1}(\mathbb{S}^n)$ if $\sigma \le 1/2$) be a positive nonconstant function and $\varphi_{P,t}$ be as in (1.13) for $P \in \mathbb{S}^n$, $1 \le t < \infty$. Suppose that there exists some constant $\varepsilon_3 = \varepsilon_3(n) \in (0,\varepsilon_1)$ such that $\|K-1\|_{L^\infty(\mathbb{S}^n)} \le \varepsilon_3$. Suppose also that for all $P \in \mathbb{S}^n$, we have*

$$\|K\circ\varphi_{P,t} - K(P)\|^2_{L^2(\mathbb{S}^n)} \le o\Big(\Big|\int_{\mathbb{S}^n} K\circ\varphi_{P,t}(x)x\,\mathrm{dvol}_{g_0}\Big|\Big) \quad \text{as } t\to\infty,$$

and

$$\deg\Big(\int_{\mathbb{S}^n} K\circ\varphi_{P,t}(x)x\,\mathrm{dvol}_{g_0}, B^{n+1}, 0\Big) \ne 0 \quad \text{for large } t.$$

Then (1.1) has at least one positive solution. Furthermore, for any $\alpha \in (0,1)$ satisfying that $\alpha + 2\sigma$ is not an integer, there exists constant $C_1 > 0$ depending only on n,α,σ, such that for all $C \ge C_1$,

$$\begin{aligned}&\deg\Big(v - (P_\sigma)^{-1}K|v|^{4\sigma/(n-2\sigma)}v, \mathcal{N}_3(t)\cap\{v\in C^{2\sigma+\alpha} : \|v\|_{C^{2\sigma+\alpha}} < C\}, 0\Big)\\ &\quad = (-1)^n\deg\Big(\int_{\mathbb{S}^n} K\circ\varphi_{P,t}(x)x\,\mathrm{dvol}_{g_0}, B^{n+1}, 0\Big).\end{aligned} \tag{3.14}$$

Proof. We initiate the proof with Proposition 3.1 since the case $\sigma \in (0,1/2)$ follows from a degree argument, see [26]. For each $p \in B_1$, let w_p is the minimizer of (3.12), set

$$\mathscr{A}_p = \frac{1}{n}\int_{\mathbb{S}^n} \langle\nabla(K\circ\varphi_{P,t}), \nabla x\rangle w_p^{2n/(n-2\sigma)}\,\mathrm{dvol}_{g_0}, \quad \mathscr{B}_p = \int_{\mathbb{S}^n} K\circ\varphi_{P,t}(x)x\,\mathrm{dvol}_{g_0}.$$

It is clear that $\mathscr{B}_p \ne 0$. We write

$$\mathscr{A}_p = \mathscr{B}_p + \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} = \fint_{\mathbb{S}^n} (K \circ \varphi_{P,t} - K(P)) x (w_p^{2n/(n-2\sigma)} - 1) \, \mathrm{dvol}_{g_0},$$
$$\mathrm{II} = -\frac{1}{n} \fint_{\mathbb{S}^n} (K \circ \varphi_{P,t} - K(P)) \langle \nabla x, \nabla (w_p^{2n/(n-2\sigma)}) \rangle \, \mathrm{dvol}_{g_0}.$$

The proof consists of 3 steps.

Step 1: Estimates of I.

By using Cauchy-Schwartz inequality and (3.2), we have, as $t \to \infty$,

$$\begin{aligned} |\mathrm{I}| &\le C \|K \circ \varphi_{P,t} - K(P)\|_{L^2(\mathbb{S}^n)} \|w_p^{2n/(n-2\sigma)} - 1\|_{L^2(\mathbb{S}^n)} \\ &\le C \|K \circ \varphi_{P,t} - K(P)\|_{L^2(\mathbb{S}^n)} \|w_p - 1\|_{H^\sigma(\mathbb{S}^n)} \\ &\le C \|K \circ \varphi_{P,t} - K(P)\|^2_{L^2(\mathbb{S}^n)} \\ &\le o(t) \mathscr{B}_p. \end{aligned} \tag{3.15}$$

Step 2: Estimates of II.

Firstly, let us claim that there exists a constant C depending on n, σ, ε_1 such that

$$\|\nabla (w_p^{2n/(n-2\sigma)})\|_{L^2(\mathbb{S}^n)} \le C \|K \circ \varphi_{P,t} - K(P)\|_{L^2(\mathbb{S}^n)}, \tag{3.16}$$

here ε_1 is given by Proposition 3.1. Once we verify it, together with Cauchy-Schwartz inequality, it will yield that, as $t \to \infty$,

$$|\mathrm{II}| \le C \|K \circ \varphi_{P,t} - K(P)\|_{L^2(\mathbb{S}^n)} \|\nabla (w_p^{2n/(n-2\sigma)})\|_{L^2(\mathbb{S}^n)} \le o(t) \mathscr{B}_p. \tag{3.17}$$

Now we give the proof of the claim.

As in (3.10), we know that

$$P_\sigma(w_p) = (\lambda_p K \circ \varphi_{P,t} - \Lambda_p \cdot x) w_p^{(n+2\sigma)/(n-2\sigma)} \quad \text{on } \mathbb{S}^n, \tag{3.18}$$

where

$$\lambda_p = \frac{\fint_{\mathbb{S}^n} w_p P_\sigma(w_p) \, \mathrm{dvol}_{g_0}}{\fint_{\mathbb{S}^n} K \circ \varphi_{P,t} w_p^{\frac{2n}{n-2\sigma}} \, \mathrm{dvol}_{g_0}} \tag{3.19}$$

and $\Lambda_p \in \mathbb{R}^{n+1}$. It is easy to see w_p solves (1.1) if and only if $\Lambda_p = 0$.

Denote $v_p = w_p - 1$, using Taylor's theorem and (3.18), we get

$$\begin{aligned} P_\sigma(v_p) =& (\lambda_p K \circ \varphi_{P,t} - \Lambda_p \cdot x) w_p^{(n+2\sigma)/(n-2\sigma)} - P_\sigma(1) \\ =& (\lambda_p - P_\sigma(1)) K \circ \varphi_{P,t} + P_\sigma(1)(K \circ \varphi_{P,t} - 1) - \Lambda_p \cdot x \\ &+ \frac{n+2\sigma}{n-2\sigma} (\lambda_p K \circ \varphi_{P,t} - \Lambda_p \cdot x) v_p \\ &+ (\lambda_p K \circ \varphi_{P,t} - \Lambda_p \cdot x) o(|v_p|) \\ =& \mathcal{R}(x) + \frac{n+2\sigma}{n-2\sigma} \mathcal{Q}(x) v_p + o(|v_p|), \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} \mathcal{R}(x) =& (\lambda_p - P_\sigma(1)) K \circ \varphi_{P,t} + P_\sigma(1)(K \circ \varphi_{P,t} - 1) - \Lambda_p \cdot x, \\ \mathcal{Q}(x) =& (\lambda_p K \circ \varphi_{P,t} - \Lambda_p \cdot x). \end{aligned}$$

By the stereographic projection and Green's representation (1.2), we have (up to a harmless constant)

$$\widetilde{v}_p(y) = \int_{\mathbb{R}^n} \frac{|J_F|^{\frac{2\sigma}{n}} \mathcal{Q}(F(y))(\frac{n+2\sigma}{n-2\sigma}\widetilde{v}_p + o(|\widetilde{v}_p|)) + |J_F|^{\frac{n-2\sigma}{2n}} \mathcal{R}(F(y))}{|z-y|^{n-2\sigma}} \,\mathrm{d}z,$$

where $\widetilde{v}_p = |J_F|^{2n/(n-2\sigma)}(v_p \circ F)$ as illustrated in the introduction. We consider

$$\begin{aligned}\widetilde{v}_p(y) &= \int_{B_3} \frac{\frac{n+2\sigma}{n-2\sigma}|J_F|^{\frac{2\sigma}{n}} \mathcal{Q}(F(z))\widetilde{v}_p(y)}{|y-z|^{n-2\sigma}} \,\mathrm{d}z \\ &\quad + \Big(\int_{\mathbb{R}^n\setminus B_3} \frac{\frac{n+2\sigma}{n-2\sigma}|J_F|^{\frac{2\sigma}{n}} \mathcal{Q}(F(z))\widetilde{v}_p(y)}{|y-z|^{n-2\sigma}} \,\mathrm{d}z \\ &\quad + \int_{\mathbb{R}^n} \frac{|J_F|^{\frac{2\sigma}{n}} \mathcal{Q}(F(z))o(|\widetilde{v}_p|) + |J_F|^{\frac{n+2\sigma}{2n}} \mathcal{R}(F(z))}{|z-y|^{n-2\sigma}} \,\mathrm{d}z\Big) \\ &=: \int_{B_3} \frac{\frac{n+2\sigma}{n-2\sigma}|J_F|^{\frac{2\sigma}{n}} \mathcal{Q}(F(z))\widetilde{v}_p(y)}{|y-z|^{n-2\sigma}} \,\mathrm{d}z + \mathcal{H}(x).\end{aligned}$$

Now we give an upper bound of $\|\mathcal{H}\|_{L^\infty(\mathbb{R}^n)}$.

It follows from Proposition 3.1 that

$$|\lambda_p - P_\sigma(1)| = \left|\frac{\|v_p\|_{H^\sigma(\mathbb{S}^n)} + 2P_\sigma(1)⨍_{\mathbb{S}^n} v_p \,\mathrm{dvol}_{g_0}}{⨍_{\mathbb{S}^n} K\circ\varphi_{P,t}|w_p|^{\frac{2n}{n-2\sigma}} \,\mathrm{dvol}_{g_0}}\right| \le O(\|K\circ\varphi_{P,t} - 1\|_{L^\infty(\mathbb{S}^n)}). \tag{3.21}$$

Multiplying (3.18) by $\Lambda_p \cdot x$ and integrating over both sides we have

$$\begin{aligned}&-\int_{\mathbb{S}^n} (\Lambda_p\cdot x)^2 w_p^{\frac{n+2\sigma}{n-2\sigma}} \,\mathrm{dvol}_{g_0} \\ &\quad = \lambda_1 \int_{\mathbb{S}^n} w_p \Lambda_p\cdot x \,\mathrm{dvol}_{g_0} - \lambda_p \int_{\mathbb{S}^n} K\circ\varphi w_p^{\frac{n+2\sigma}{n-2\sigma}} \Lambda_p\cdot x \,\mathrm{dvol}_{g_0}.\end{aligned} \tag{3.22}$$

It is easy to see that

$$|\Lambda_p| = O(\|K\circ\varphi_{P,t} - 1\|_{L^\infty(\mathbb{S}^n)}). \tag{3.23}$$

Meanwhile, we get from (3.21) and (3.23) that

$$\|\mathcal{H}\|_{L^\infty(\mathbb{S}^n)} \le O(\|K\circ\varphi_{P,t} - 1\|_{L^\infty(\mathbb{S}^n)}). \tag{3.24}$$

Thanks to (3.24) and [27, Corollary 2.1], we obtain

$$\|v_p\|_{L^\infty(\mathbb{S}^n)} \le O(\|K\circ\varphi_{P,t} - 1\|_{L^\infty(\mathbb{S}^n)}). \tag{3.25}$$

Putting the above estimate and (3.25) together, we find

$$|\Lambda_p| \le C(n,\sigma)\|K\circ\varphi_{P,t} - K(P)\|_{L^2(\mathbb{S}^n)}. \tag{3.26}$$

Using Proposition 3.1, (3.25) and $\|K-1\|_{L^\infty} \le \varepsilon_3 < \varepsilon_1$, we have

$$\begin{aligned}
&\left|\lambda_p - \frac{P_\sigma(1)}{K(P)}\right| \\
&\le C(n,\sigma,\varepsilon_1)\Big(\frac{\|w_p-1\|_{H^\sigma(\mathbb{S}^n)} + \|w_p-1\|_{L^1(\mathbb{S}^n)} + \fint_{\mathbb{S}^n} |K\circ\varphi_{P,t} - K(P)| w_p^{\frac{2n}{n-2\sigma}} \,\mathrm{dvol}_{g_0}}{\fint_{\mathbb{S}^n} K\circ\varphi_{P,t} w_p^{\frac{2n}{n-2\sigma}} \,\mathrm{dvol}_{g_0}}\Big) \\
&\le C(n,\sigma,\varepsilon_1)\|K\circ\varphi_{P,t} - K(P)\|_{L^2(\mathbb{S}^n)}.
\end{aligned} \tag{3.27}$$

Then, by (3.27) and (3.26), we have

$$\|(\lambda_p K\circ\varphi_{P,t} - \Lambda_p\cdot x) w_p^{\frac{n+2\sigma}{n-2\sigma}} - P_\sigma(1)\|_{L^2(\mathbb{S}^n)} \le C(n,\sigma,\varepsilon_1)\|K\circ\varphi_{P,t} - K(P)\|_{L^2(\mathbb{S}^n)}. \tag{3.28}$$

Combining (3.20), (3.28), (3.3), and the spherical expansion of $w_p - 1$, we arrive at

$$\|w_p - 1\|^2_{H^1(\mathbb{S}^n)} \le \int_{\mathbb{S}^n} (P_\sigma(w_p-1))^2 \,\mathrm{dvol}_{g_0} \le C(n,\sigma,\varepsilon_1)\|K\circ\varphi_{P,t} - K(P)\|_{L^2(\mathbb{S}^n)}.$$

This justifies the claim.

Step 3: Complete the proof.

From (3.15) and (3.17), we know that for sufficiently large t, there exists $0 < C_0 < 1$ such that

$$\mathscr{A}_p \cdot \mathscr{B}_p \ge (1-C_0)|\mathscr{B}_p|^2.$$

As a consequence of the homotopy invariance property of the degree,

$$\deg(\mathscr{A}_p, B^{n+1}, 0) = \deg(\mathscr{B}_p, B^{n+1}, 0).$$

We note that Λ_p can also be computed more directly from the function K as follows. In view of (3.18) and the Kazdan-Warner identity (see [27]), we have

$$\int_{\mathbb{S}^n} \langle \nabla(\lambda_p K\circ\varphi_{P,t} - \Lambda_p\cdot x), \nabla x_i\rangle w_p^{\frac{2n}{n-2\sigma}} \,\mathrm{dvol}_{g_0} = 0, \quad 1 \le i \le n+1.$$

It follows that, for $1 \le i \le n+1$,

$$\sum_{j=1}^{n+1} \Lambda_p^j \int_{\mathbb{S}^n} \langle \nabla x_j, \nabla x_i\rangle w_p^{\frac{2n}{n-2\sigma}} \,\mathrm{dvol}_{g_0} = \lambda_p \int_{\mathbb{S}^n} \langle \nabla(K\circ\varphi_{P,t}), \nabla x_i\rangle w_p^{\frac{2n}{n-2\sigma}} \,\mathrm{dvol}_{g_0}. \tag{3.29}$$

Note that, as $\varepsilon_3 \to 0$, by Proposition 3.1, we have $w_p \to 1$ uniformly for small ε_3. This implies that the coefficient matrix on the left hand side of (3.29) is positive definite:

$$\Big(\int_{\mathbb{S}^n} \langle \nabla x_i, \nabla x_j\rangle w_p^{\frac{2n}{n-2\sigma}} \,\mathrm{dvol}_{g_0}\Big)_{1\le i,j\le n+1} > 0.$$

It follows that, for t large with $s = (t-1)/t$,

$$\deg\left(\Lambda_p, B_s^{n+1}, 0\right) = \deg\left(\mathscr{A}_p, B^{n+1}, 0\right) = \deg\left(\mathscr{B}_p, B^{n+1}, 0\right), \tag{3.30}$$

where B_s^{n+1} denotes the open ball in $\mathbb{R}^{n+1}$ with centered at the origin and s as the radius. Therefore, Λ_p has to have a zero inside B^{n+1} which immediately implies that (1.1) has at least one positive solution.

Let $I(w,p)$ be as in (3.11), the same argument in [26, Theorem 3.1] can be applied to obtain that, for any $p_0 \in B^{n+1}$,

$$\partial_p I(w_{p_0},p)|_{p=p_0} = -\frac{n-2\sigma}{n}\Big(\fint_{\mathbb{S}^n} K(T^{-1}_{\varphi_{p_0}} w_{p_0})^{2n/(n-2\sigma)}\,\mathrm{dvol}_{g_0}\Big)^{\frac{2\sigma-n}{n}} \times \partial_p\Big(\fint_{\mathbb{S}^n} \Lambda_{p_0}\cdot\varphi^{-1}_{p_0}\circ\varphi_{P,t} w_{p_0}^{\frac{2n}{n-2\sigma}}\,\mathrm{dvol}_{g_0}\Big)\Big|_{p=p_0}.$$

By Li [33, Appendix A], we get the matrix

$$\partial_p\Big(\fint_{\mathbb{S}^n} \Lambda_{p_0}\cdot\varphi^{-1}_{p_0}\circ\varphi_{P,t} w_{p_0}^{\frac{2n}{n-2\sigma}}\,\mathrm{dvol}_{g_0}\Big)\Big|_{p=p_0}$$

is invertible with positive determinant. Therefore, for t large with $s=(t-1)/t$, we have

$$(-1)^{n+1}\deg(\Lambda_p, B_s^{n+1}, 0) = \deg(\partial_p I(w_p,p), B_s^{n+1}, 0).$$

The rest of the proof of (3.14) is similar to that in [33, page 386] and we omit them here. □

4. Proof of Theorem 1.1 and Corollary 1.1

In this section, we give the proof of Theorem 1.1 and Corollary 1.1.

Proof of Theorem 1.1. The proof is divided into three steps.

Step 1: Proof of (1.14). Suppose the contrary that the solution v to (1.1) has at least one isolated simple blow up point. In the case of $\sharp\mathscr{K}^-_{n-2\sigma} \le 1$, it follows from Theorem 2.1 that v has only one blow up point, and then we obtain from [27, Theorem 3.5] that there exists a constant $C>0$ such that

$$1/C < v < C \quad \text{on } \mathbb{S}^n.$$

In another case, we assume that v corresponding to (1.1) blow up at $\{q^{(1)},\ldots,q^{(k)}\}$ with $k\ge 2$. By Theorem 2.1 we know that equation

$$M(q^{(1)},\ldots,q^{(k)},\eta^{(1)},\ldots,\eta^{(k)})\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_k\end{pmatrix} = 0$$

has at least one solution $\lambda_1,\ldots,\lambda_k>0$. By Cramer's Rule we can deduce a contradiction. Therefore, in both cases we prove that (1.14) holds.

Step 2: We now prove the following claim: let ε_3 be as in Proposition 3.1, and $\mathcal{N}_1$ be as in (3.13), then there exists a constant $\varepsilon_4>0$ such that, for $0\le\mu\le\varepsilon_4$, we have $\|K_\mu - 1\|_{L^\infty(\mathbb{S}^n)} < \varepsilon_3$, where $K_\mu := \mu K + (1-\mu)$. Furthermore, if v is any solution to (1.1) with $K=K_\mu$, and there exists $(w,p)\in\mathscr{S}_0\times B^{n+1}$ such that $v=\pi(w,p)$, then $w\in\mathcal{N}_1$. The proof of the claim is similar to that in [26, page 1529], and we omit it here.

Step 3: Proofs of (1.15) and (1.16). Equation (1.15) follows from [33, Lemma 6.7]. Next we prove (1.16).

From the proof of Step 1, it is known that there exists some constant $C_0 > 1$ such that for all $\varepsilon_4 \le \mu < 1$,

$$1/C_0 < v_\mu < C_0,$$

where v_μ is any solution of (1.1) with $K = K_\mu$.

For $\sigma \in [1, n/2)$, it follows from the homotopy property of the Leray Schauder degree and Proposition 3.1 that

$$\begin{aligned}
&\deg\Big(v - (P_\sigma)^{-1} K v^{\frac{(n+2\sigma)}{(n-2\sigma)}}, C^{2\sigma+\alpha}(\mathbb{S}^n) \cap \{1/C_0 \le v_\mu \le C_0\}, 0\Big)\\
&\quad = \deg\Big(v - (P_\sigma)^{-1} K_{\varepsilon_4} v^{\frac{(n+2\sigma)}{(n-2\sigma)}}, C^{2\sigma+\alpha}(\mathbb{S}^n) \cap \{1/C_0 \le v_\mu \le C_0\}, 0\Big)\\
&\quad = (-1)^n \deg\Big(\int_{\mathbb{S}^n} K_{\varepsilon_4} \circ \varphi_{P,t}(x) x \,\mathrm{dvol}_{g_0}, B^{n+1}, 0\Big)\\
&\quad = (-1)^n \deg\Big(\int_{\mathbb{S}^n} K \circ \varphi_{P,t}(x) x \,\mathrm{dvol}_{g_0}, B^{n+1}, 0\Big).
\end{aligned}$$

For $\sigma \in (0, 1)$, Eq. (1.16) follows from [26, Theorem 3.2].

If

$$\deg\Big(\int_{\mathbb{S}^n} K \circ \varphi_{P,t}(x) x \,\mathrm{dvol}_{g_0}, B^{n+1}, 0\Big) \ne 0$$

for large t, then (1.1) has at least one solution. □

Proof of Corollary 1.1. Corollary 1.1 follows from Proposition 3.1 and [33, Lemma 6.7], the proof of Theorem 1.1 and [33, Corollary 6.2]. □

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Declarations

Conflict of interest statement The authors declared that they have no conflict of interest.

Appendix A

In this section, we review some results about the blow up profiles for integral equations obtained in Jin-Li-Xiong [27]. For any $x \in \mathbb{R}^n$ and $r > 0$, $B_r(x)$ denotes the ball in $\mathbb{R}^n$ with radius r and center x, and $B_r := B_r(0)$.

Let Ω be a domain in $\mathbb{R}^n$ and K_i are nonnegative bounded functions in $\mathbb{R}^n$. Let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence of nonnegative constants satisfying $\lim_{i\to\infty} \tau_i = 0$, and set

$$p_i = \frac{n+2\sigma}{n-2\sigma} - \tau_i.$$

Suppose that $0 \le u_i \in L^{\infty}_{loc}(\mathbb{R}^n)$ satisfies the nonlinear integral equation

$$u_i(x) = \int_{\mathbb{R}^n} \frac{K_i(y)u_i(y)^{p_i}}{|x-y|^{n-2\sigma}}\,\mathrm{d}y \quad \text{in } \Omega. \tag{A.1}$$

We assume that $K_i \in C^1(\Omega)$ ($K_i \in C^{1,1}(\Omega)$ if $\sigma \le 1/2$) and, for some positive constants A_1 and A_2,

$$1/A_1 \le K_i,\ \|K_i\|_{C^1(\Omega)} \le A_2,\ \left(\|K_i\|_{C^{1,1}(\Omega)} \le A_2 \text{ if } \sigma \le \frac{1}{2}\right). \tag{A.2}$$

Proposition A.1. (Pohozaev type identity) *Let $u \ge 0$ in $\mathbb{R}^n$, and $u \in C(\overline{B}_R)$ be a solution of*

$$u(x) = \int_{B_R} \frac{K(y)u(y)^p}{|x-y|^{n-2\sigma}}\,\mathrm{d}y + h_R(x),$$

where $1 < p \le \frac{n+2\sigma}{n-2\sigma}$, and $h_R(x) \in C^1(B_R)$, $\nabla h_R \in L^1(B_R)$. Then

$$\begin{aligned}
&\Big(\frac{n-2\sigma}{2} - \frac{n}{p+1}\Big)\int_{B_R} K(x)u(x)^{p+1}\,\mathrm{d}x - \frac{1}{p+1}\int_{B_R} x\nabla K(x)u(x)^{p+1}\,\mathrm{d}x\\
&\quad= \frac{n-2\sigma}{2}\int_{B_R} K(x)u(x)^p h_R(x)\,\mathrm{d}x + \int_{B_R} x\nabla h_R(x)K(x)u(x)^p\,\mathrm{d}x\\
&\qquad - \frac{R}{p+1}\int_{\partial B_R} K(x)u(x)^{p+1}\,\mathrm{d}s.
\end{aligned}$$

Proposition A.2. *Suppose that $0 \le u_i \in L^{\infty}_{loc}(\mathbb{R}^n)$ satisfies (A.1) with K_i satisfying (A.2). Suppose that $x_i \to 0$ is an isolated blow up point of $\{u_i\}$, i.e., for some positive constants A_3 and $\bar r$ independent of i,*

$$|x-x_i|^{2\sigma/(p_i-1)}u_i(x) \le A_3 \quad \textit{for all } x \in B_{\bar r} \subset \Omega.$$

Then for any $0 < r < \bar r/3$, we have the following Harnack inequality

$$\sup_{B_{2r}(x_i)\setminus\overline{B_{r/2}(x_i)}} u_i \le C \inf_{B_{2r}(x_i)\setminus\overline{B_{r/2}(x_i)}} u_i,$$

where C is a positive constant depending only on $\sup_i \|K_i\|_{L^\infty(B_{\bar r}(x_i))}, n, \sigma, \bar r$ and A_3.

Proposition A.3. *Under the hypotheses in Proposition A.2. Then for every $R_i \to \infty$, $\varepsilon_i \to 0^+$, we have, after passing to a subsequence (still denoted as $\{u_i\}$, $\{x_i\}$, etc.), that*

$$\|m_i^{-1}u_i(m_i^{-(p_i-1)/2\sigma}\cdot+x_i)-(1+k_i|\cdot|^2)^{(2\sigma-n)/2}\|_{C^2(B_{2R_i}(0))}\leq\varepsilon_i,$$

$$r_i:=R_i m_i^{-(p_i-1)/2\sigma}\to 0 \quad \textit{as } i\to\infty,$$

where $m_i:=u_i(x_i)$ and $k_i:=(K_i(x_i)\pi^{n/2}\Gamma(\sigma)/\Gamma(\frac{n}{2}+\sigma))^{1/\sigma}$.

Proposition A.4. *Under the hypotheses of Proposition A.2, there exists a positive constant $C=C(n,\sigma,A_1,A_2,A_3)$ such that,*

$$u_i(x)\geq C^{-1}m_i(1+k_i m_i^{(p_i-1)/\sigma}|x-x_i|^2)^{(2\sigma-n)/2} \qquad \textit{for all } |x-x_i|\leq 1.$$

In particular, for any $e\in\mathbb{R}^n$, $|e|=1$, we have

$$u_i(x_i+e)\geq C^{-1}m_i^{-1+((n-2\sigma)/2\sigma)\tau_i},$$

where $\tau_i=(n+2\sigma)/(n-2\sigma)-p_i$.

Proposition A.5. *Under the hypotheses of Proposition A.2 with $\bar{r}=2$, and in addition that $x_i\to 0$ is also an isolated simple blow up point with constant ρ, we have*

$$\tau_i=O(u_i(x_i)^{-c_1+o(1)}) \quad \textit{and} \quad u_i(x_i)^{\tau_i}=1+o(1),$$

where $c_1=\min\{2,2/(n-2\sigma)\}$. Moreover,

$$u_i(x)\leq Cu_i^{-1}(x_i)|x-x_i|^{2\sigma-n} \qquad \textit{for all } |x-x_i|\leq 1.$$

Proposition A.6. *Under the hypotheses of Proposition A.5, let*

$$\begin{aligned} T_i(x) :=& u_i(x_i)\int_{B_1(x_i)}\frac{K_i(y)u_i(y)^{p_i}}{|x-y|^{n-2\sigma}}\,\mathrm{d}y+u_i(x_i)\int_{\mathbb{R}^n\backslash B_1(x_i)}\frac{K_i(y)u_i(y)^{p_i}}{|x-y|^{n-2\sigma}}\,\mathrm{d}y\\ =:& T_i'(x)+T_i''(x). \end{aligned}$$

Then, after passing to a subsequence,

$$T_i'(x)\to a|x|^{2\sigma-n} \qquad \textit{in } C^2_{loc}(B_1\backslash\{0\})$$

and

$$T_i''(x)\to h(x) \qquad \textit{in } C^2_{loc}(B_1)$$

for some $h(x)\in C^2(B_2)$, where

$$a=\Big(\frac{\pi^{n/2}\Gamma(\sigma)}{\Gamma(\frac{n}{2}+\sigma)}\Big)^{-\frac{n}{2\sigma}}\int_{\mathbb{R}^n}\Big(\frac{1}{1+|y|^2}\Big)^{\frac{n+2\sigma}{2}}\,\mathrm{d}y\lim_{i\to\infty}K_i(0)^{\frac{2\sigma-n}{2\sigma}}.$$

Consequently, we have

$$u_i(x_i)u_i(x)\to a|x|^{2\sigma-n}+b(x) \qquad \textit{in } C^2_{loc}(B_1\backslash\{0\}).$$

Proposition A.7. *Under the hypotheses of Proposition A.2, we have*

$$\int_{|y-y_i|\leq r_i}|y-y_i|^s u_i(y)^{p_i+1}\,\mathrm{d}y=\begin{cases}O(u_i(y_i)^{-2s/(n-2\sigma)}), & -n<s<n,\\ O(u_i(y_i)^{-2n/(n-2\sigma)}\log u_i(y_i)), & s=n,\\ o(u_i(y_i)^{-2n/(n-2\sigma)}), & s>n,\end{cases}$$

and

$$\int_{r_i<|y-y_i|\le 1} |y-y_i|^s u_i(y)^{p_i+1}\,\mathrm{d}y = \begin{cases} o(u_i(y_i)^{-2s/(n-2\sigma)}), & -n<s<n,\\ O(u_i(y_i)^{-2n/(n-2\sigma)}\log u_i(y_i)), & s=n,\\ O(u_i(y_i)^{-2n/(n-2\sigma)}), & s>n. \end{cases}$$

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Gradient estimates for elliptic systems from composite materials with closely spaced stiff $C^{1,\gamma}$ inclusions

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This paper is devoted to establishing the pointwise upper and lower bound estimates of the gradient of the solutions to a class of general elliptic systems with Hölder continuous coefficients in a narrow region where the upper and lower boundaries are $C^{1,\gamma}$, $0 < \gamma < 1$, weaker than the previous $C^{2,\gamma}$ assumption. These estimates play a key role in the damage analysis of composite materials. From our results, the damage may initiate from the narrowest place.

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1. Introduction

1.1. Background and problem formulation

Damage analysis of composite materials is of great significance in engineering, which is also an important application of gradient estimates of second-order elliptic systems in partial differential equations [16,27,28]. Babuška et al. [7] computationally analyzed the damage and fracture in fiber composite materials, and some numerical results of stress concentration are given, which plays a key role in the problem of stress concentration in materials. At present, a lot of progress has been made in precisely studying this field concentration phenomenon, see e.g. [2,9,15,16,24–26,28].

From the viewpoint of mathematics, Li and Vogelius [29] described this stress concentration by using the gradient of the solution to a specific class of elliptic systems with partially degenerate coefficients. Before studying the gradient estimates of elliptic systems, Bonnetier et al. [14] considered the simplified scalar equation:

$$\nabla \cdot \big((1+(a-1)\chi_{\cup_{i=1}^N D_i})\nabla v\big) = 0, \quad \text{in} \;\; \Omega, \tag{1.1}$$

to model a problem of electric conduction, where $a \neq 1$, $\Omega \subset \mathbb{R}^n$ represents a bounded domain, $N \in \mathbb{Z}_+$ represents the number of the inclusions and $D_i \subset \Omega$ represents the inclusions in the matrix material which are close to each other. Bonnetier and Vogelius [14] proved rigorously that the gradient of the solution to (1.1) is indeed bounded when $N = 2$ and D_1, D_2 are two touching disks with comparable radii in $\mathbb{R}^2$. Li and Vogelius [29] extended the result to a large class of divergence form second order elliptic equations with piecewise Hölder continuous coefficients in $\mathbb{R}^2$, when $N \geq 2$ and the inclusions are $C^{1,\gamma}, (0 < \gamma < 1)$ (see Definition 1.2 below). Li and Nirenberg [28] further extended such results to general divergence form elliptic systems with Hölder continuous coefficients satisfying the strong elliptic condition.

When the coefficients degenerate to infinity in D_i, the gradient of the solution is no longer bounded but blows up. For the scalar case, we call it perfect conductivity problem. Let ε be the distance between the two inclusions. The blow-up rate of $|\nabla u|$ is respectively $\varepsilon^{-1/2}$ in two dimensions, $(\varepsilon|\ln\varepsilon|)^{-1}$ in three dimensions and ε^{-1} in four dimensions and higher dimensions. See Ammari, Kang and Lim [6], Ammari, Kang, Lee, Lee and Lim [4], Bao, Li and Yin [11], and Yun [33]. There have been many papers on the problem and related ones: see e.g. [2,3,5,12,13,24,27,30] and the references therein.

However, when considering the gradient estimates of the solution to the linear elasticity problem, namely the Lamé system, the method of scalar equation is no longer suitable for using. Under the assumption that the smoothness of the inclusion boundary is $C^{2,\gamma}$ $(0 < \gamma < 1)$, Bao, Li and Li [9,10] applied an energy method and an iteration technique, which was first used in [27], to obtain a pointwise upper bound of $|\nabla u|$ in the narrow region between inclusions. Kang and Yu [25] proved that the blow up rate $\varepsilon^{-1/2}$ is optimal in some two-dimensional cases when the smoothness of inclusion boundary is $C^{3,\gamma}$. Ju, Li and Xu [23] established the pointwise upper and lower bounds of the gradient of the solutions to a class of general elliptic systems in the narrow region between two $C^{2,\gamma}$ inclusions. For more work on elliptic equations and systems related to the study of composites, see [8,14,16–18,21,22,24,26,32] and the references therein.

Under a weaker smoothness assumption on the inclusion boundary, namely, $C^{1,\gamma}$, Chen and Li [15] proved that the blow up rate of the gradient for the Lamé system of linear elasticity with partially infinite coefficients is $\varepsilon^{-1/(1+\gamma)}$ in two dimensions and ε^{-1} in $n \geq 3$ dimensions.

Contrary to the case where the smoothness of the inclusion boundary is $C^{2,\gamma}$ or higher, less is known on such blow up phenomenon for the case of weaker smoothness $C^{1,\gamma}$. Based on the classical elliptic theory, a natural question is whether it is possible to obtain gradient estimates of the solutions to a class of general elliptic systems (see Definition 1.4 below), under a weaker smoothness assumption on the inclusions, namely, $C^{1,\gamma}$. In addition, we want to obtain more information on what factors $|\nabla u|$ depends on, which plays an important role in the study of the perfect conductivity problem and Lamé system with partially infinite coefficients.

In this paper, we investigate the gradient estimates of the solutions to a class of general elliptic systems with Hölder continuous coefficients in a general narrow region between the two $C^{1,\gamma}$ inclusions. This is a generalization of the stress concentration problem in two-phase high-contrast elastic composites with densely packed $C^{1,\gamma}$ inclusions. These estimates have a wide range of applications and play a key role in the damage analysis of composite materials. When we apply these results to the Lamé systems of linear elasticity, under the assumption of the $C^{1,\gamma}$ regularity of the boundary, the results can present more dependency information about gradient. Our results show that the damage may initiate from the narrowest place.

Before stating our results, we first introduce some definitions and notations, as well as fix our domain.

Let U be any domain in $\mathbb{R}^n$. Denote by the *symbol* $C(U)$ the set of all continuous functions on U. For every $0<\gamma\leq 1$, a function $u\in C(U)$ is said to be *Hölder continuous with exponent* γ if $|u(x)-u(y)|\leq C|x-y|^{\gamma}$, $x,y\in U$ for some constant C. If $u:U\to\mathbb{R}$ is bounded and continuous, we write $\|u\|_{C(U)}:=\sup_{x\in U}|u(x)|$. We use the *symbol* $C^k(U)$ to denote the set of all k-th continuously differentiable functions on U for integer $k\geq 0$. For every $0<\gamma\leq 1$, the γ^{th} -Hölder semi-norm of $u:U\to\mathbb{R}$ is

$$[u]_{C^{0,\gamma}(U)}:=\sup_{\substack{x,y\in U\\ x\neq y}}\left\{\frac{|u(x)-u(y)|}{|x-y|^{\gamma}}\right\}. \tag{1.2}$$

Definition 1.1. Let $0<\gamma\leq 1$, $k\in\mathbb{Z}_+$, and $\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathbb{Z}_+^n$ be a multiindex of order $|\alpha|=\alpha_1+\cdots+\alpha_n$. The *Hölder space* $C^{k,\gamma}(U)$ is defined to be the set of all k-th continuous differentiable real valued functions satisfying that the k-th order derivatives are Hölder continuous with exponent γ and

$$\|u\|_{C^{k,\gamma}(U)}:=\sum_{|\alpha|\leq k}\|\partial^{\alpha}u\|_{C(U)}+\sum_{|\alpha|=k}[\partial^{\alpha}u]_{C^{0,\gamma}(U)}<\infty,$$

where $\partial^{\alpha}u:=\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}u$. In particular, for $0<\gamma<1$, we often use the *symbol* $C^{\gamma}(U)$ to denote $C^{0,\gamma}(U)$.

Definition 1.2. Let U be any domain in $\mathbb{R}^n$, the integer $k>0$ and $0<\gamma<1$, the boundary ∂U is said to be C^k or $C^{k,\gamma}$ if for each point $x_0\in\partial U$ there exist $r>0$ and a C^k or $C^{k,\gamma}$ function $T:\mathbb{R}^{n-1}\to\mathbb{R}$ such that (upon relabeling and reorienting the coordinates axes if necessary) we have

$$U\cap B_r(x_0)=\{x\in B_r(x_0)\mid x_n>T(x_1,\ldots,x_{n-1})\}.$$

Furthermore, the inclusion is said to be $C^k(U)$ or $C^{k,\gamma}(U)$ if its boundary is $C^k(U)$ or $C^{k,\gamma}(U)$.

Definition 1.3. Let $1\leq p\leq\infty$, $k\in\mathbb{Z}_+$ and $\alpha=(\alpha_1,\cdots,\alpha_n)\in\mathbb{Z}_+^n$ be a multiindex of order $|\alpha|=\alpha_1+\cdots+\alpha_n$. The *Sobolev space* $W^{k,p}(U)$ is defined to be the set of all locally integrable functions $u:U\to\mathbb{R}$ such that for each multiindex $\alpha\in\mathbb{Z}_+^n$ with $|\alpha|\leq k$, $\partial_{\alpha}u$ exists in the weak sense and belongs to the standard Lebesgue spaces $L^p(U)$ and

$$\|u\|_{W^{k,p}(U)}:=\begin{cases}\Big(\sum\limits_{|\alpha|\leq k}\int_U|\partial^{\alpha}u|^p\mathrm{d}x\Big)^{1/p} & (1\leq p<\infty)\\ \sum\limits_{|\alpha|\leq k}\operatorname{ess\,sup}_U|\partial^{\alpha}u| & (p=\infty)\end{cases}<\infty.$$

Denote by the *symbol* $C_c^{\infty}(U)$ the set of all infinitely continuously differentiable functions on U with compact support. We denote by $W_0^{k,p}(U)$ the closure of $C_c^{\infty}(U)$ in $W^{k,p}(U)$.

Next, we fix our domain. Let D be a domain in $\mathbb{R}^n$, and D_1, D_2 be a pair of convex subdomains of $D \subset \mathbb{R}^n$. Let the distance between D_1 and D_2 be $\varepsilon > 0$ (sufficiently small positive number). Denote $P_1 := (\vec{0}_{n-1}, \varepsilon/2)$, $P_2 := (\vec{0}_{n-1}, -\varepsilon/2)$ the nearest points between ∂D_1 and ∂D_2 such that

$$\mathrm{dist}(P_1, P_2) = \mathrm{dist}(\partial D_1, \partial D_2) = \varepsilon, \tag{1.3}$$

where for any $x' \in \mathbb{R}^{n-1}$, $x := (x', x_n) \in \mathbb{R}^n$. Let B'_r be the ball with $\vec{0}_{n-1}$ as the center and $r \in (0, 1]$ as the radius in $\mathbb{R}^{n-1}$.

Let ε be as in (1.3), $h_1, h_2 \in C^{1,\gamma}(B'_1)$, $0 < \gamma < 1$ satisfy

$$-\frac{\varepsilon}{2} + h_2(x') < \frac{\varepsilon}{2} + h_1(x'), \quad \text{for} \quad |x'| \leq 1, \tag{1.4}$$

$$h_1(\vec{0}_{n-1}) = h_2(\vec{0}_{n-1}) = 0, \quad \nabla h_1(\vec{0}_{n-1}) = \nabla h_2(\vec{0}_{n-1}) = 0, \tag{1.5}$$

and there exist some constants $0 < \kappa_0 < \kappa_1$ such that

$$\kappa_0 |x'|^\gamma \leq |\nabla h_1(x')|, \quad |\nabla h_2(x')| \leq \kappa_1 |x'|^\gamma, \quad \text{for} \quad |x'| \leq 1. \tag{1.6}$$

To be precise, we define general narrow region in $\mathbb{R}^n$: for $r \leq 1$,

$$\Omega_r := \Big\{(x', x_n) \in D : -\frac{\varepsilon}{2} + h_2(x') < x_n < \frac{\varepsilon}{2} + h_1(x'), \ |x'| \leq r\Big\}. \tag{1.7}$$

We here assume that ∂D_1 and ∂D_2 are $C^{1,\gamma}$, $0 < \gamma < 1$ as in Definition 1.2, and the top and bottom boundaries of the narrow region Ω_1 between ∂D_1 and ∂D_2 satisfy

$$\begin{aligned}
&\{(x', x_n) \in \mathbb{R}^n : x_n = \frac{\varepsilon}{2} + h_1(x'), \, |x'| \leq 1\} \subset \partial D_1, \\
&\{(x', x_n) \in \mathbb{R}^n : x_n = -\frac{\varepsilon}{2} + h_2(x'), \, |x'| \leq 1\} \subset \partial D_2, \\
&\text{and} \\
&\{(x', x_n) \in \mathbb{R}^n : -\frac{\varepsilon}{2} + h_2(x') < x_n < \frac{\varepsilon}{2} + h_1(x'), \, |x'| \leq 1\} \cap (\partial D_1 \cup \partial D_2) = \emptyset.
\end{aligned} \tag{1.8}$$

Furthermore, we denote the top and bottom boundaries of Ω_r as

$$\begin{aligned}
\Gamma_r^+ &:= \{x \in \partial D_1 : \, x_n = \frac{\varepsilon}{2} + h_1(x'), \, |x'| \leq r\}, \\
\Gamma_r^- &:= \{x \in \partial D_2 : \, x_n = -\frac{\varepsilon}{2} + h_2(x'), \, |x'| \leq r\}.
\end{aligned} \tag{1.9}$$

We now introduce the definition of general elliptic system with Hölder continuous coefficients in a narrow region Ω_1 in this paper.

Definition 1.4. Let $0 < \gamma < 1$, $m, n \in \mathbb{Z}_+$, $A_{ij}^{\alpha\beta}, B_{ij}^{\alpha}, C_{ij}^{\beta}, D_{ij} \in C^\gamma(\Omega_1)$ for any integer $0 \leq \alpha, \beta \leq n$, $0 \leq i, j \leq m$, and the matrix of coefficients $(A_{ij}^{\alpha\beta})_{1\leq i,j\leq m}^{1\leq\alpha,\beta\leq n}$ satisfy the *strong ellipticity condition* in Ω_1, namely, there exists a constant $\lambda > 0$ such that

$$\sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta}(x) \xi_\alpha \xi_\beta \eta_i \eta_j \geq \lambda |\xi|^2 |\eta|^2, \quad \forall \, \xi \in \mathbb{R}^n, \, \eta \in \mathbb{R}^m, \, x \in \Omega_1. \tag{1.10}$$

Let

$$\begin{aligned} \varphi &:= \left(\varphi^{(1)}, \varphi^{(2)}, \cdots, \varphi^{(m)}\right) \in C^{1,\gamma}(\Gamma_1^+; \mathbb{R}^m), \\ \psi &:= \left(\psi^{(1)}, \psi^{(2)}, \cdots, \psi^{(m)}\right) \in C^{1,\gamma}(\Gamma_1^-; \mathbb{R}^m). \end{aligned} \tag{1.11}$$

Then the following system

$$\begin{cases} \sum_{\alpha,\beta,j} \partial_\alpha \left(A_{ij}^{\alpha\beta} \partial_\beta u^{(j)} + B_{ij}^{\alpha} u^{(j)}\right) + C_{ij}^{\beta} \partial_\beta u^{(j)} + D_{ij} u^{(j)} = 0 & \text{in } \ \Omega_1, \\ \mathsf{u} = \varphi, & \text{on } \ \Gamma_1^+, \\ \mathsf{u} = \psi, & \text{on } \ \Gamma_1^-, \end{cases} \tag{1.12}$$

is called a *general elliptic system*, where Γ_1^+ and Γ_1^- are as in (1.9).

A function $\mathsf{u} := (u^{(1)}, u^{(2)}, \cdots, u^{(m)}) \in W^{1,2}(\Omega_1 \subset \mathbb{R}^n; \mathbb{R}^m)$ is said to be a *weak solution* to the *general elliptic system* defined as in Definition 1.4 if, for every vector-valued function $\phi \in W_0^{1,2}(\Omega_1 \subset \mathbb{R}^n; \mathbb{R}^m)$,

$$\begin{aligned} \sum_{\alpha,\beta,j} \int_{\Omega_1} & \left(A_{ij}^{\alpha\beta}(x) \partial_\beta u^{(j)}(x) + B_{ij}^{\alpha}(x) u^{(j)}(x)\right) \partial_\alpha \phi^{(i)}(x) \\ & - C_{ij}^{\beta}(x) \partial_\beta u^{(j)}(x) \phi^{(i)}(x) - D_{ij}(x) u^{(j)}(x) \phi^{(i)}(x)\, dx = 0 \end{aligned} \tag{1.13}$$

holds true, for any integer $i = 1, \cdots, m$.

Remark 1.1. It is clear that hypotheses (1.10) and (1.18) are satisfied by the following Lamé system, (see [31]),

$$\lambda_1 \Delta \mathsf{u} + (\lambda_1 + \mu_1) \nabla(\nabla \cdot \mathsf{u}) = 0,$$

where Lamé constants (λ_1, μ_1) satisfy the ellipticity condition: $\mu_1 > 0$, $\lambda_1 + \mu_1 > 0$. Therefore, the gradient estimates results in this paper include the case of Lamé systems.

1.2. Main results

Now we are going to present our main result about pointwise gradient estimates of the weak solution to the *general elliptic system*, which is:

Theorem 1.1. *Let ε be as in (1.3), $0 < \gamma < 1$, $h_1, h_2 \in C^{1,\gamma}(B_1')$ satisfy (1.4)–(1.19), and Ω_r, $r \in \{1/2, 1\}$ be as in (1.7). Let φ and ψ be as in (1.11) and $\mathsf{u} \in W^{1,2}(\Omega_1 \subset \mathbb{R}^n; \mathbb{R}^m)$ be a weak solution to* general elliptic system *as in Definition 1.4. Then there exists a positive constant C independent of ε, such that, for any $x = (x', x_n) \in \Omega_{1/2}$,*

$$\begin{aligned} |\nabla \mathsf{u}(x', x_n)| \le & \frac{C}{\varepsilon + |x'|^{1+\gamma}} \left|\varphi(x', \varepsilon/2 + h_1(x')) - \psi(x', -\varepsilon/2 + h_2(x'))\right| \\ & + C \left(\|\varphi\|_{C^{1,\gamma}(\Gamma_1^+)} + \|\psi\|_{C^{1,\gamma}(\Gamma_1^-)} + \|\mathsf{u}\|_{L^2(\Omega_1)}\right), \end{aligned} \tag{1.14}$$

where Γ_1^+ and Γ_1^- are as in (1.9).

Moreover, if $\varphi^{(\ell)}(\vec{0}_{n-1}, \varepsilon/2) \neq \psi^{(\ell)}(\vec{0}_{n-1}, -\varepsilon/2)$ for some integer $\ell \in \{1, \cdots, m\}$, then there exists a positive constant C independent of ε, such that, for any $x_n \in (-\varepsilon/2, \varepsilon/2)$,

$$|\nabla \mathsf{u}(\vec{0}_{n-1}, x_n)| \ge C \frac{|\varphi^{(\ell)}(\vec{0}_{n-1}, \varepsilon/2) - \psi^{(\ell)}(\vec{0}_{n-1}, -\varepsilon/2)|}{\varepsilon}.$$

For the convenience of further applications, we list the analog result about the Lamé system in the narrow region, which is:

Corollary 1.1. *Let ε be as in (1.3), Ω_r, $r \in \{\frac{1}{2}, 1\}$ be as in (1.7), Γ_1^+, Γ_1^- be as in (1.9), and $\varphi \in C^{1,\gamma}(\Gamma_1^+; \mathbb{R}^n)$, $\psi \in C^{1,\gamma}(\Gamma_1^-; \mathbb{R}^n)$ be as in (1.11). Let λ_1, $\mu_1 \in \mathbb{R}$ be the pair of Lamé constants which satisfy the strong ellipticity condition: $\mu_1 > 0$, $\lambda_1 + \mu_1 > 0$. Let $\mathsf{u} = (u^{(1)}, \cdots, u^{(n)}) \in W^{1,2}(\Omega_1)$ be the weak solution to*

$$\begin{cases} \mathcal{L}_{\lambda_1,\mu_1}\mathsf{u} := \nabla \cdot (\mathbb{C}^0 e(\mathsf{u})) = \lambda_1 \Delta \mathsf{u} + (\lambda_1 + \mu_1)\nabla(\nabla \cdot \mathsf{u}) = 0, & \text{in } \Omega_1, \\ \mathsf{u} = \varphi, & \text{on } \Gamma_1^+, \\ \mathsf{u} = \psi, & \text{on } \Gamma_1^-, \end{cases} \tag{1.15}$$

where $e(\mathsf{u}) = \frac{1}{2}(\nabla \mathsf{u} + (\nabla \mathsf{u})^{\mathsf{T}})$ and the elastic tensor $\mathbb{C}^0 = \mathbb{C}(\lambda_1, \mu_1)$ consists of elements

$$C_{ijkl}(\lambda_1, \mu_1) = \lambda_1 \delta_{ij}\delta_{kl} + \mu_1(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad i, j, k, l = 1, 2, \cdots, n, \tag{1.16}$$

and δ_{ij} is Kronecker symbol: $\delta_{ij} = 0$ for $i \neq j$, $\delta_{ij} = 1$ for $i = j$.

Then there exists a positive constant C independent of ε, for any $x = (x_1, x_2) \in \Omega_{1/2}$,

$$\begin{aligned} |\nabla \mathsf{u}(x', x_n)| \leq & \frac{C}{\varepsilon + |x'|^{1+\gamma}} |\varphi(x', \varepsilon/2 + h_1(x')) - \psi(x', -\varepsilon/2 + h_2(x'))| \\ & + C\left(\|\varphi\|_{C^{1,\gamma}(\Gamma_1^+)} + \|\psi\|_{C^{1,\gamma}(\Gamma_1^-)} + \|\mathsf{u}\|_{L^2(\Omega_1)}\right). \end{aligned} \tag{1.17}$$

Moreover, if $\varphi^{(\ell)}(\vec{0}_{n-1}, \varepsilon/2) \neq \psi^{(\ell)}(\vec{0}_{n-1}, -\varepsilon/2)$ for some integer $\ell \in \{1, \cdots, n\}$, then there exists a positive constant C independent of ε, such that, for any $x_n \in (-\varepsilon/2, \varepsilon/2)$,

$$|\nabla \mathsf{u}(\vec{0}_{n-1}, x_n)| \geq C \frac{\left|\varphi^{(\ell)}(\vec{0}_{n-1}, \varepsilon/2) - \psi^{(\ell)}(\vec{0}_{n-1}, -\varepsilon/2)\right|}{\varepsilon}.$$

Indeed, the proof of this result is nontrivial. Under the assumption that the smoothness of the inclusions boundaries is $C^{1,\gamma}$, we need to estimate the Hölder semi-norm of the gradient of the constructed auxiliary function when we use the iteration method to prove that the gradient of the auxiliary function is the major singular term of the gradient of the solution to (1.12). In this paper, we consider the general elliptic system as in Definition 1.4, so the complexity of the constructed auxiliary function makes it more cumbersome to deal with some parameters in estimating the Hölder semi-norm, see Proposition 2.1 below. In addition, compared with [15], the coefficients of the elliptic system here are no longer constant, and the right-hand side is no longer in divergence form, which leads to the iteration process more complex, see Lemmas 2.1, 2.2, and 2.3 below.

To be precise, this article is organized as follows.

In Section 2, we devoted to proving the Theorem 1.1. Firstly, we give the $C^{1,\gamma}$ estimates and $W^{1,p}$ estimates required in the iteration process. Then, we obtain the solutions v_ℓ, $\ell = 1, \cdots, m$ (see (2.5) below) of m elliptic systems with relatively simple boundary conditions (see (2.4) below) by decomposing the solution u to *general elliptic system*. Next, we use a scalar auxiliary function $\bar{u}$ (see (2.6) below) to generate a family of vector-valued auxiliary functions $\widetilde{\mathsf{u}}_\ell$, whose value is same as v_ℓ on Γ_1^+ and Γ_1^- (see (2.9) below). In order to prove that the $\nabla\widetilde{\mathsf{u}}_\ell$ is the major singular term of $\nabla \mathsf{v}_\ell$ by using iteration method in [9,10,27], it is very important to consider the estimates of the Hölder semi-norm of $\nabla\widetilde{\mathsf{u}}_\ell$ in a small region (see Proposition 2.1 below). Finally, by using semi-norm estimates, $C^{1,\gamma}$ estimates and $W^{1,p}$ estimates, we complete the proof of Theorem 1.1.

In Section 3, our main task is to prove Corollary 1.1. When the general elliptic system is simplified to the Lamé system, the pointwise estimate of the gradient is obtained under the assumptions in Definition 1.4.

In Section 4, we show the proofs of the $C^{1,\gamma}$ estimates (Theorem 2.1) and $W^{1,p}$ estimates (Theorem 2.2), which play a key role in the proof of Theorem 1.1. Different from the Theorem 2.3 and 2.4 in [15], the elliptic systems considered in this paper is no longer simple constant coefficients, and the right-hand side of the system is no longer in divergence form. This section can be regarded as a generalization of [15, Theorem 2.3 and 2.4]. To prove Theorem 2.1, we first give the interior $C^{1,\gamma}$ estimates (see Lemma 4.2 below) of the solution to the elliptic system with non divergence form at the right-hand side, with the help of the Campanato's approach, Schauder estimates in [19, Theorem 5.14]. Then we can obtain the boundary $C^{1,\gamma}$ estimates (see Corollary 4.1 below) on half space by using the [19, Theorem 5.21] and Lemma 4.2. Finally, we use these estimates and the technology of locally flattening the boundary to obtain the estimates near the $C^{1,\gamma}$ boundary Γ. We prove the $W^{1,p}$ estimates by applying the interior $W^{1,p}$ estimates and the boundary $W^{1,p}$ estimates of the upper half space.

Finally, we make some conventions on notation. Let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. We always denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. If E is a subset of $\mathbb{R}^n$, we denote by χ_E its characteristic function. Let U and V be the open subsets of $\mathbb{R}^n$, we write $V \subset\subset U$ if $V \subset \bar{V} \subset U$ and $\bar{V}$ is compact, and say V is compactly contained in U. The *symbol* ∂D_i denotes the boundary of D_i. For any $x_0 \in \mathbb{R}^n$ and $r > 0$, the *symbol* $B_r(x_0)$ denotes the open ball with center x_0, radius r in $\mathbb{R}^n$.

Throughout this article, let $\Lambda > 0$ satisfy for $\alpha, \beta = 1, \cdots, n$, $i, j = 1, \cdots, m$,

$$\left|A_{ij}^{\alpha\beta}(x)\right| \leq \Lambda, \quad \forall\, x \in \Omega_1, \tag{1.18}$$

and κ_2, $\kappa_3 > 0$ such that

$$\|h_1\|_{C^{1,\gamma}(B_1')} + \|h_2\|_{C^{1,\gamma}(B_1')} \leq \kappa_2, \tag{1.19}$$

$$\|\mathsf{A}\|_{C^{\gamma}(\Omega_1)} + \|\mathsf{B}\|_{C^{\gamma}(\Omega_1)} + \|\mathsf{C}\|_{C^{\gamma}(\Omega_1)} + \|\mathsf{D}\|_{C^{\gamma}(\Omega_1)} \leq \kappa_3, \tag{1.20}$$

where

$$\|\mathsf{A}\|_{C^{\gamma}(\Omega_1)} := \max_{\alpha,\beta,i,j} \|A_{ij}^{\alpha\beta}(\cdot)\|_{C^{\gamma}(\Omega_1)}, \quad \|\mathsf{B}\|_{C^{\gamma}(\Omega_1)} := \max_{\alpha,i,j} \|B_{ij}^{\alpha}(\cdot)\|_{C^{\gamma}(\Omega_1)},$$

$$\|\mathsf{C}\|_{C^{\gamma}(\Omega_1)} := \max_{\beta,i,j} \|C_{ij}^{\beta}(\cdot)\|_{C^{\gamma}(\Omega_1)}, \quad \|\mathsf{D}\|_{C^{\gamma}(\Omega_1)} := \max_{i,j} \|D_{ij}(\cdot)\|_{C^{\gamma}(\Omega_1)}.$$

2. The gradient estimates for the general elliptic systems

In this section, our main task is to prove Theorem 1.1. By using the idea of iteration in [9,10,27] and the treatment of $C^{1,\gamma}$ boundary in [15], we establish the pointwise upper and lower bound estimates of the gradient of the solution to general elliptic systems, under the assumption that the smoothness of partial boundary of the region is $C^{1,\gamma}$. Before that, we first give the $C^{1,\gamma}$ estimates and $W^{1,p}$ estimates required in the iteration process.

2.1. $C^{1,\gamma}$ estimates and $W^{1,p}$ estimates

Firstly, we give the definition of *general elliptic type system* in a more general region.

Definition 2.1. Let $0 < \gamma < 1$ and Q be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with a $C^{1,\gamma}$ boundary portion $\Gamma \subset \partial Q$. Let $A_{ij}^{\alpha\beta}, B_{ij}^{\alpha}, C_{ij}^{\beta}, D_{ij}, F_i^{(\alpha)} \in C^{\gamma}(Q)$ and $H^{(i)} \in L^{\infty}(Q)$ for any $\alpha, \beta = 1, \cdots, n$, $i, j = 1, \cdots, m$. Let the matrix of coefficients $(A_{ij}^{\alpha\beta})$ satisfies (1.10) and (1.18), and B_{ij}^{α}, C_{ij}^{β}, D_{ij} satisfy (1.20). Then the following system

$$\begin{cases} \sum\limits_{\alpha,\beta,j} \partial_\alpha \left(A_{ij}^{\alpha\beta} \partial_\beta w^{(j)} + B_{ij}^{\alpha} w^{(j)} \right) + C_{ij}^{\beta} \partial_\beta w^{(j)} + D_{ij} w^{(j)} = H^{(i)} - \partial_\alpha F_i^{(\alpha)}, & \text{in } Q, \\ \mathsf{w} := (w^{(1)}, \cdots, w^{(m)}) = 0, & \text{on } \Gamma, \end{cases} \tag{2.1}$$

is called a *general elliptic type system.*

Theorem 2.1. *($C^{1,\gamma}$ estimates) Let κ_3 be as in (1.20) and $\mathsf{w} = (w^{(1)}, \cdots, w^{(m)}) \in W^{1,2}(Q \subset \mathbb{R}^n; \mathbb{R}^m) \cap C^1(Q \cup \Gamma; \mathbb{R}^m)$ be the weak solution to the* general elliptic type system *as in Definition 2.1. Then, for any $Q' \subset\subset Q \cup \Gamma$, there exists a positive constant C depending on $n, m, \kappa_3, \gamma, Q', Q$, such that,*

$$\|\mathsf{w}\|_{C^{1,\gamma}(Q')} \le C \left(\|\mathsf{w}\|_{L^\infty(Q)} + [\mathsf{F}]_{\gamma,Q} + \|\mathsf{H}\|_{L^\infty(Q)} \right). \tag{2.2}$$

Here and thereafter, the Hölder semi-norm of the matrix-valued function $\mathsf{F} = (F_i^{(\alpha)})_{i,\alpha}$ is defined as $[\mathsf{F}]_{\gamma,Q} := \max_{i,\alpha} [F_i^{(\alpha)}]_{\gamma,Q}$, the L^∞ norm of the vector-valued function $\mathsf{H} := (H^{(1)}, \cdots, H^{(m)})$ is defined as $\|\mathsf{H}\|_{L^\infty(Q)} := \max_i \|H^{(i)}\|_{L^\infty(Q)}$.

Theorem 2.2. *($W^{1,p}$ estimates) Let $0 < \gamma < 1$ and $\mathsf{w} = (w^{(1)}, \cdots, w^{(m)}) \in W^{1,2}(Q \subset \mathbb{R}^n; \mathbb{R}^m) \cap C^1(Q \cup \Gamma; \mathbb{R}^m)$ be the weak solution to the* general elliptic type system *as in Definition 2.1. Then, for any $2 \le p < \infty$ and any domain $Q' \subset\subset Q \cup \Gamma$, there exists a positive constant C depending on $\lambda, \kappa_3, p, Q', Q$, such that,*

$$\|\mathsf{w}\|_{W^{1,p}(Q')} \le C(\|\mathsf{w}\|_{L^2(Q)} + [\mathsf{F}]_{\gamma,Q} + \|\mathsf{H}\|_{L^\infty(Q)}). \tag{2.3}$$

In particular, if $p > n$, there exists a positive constant C depending on $\lambda, \tau, \kappa_3, p, Q', Q$, such that, for any $0 < \tau \le 1 - n/p$,

$$\|\mathsf{w}\|_{C^\tau(Q')} \le C(\|\mathsf{w}\|_{L^2(Q)} + [\mathsf{F}]_{\gamma,Q} + \|\mathsf{H}\|_{L^\infty(Q)}).$$

For readers' convenience, the proofs of Theorem 2.1 and Theorem 2.2 are given in Section 4.

2.2. Proof of Theorem 1.1

Definition 2.2. Let $A_{ij}^{\alpha\beta}, B_{ij}^{\alpha}, C_{ij}^{\beta}, D_{ij}$, $\alpha, \beta = 1, \cdots, n$, $i, j = 1, \cdots, m$, be as in Definition 1.4 and φ, ψ be as in (1.11). Let

$$\mathsf{v}_\ell = (v_\ell^{(1)}, v_\ell^{(2)}, \cdots, v_\ell^{(m)}), \qquad \ell = 1, 2, \cdots, m$$

with $v_\ell^{(j)} = 0$ for $j \ne \ell$, $v_\ell^{(j)} = u^{(\ell)}$ for $j = \ell$, be a weak solution to the following boundary value problem:

$$\begin{cases} \sum_{\alpha,\beta,j} \partial_\alpha \left(A_{ij}^{\alpha\beta} \partial_\beta v_\ell^{(j)} + B_{ij}^{\alpha} v_\ell^{(j)} \right) + C_{ij}^{\beta} \partial_\beta v_\ell^{(j)} + D_{ij} v_\ell^{(j)} = 0, & \text{in } \Omega_1, \\ \mathsf{v}_\ell = \left(0, \cdots, 0, \varphi^{(\ell)}, 0, \cdots, 0\right), & \text{on } \Gamma_1^+, \\ \mathsf{v}_\ell = \left(0, \cdots, 0, \psi^{(\ell)}, 0, \cdots, 0\right), & \text{on } \Gamma_1^-. \end{cases} \tag{2.4}$$

It follows from Definition 1.4 that for the solution $\mathsf{u} = (u^{(1)}, \cdots, u^{(m)})$ to the *general elliptic system* as in Definition 1.4, we have

$$\mathsf{u} = \mathsf{v}_1 + \mathsf{v}_2 + \cdots + \mathsf{v}_\mathsf{m} \quad \text{and} \quad \nabla \mathsf{u} = \sum_{\ell=1}^{\mathsf{m}} \nabla \mathsf{v}_\ell \ \text{ in } \Omega_1. \tag{2.5}$$

In order to estimate $|\nabla \mathsf{v}_\ell|$, $\ell = 1, \cdots, m$, we introduce a scalar function $\bar{u} \in C^{1,\gamma}(\mathbb{R}^n)$ such that $\bar{u} = 1$ on Γ_1^+, $\bar{u} = 0$ on Γ_1^- and

$$\bar{u}(x) := \frac{x_n - h_2(x') + \varepsilon/2}{\varepsilon + h_1(x') - h_2(x')}, \quad x \in \Omega_1, \tag{2.6}$$

where $h_1,\ h_2 \in C^{1,\gamma}(B_1')$ satisfy (1.4)–(1.6).

By a direct calculation, we obtain that for $x := (x', x_n) \in \Omega_1$,

$$|\partial_\alpha \bar{u}(x)| \le \frac{C|x'|^\gamma}{\varepsilon + |x'|^{1+\gamma}},\ \alpha = 1, 2, \cdots, n-1, \quad \partial_n \bar{u}(x) = \frac{1}{\delta(x')}, \tag{2.7}$$

where

$$\delta(x') := \varepsilon + h_1(x') - h_2(x'), \quad \text{in}\, \Omega_1. \tag{2.8}$$

Using $\bar{u}$ to define a family of vector-valued auxiliary functions, for $\ell = 1, 2, \cdots, m$, we define

$$\widetilde{\mathsf{u}}_\ell := \Big(0, \cdots, 0, \varphi^{(\ell)}(x', \frac{\varepsilon}{2} + h_1(x'))\bar{u}(x) + \psi^{(\ell)}(x', -\frac{\varepsilon}{2} + h_2(x'))(1 - \bar{u}(x)), 0, \cdots, 0\Big). \tag{2.9}$$

It is obvious that $\widetilde{\mathsf{u}}_\ell = (0, \cdots, 0, \varphi^{(\ell)}, 0, \cdots, 0)$ on Γ_1^+ and $\widetilde{\mathsf{u}}_\ell = (0, \cdots, 0, \psi^{(\ell)}, 0, \cdots, 0)$ on Γ_1^-.

In view of (2.7) and (2.9), for any $x \in \Omega_1$ and $\alpha = 1, 2, \cdots, n-1$, we have

$$\begin{aligned} |\partial_\alpha \tilde{\mathsf{u}}_\ell(x)| \le & \frac{C|x'|^\gamma}{\varepsilon + |x'|^{1+\gamma}} \left|\varphi^{(\ell)}(x', \varepsilon/2 + h_1(x')) - \psi^{(\ell)}(x', -\varepsilon/2 + h_2(x'))\right| \\ & + C\left(\|\nabla\varphi^{(\ell)}\|_{L^\infty(\Gamma_1^+)} + \|\nabla\psi^{(\ell)}\|_{L^\infty(\Gamma_1^-)}\right), \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} & \frac{\left|\varphi^{(\ell)}(x', \varepsilon/2 + h_1(x')) - \psi^{(\ell)}(x', -\varepsilon/2 + h_2(x'))\right|}{C(\varepsilon + |x'|^{1+\gamma})} \\ & \le |\partial_n \widetilde{\mathsf{u}}_\ell(x)| \le \frac{C\left|\varphi^{(\ell)}(x', \varepsilon/2 + h_1(x')) - \psi^{(\ell)}(x', -\varepsilon/2 + h_2(x'))\right|}{\varepsilon + |x'|^{1+\gamma}}. \end{aligned} \tag{2.11}$$

Next, we estimate $|\nabla \mathsf{v}_\ell|$, $\ell = 1, \cdots, m$. Let

$$\mathsf{w}_\ell = \mathsf{v}_\ell - \widetilde{\mathsf{u}}_\ell, \quad \ell = 1, 2, \cdots, m, \tag{2.12}$$

then, by (2.4) and (2.9), $\mathsf{w}_\ell = (w_\ell^{(1)}, w_\ell^{(2)}, \cdots, w_\ell^{(m)})$ satisfies

$$\begin{cases} \sum\limits_{\alpha,\beta,j} \left(\partial_\alpha\left(A_{ij}^{\alpha\beta}\partial_\beta w_\ell^{(j)} + B_{ij}^\alpha w_\ell^{(j)}\right) + C_{ij}^\beta \partial_\beta w_\ell^{(j)} + D_{ij} w_\ell^{(j)}\right) = H^{(i)} - \sum\limits_\alpha \partial_\alpha F_i^\alpha, & \text{in}\, \Omega_1 \\ \mathsf{w}_\ell = 0, & \text{on}\, \Gamma_1^\pm, \end{cases} \tag{2.13}$$

where for any $1 \le \alpha \le n$, $1 \le i \le m$,

$$\begin{aligned} F_i^{(\alpha)}(x) : &= \sum_{\beta,j} \left(A_{ij}^{\alpha\beta}(x)\partial_\beta \widetilde{u}_\ell^{(j)}(x) + B_{ij}^\alpha(x)\widetilde{u}_\ell^{(j)}(x)\right), \\ H^{(i)}(x) : &= \sum_{\beta,j} \left(-C_{ij}^\beta \partial_\beta \widetilde{u}_\ell^{(j)}(x) - D_{ij}(x)\widetilde{u}_\ell^{(j)}(x)\right). \end{aligned} \tag{2.14}$$

Lemma 2.1. *Let $\mathsf{v}_\ell \in W^{1,2}(\Omega_1 \subset \mathbb{R}^n; \mathbb{R}^m)$, $1 \le \ell \le m$, be a weak solution to (2.4) in the sense of (1.13), thus w_ℓ as in (2.12) satisfies (2.13). Let φ, ψ be as in (1.11), Γ_1^+, Γ_1^- be as in (1.9) and Ω_r, $r \in \{1/2, 1\}$ be as in (1.7). Then there exists a positive constant C independent of ε, ℓ, such that, for any $\ell = 1, \cdots, m$,*

$$\int_{\Omega_{1/2}} |\nabla \mathsf{w}_\ell|^2 \, dx \le C \left(\|\mathsf{w}_\ell\|^2_{L^2(\Omega_1)} + \|\varphi^{(\ell)}\|^2_{C^{1,\gamma}(\Gamma_1^+)} + \|\psi^{(\ell)}\|^2_{C^{1,\gamma}(\Gamma_1^-)} \right).$$

Proof. For simplicity, we assume that $\psi \equiv 0$. Multiply (2.13) by w_ℓ, making use of the integration by parts in $\Omega_{1/2}$,

$$\begin{aligned}
&\sum_{\alpha\beta,i,j} \int_{\Omega_{1/2}} \left(A_{ij}^{\alpha,\beta}(x) \partial_\beta w_\ell^{(j)} \right) \partial_\alpha w_\ell^{(i)} \, dx \\
&= \sum_{\alpha,\beta,i,j} \left(- \int_{\Omega_{1/2}} B_{ij}^{\alpha}(x) w_\ell^{(j)} \partial_\alpha w_\ell^{(i)} \, dx + \int_{\Omega_{1/2}} C_{ij}^{\beta}(x) \partial_\beta w_\ell^{(j)} w_\ell^{(i)} \, dx \right) \\
&\quad + \sum_{\alpha,\beta,i,j} \left(\int_{\Omega_{1/2}} D_{ij}(x) w_\ell^{(j)} w_\ell^{(i)} \, dx - \int_{\Omega_{1/2}} H^{(i)} w_\ell^{(i)} + \int_{\Omega_{1/2}} \partial_\alpha F_i^\alpha w_\ell^{(i)} \, dx \right) \\
&\quad + \sum_{\alpha,\beta,i,j} \left(\int\limits_{\substack{|x'|=1/2, \\ -\frac{\varepsilon}{2}+h_2(x')<x_n<\frac{\varepsilon}{2}+h_1(x')}} \left(A_{ij}^{\alpha\beta}(x) \partial_\beta w_\ell^{(j)} + B_{ij}^{\alpha}(x) w_\ell^{(j)} \right) w_\ell^{(i)} \nu^{(\alpha)} \, ds \right),
\end{aligned}$$

where $\vec{\nu} := (\nu^{(1)}, \cdots, \nu^{(n)}) \in \mathbb{R}^n$ is the unit outer normal vector of vertical boundary on both sides of $\Omega_{1/2}$.

From the strong ellipticity condition (1.10) and Cauchy's inequality, we have

$$\begin{aligned}
\lambda \int_{\Omega_{1/2}} |\nabla \mathsf{w}_\ell|^2 \, dx &\le \sum_{\alpha,\beta,i,j} \left(\int_{\Omega_{1/2}} \left(A_{ij}^{\alpha\beta}(x) \partial_\beta w_\ell^{(j)} \right) \partial_\alpha w_\ell^{(i)} \, dx \right) \\
&\le \frac{\lambda}{4} \int_{\Omega_{1/2}} |\nabla \mathsf{w}_\ell|^2 \, dx + C \int_{\Omega_{1/2}} |\mathsf{w}_\ell|^2 \, dx + \sum_i \left| \int_{\Omega_{1/2}} H^{(i)} w_\ell^{(i)} \, dx \right| \\
&\quad + C \int\limits_{\substack{|x'|=1/2, \\ -\frac{\varepsilon}{2}+h_2(x')<x_n<\frac{\varepsilon}{2}+h_1(x')}} \left(|\nabla \mathsf{w}_\ell|^2 + |\mathsf{w}_\ell|^2 \right) \, dx + \sum_{\alpha,i} \left| \int_{\Omega_{1/2}} \partial_\alpha F_i^\alpha w_\ell^{(i)} \, dx \right|. \qquad (2.15)
\end{aligned}$$

By using Cauchy's inequality again, (2.9), and (2.14), we have

$$\begin{aligned}
\sum_i \left| \int_{\Omega_{1/2}} H^{(i)} w_\ell^{(i)} \, dx \right| &\le C \int_{\Omega_1} \left(|\nabla \tilde{\mathsf{u}}_\ell| \, |\mathsf{w}_\ell| + |\tilde{\mathsf{u}}_\ell| |\mathsf{w}_\ell| \right) \, dx \\
&\le C \left(\int_{\Omega_1} (|\nabla \tilde{\mathsf{u}}_\ell|^2 + |\tilde{\mathsf{u}}_\ell|^2) \, dx + \int_{\Omega_1} |\mathsf{w}_\ell|^2 \, dx \right) \\
&\le C \left(\|\mathsf{w}_\ell\|^2_{L^2(\Omega_1)} + \|\varphi^{(\ell)}\|^2_{C^{1,\gamma}(\Gamma_1^+)} \right). \qquad (2.16)
\end{aligned}$$

Note that $\Omega_{2/3} \backslash \overline{\Omega_{1/3}} \subset\subset \Omega_1 \backslash \overline{\Omega_{1/4}}$, applying Theorem 2.1 and (2.3) for (2.13) with (2.14), one has,

$$\|\nabla \mathsf{w}_\ell\|_{L^\infty(\Omega_{2/3}\setminus\overline{\Omega_{1/3}})} \le \|\mathsf{w}_\ell\|_{C^{1,\gamma}(\Omega_{2/3}\setminus\overline{\Omega_{1/3}})} \le C\left(\|\mathsf{w}_\ell\|_{L^\infty(\Omega_1\setminus\overline{\Omega_{1/4}})} + \|\varphi^{(\ell)}\|_{C^{1,\gamma}(\Gamma_1^+)}\right)$$
$$\le C\left(\|\mathsf{w}_\ell\|_{L^2(\Omega_1)} + \|\varphi^{(\ell)}\|_{C^{1,\gamma}(\Gamma_1^+)}\right).$$

This implies that for $x = (x', x_n) \in \Omega_{2/3}\setminus\overline{\Omega_{1/3}}$,

$$|\mathsf{w}_\ell(x', x_n)| \le C\left(\|\mathsf{w}_\ell\|_{L^2(\Omega_1)} + \|\varphi^{(\ell)}\|_{C^{1,\gamma}(\Gamma_1^+)}\right),$$

it follows that

$$\int\limits_{\substack{|x'|=1/2,\\ -\frac{\varepsilon}{2}+h_2(x')<x_n<\frac{\varepsilon}{2}+h_1(x')}} \left(|\nabla \mathsf{w}_\ell|^2 + |\mathsf{w}_\ell|^2\right) dx \le C\left(\|\mathsf{w}_\ell\|^2_{L^2(\Omega_1)} + \|\varphi^{(\ell)}\|^2_{C^{1,\gamma}(\Gamma_1^+)}\right). \tag{2.17}$$

By (2.10), we deduce that

$$\begin{aligned}
&\sum_{\alpha=1}^{n-1}\int\limits_{\Omega_1} |\partial_\alpha \widetilde{\mathsf{u}}_\ell|^2\, dx \\
\le& C\int\limits_{|x'|<1} \delta(x')\left(\frac{|x'|^{2\gamma}}{(\varepsilon+|x'|^{1+\gamma})^2}\left|\varphi^\ell(x', \frac{\varepsilon}{2}+h_1(x'))\right|^2 + \|\nabla\varphi^{(\ell)}\|^2_{L^\infty(\Gamma_1^+)}\right) dx \\
\le& C\left\|\varphi^{(\ell)}\right\|^2_{C^{1,\gamma}(\Gamma_1^+)}.
\end{aligned} \tag{2.18}$$

In view of $\partial_{nn}\widetilde{\mathsf{u}}_\ell = 0$ and (2.17), (2.18), by applying the integration by parts in $\Omega_{1/2}$ we have

$$\begin{aligned}
\sum_{\alpha,i}\left|\int\limits_{\Omega_{1/2}} \partial_\alpha F_i^{(\alpha)} w_\ell^{(i)}\, dx\right| \le& \sum_{\alpha,i}\left|\int\limits_{\Omega_{1/2}} F_i^{(\alpha)}\partial_\alpha w_\ell^{(i)}\, dx\right| + \left|\int\limits_{\substack{|x'|=1/2,\\ -\frac{\varepsilon}{2}+h_2(x')<x_n<\frac{\varepsilon}{2}+h_1(x')}} F_i^{(\alpha)} w_\ell^{(i)}\nu^\alpha\, dx\right| \\
\le& C\int\limits_{\Omega_{1/2}}\left(\sum_{\alpha=1}^{n-1}|\partial_\alpha\widetilde{\mathsf{u}}_\ell|\,|\nabla\mathsf{w}_\ell| + |\widetilde{\mathsf{u}}_\ell||\nabla\mathsf{w}_\ell|\right) dx \\
&+ C\int\limits_{\substack{|x'|=1/2,\\ -\frac{\varepsilon}{2}+h_2(x')<x_n<\frac{\varepsilon}{2}+h_1(x')}}\left(\sum_{\alpha=1}^{n-1}|\partial_\alpha\widetilde{\mathsf{u}}_\ell|\,|\mathsf{w}_\ell| + |\widetilde{\mathsf{u}}_\ell||\mathsf{w}_\ell|\right) ds \\
\le& \frac{\lambda}{4}\int\limits_{\Omega_{1/2}} |\nabla\mathsf{w}_\ell|^2\, dx + C\left(\|\varphi^{(\ell)}\|^2_{C^{1,\gamma}(\Gamma_1^+)} + \|\mathsf{w}_\ell\|^2_{L^2(\Omega_1)}\right),
\end{aligned}$$

this, combining with (2.15), (2.16), and (2.17) we obtain

$$\int\limits_{\Omega_{1/2}} |\nabla\mathsf{w}_\ell|^2\, dx \le C\left(\|\mathsf{w}_\ell\|^2_{L^2(\Omega_1)} + \|\varphi^{(\ell)}\|^2_{C^{1,\gamma}(\Gamma_1^+)}\right).$$

We have completed the proof of Lemma 2.1. $\square$

Let $1 \leq s \leq 1/2$ and $\Omega_{1/2}$ be as in (1.7). Let $h_1, h_2 \in C^{1,\gamma}(B_1')$ satisfy (1.4)–(1.19) and ε be as in (1.3), we define set as

$$\widehat{\Omega}_s(z) := \{(x', x_n) \in \Omega_{1/2} \,|\, -\frac{\varepsilon}{2} + h_2(x') < x_n < \frac{\varepsilon}{2} + h_1(x'),\ |x' - z'| < s\}. \tag{2.19}$$

In order to use iteration as in [9,15] to prove that the gradients of the auxiliary functions $\widetilde{\mathrm{u}}_\ell$ are the major singular terms of $|\nabla v_\ell|$, we need the following estimates: namely, for a fixed point

$$z = (z', z_n) \in \Omega_{1/2}, \tag{2.20}$$

we consider the Hölder semi-norm estimates of $\nabla \widetilde{\mathrm{u}}_\ell$ in $\widehat{\Omega}_s(z)$. In the following, we always assume that ε and $|z'|$ are sufficiently small.

Proposition 2.1. *Let h_1, h_2 satisfy (1.4)–(1.19), Γ_1^+, Γ_1^- be as in (1.9) and φ, ψ be as in (1.11). Let $z = (z', z_n)$ be as in (2.20), $\delta(z')$ be as in (2.8), and $\widehat{\Omega}_s(z)$ be as in (2.19) for $0 < s \leq C\delta(z')$. Then there exists a positive constant C independent of ε, such that, for any $\ell = 1, \cdots, m$,*

$$\begin{aligned}[\nabla \widetilde{\mathrm{u}}_\ell]_{\gamma, \widehat{\Omega}_s(z)} \leq & C \left| \varphi^{(\ell)}(z', \frac{\varepsilon}{2} + h_1(z')) - \psi^{(\ell)}(z', -\frac{\varepsilon}{2} + h_2(z')) \right| \\ & \left(\delta(z')^{-1-\frac{1}{1+\gamma}} s^{1-\gamma} + \delta(z')^{-\gamma-\frac{1}{1+\gamma}} \right) + C \left(\|\varphi^{(\ell)}\|_{C^{1,\gamma}(\Gamma_1^+)} + \|\psi^{(\ell)}\|_{C^{1,\gamma}(\Gamma_1^-)} \right) \\ & \left(\delta(z')^{-1-\frac{1}{1+\gamma}} s^{2-\gamma} + \delta(z')^{-1} s^{1-\gamma} + \delta(z')^{-\gamma-\frac{1}{1+\gamma}} s + \delta(z')^{-\gamma} \right). \end{aligned} \tag{2.21}$$

Proof. Since $s \leq C\delta(z')$ and $|z'| \leq C\delta(z')^{\frac{1}{1+\gamma}}$, for any $x = (x', x_n) \in \widehat{\Omega}_s(z)$, we have

$$|x'| \leq |x' - z'| + |z'| < s + |z'| \leq C\delta(z')^{\frac{1}{1+\gamma}},$$

and combining with mean value theorem, we have

$$|h_i(x') - h_i(\tilde{x}')| \leq C|x'_{\theta_i}|^\gamma |x' - \tilde{x}'| \leq C\delta(z')^{\frac{\gamma}{1+\gamma}} |x' - \tilde{x}'|, \quad i = 1, 2, \tag{2.22}$$

for any $x', \tilde{x}' \in \widehat{\Omega}_s(z)$, $x' \neq \tilde{x}'$. In view of (2.8), we obtain

$$|\delta(x') - \delta(\tilde{x}')| \leq |h_1(x') - h_1(\tilde{x}')| + |h_2(x') - h_2(\tilde{x}')| \leq C\delta(z')^{\frac{\gamma}{1+\gamma}} |x' - \tilde{x}'|. \tag{2.23}$$

In particular, taking $\tilde{x}' = z'$, and recalling that $|x' - z'| < s \leq C\delta(z')$, we have

$$\delta(x') \leq \delta(z') + |h_1(x') - h_1(z')| + |h_2(x') - h_2(z')| \leq C\delta(z'), \tag{2.24}$$

and for sufficiently small ε and $|z'|$,

$$\delta(x') \geq \delta(z') - |h_1(x') - h_1(z')| - |h_2(x') - h_2(z')| \geq \frac{1}{2}\delta(z'). \tag{2.25}$$

Next we estimate $|\partial_\alpha \bar{u}(x) - \partial_\alpha \bar{u}(\tilde{x})|$, $x, \tilde{x} \in \widehat{\Omega}_s(z)$, $x \neq \tilde{x}$, for $\alpha = 1, 2, \cdots, n$. For $\alpha = 1, 2, \cdots, n-1$, recalling (2.6), we have

$$\begin{aligned}\partial_\alpha \bar{u}(x', x_n) &= \frac{-\partial_\alpha h_2(x')}{\delta(x')} + \frac{-x_n}{\delta^2(x')} + \frac{(h_2(x') - \varepsilon/2)\partial_\alpha \delta(x')}{\delta^2(x')} \\ &=: \Pi_1(x) + \Pi_2(x) + \Pi_3(x). \end{aligned} \tag{2.26}$$

Since $h_1(x'), h_2(x') \in C^{1,\gamma}(B_1')$, (2.23), and (2.25), we obtain

$$\begin{aligned}|\Pi_1(x) - \Pi_1(\tilde{x})| &\le \frac{|\partial_\alpha h_2(x') - \partial_\alpha h_2(\tilde{x}'|)}{\delta(x')} + |\partial_\alpha h_2(\tilde{x}')| \left| \frac{1}{\delta(\tilde{x}')} - \frac{1}{\delta(x')} \right| \\ &\le C \left(\delta(z')^{-1}|x' - \tilde{x}'|^\gamma + \delta(z')^{-\frac{2}{1+\gamma}}|x' - \tilde{x}'| \right), \end{aligned} \tag{2.27}$$

for any $x, \tilde{x} \in \widehat{\Omega}_s(z)$, $x \neq \tilde{x}$. By using $|\tilde{x}_n| \le \delta(z')$, we can obtain

$$\begin{aligned}|\Pi_2(x) - \Pi_2(\tilde{x})| &\le \frac{|\partial_\alpha \delta(x')|}{\delta^2(x')}|x_n - \tilde{x}_n| + \frac{|\tilde{x}_n|}{\delta^2(x')}|\partial_\alpha \delta(x') - \partial_\alpha \delta(\tilde{x}')| \\ &\le C \left(\delta(z')^{-1-\frac{1}{1+\gamma}}|x_n - \tilde{x}_n| + \delta(z')^{-1}|x' - \tilde{x}'|^\gamma \right). \end{aligned} \tag{2.28}$$

It follows from (2.22), (2.24), and (2.25) that

$$\begin{aligned}|\Pi_3(x) - \Pi_3(\tilde{x})| \le& \frac{|\partial_\alpha \delta(x')|}{\delta(x')^2}|h_2(x') - h_2(\tilde{x}')| + \frac{\delta(\tilde{x}')}{\delta(x')^2}|\partial_\alpha \delta(x') - \partial_\alpha \delta(\tilde{x}')| \\ &+ \delta(\tilde{x}')|\partial_\alpha \delta(\tilde{x}')| \left| \frac{1}{\delta(x')^2} - \frac{1}{\delta(\tilde{x}')^2} \right| \\ &\le C \left(\delta(z')^{-\frac{2}{1+\gamma}}|x' - \tilde{x}'| + \delta(z')^{-1}|x' - \tilde{x}'|^\gamma \right). \end{aligned}$$

This, combining (2.26), (2.27), and (2.28), we get for $\alpha = 1, 2, \cdots, n-1$,

$$\begin{aligned}&|\partial_\alpha \bar{u}(x) - \partial_\alpha \bar{u}(\tilde{x})| \\ \le& C \left(\delta(z')^{-\frac{2}{1+\gamma}}|x' - \tilde{x}'| + \delta(z')^{-1}|x' - \tilde{x}'|^\gamma + \delta(z')^{-1-\frac{1}{1+\gamma}}|x_n - \tilde{x}_n| \right), \end{aligned} \tag{2.29}$$

and for $\alpha = n$, by using (2.23) and (2.25), we obtain

$$|\partial_n \bar{u}(x) - \partial_n \bar{u}(\tilde{x})| \le C \left| \frac{1}{\delta(x')} - \frac{1}{\delta(\tilde{x}')} \right| \le C\delta(z')^{-1-\frac{1}{1+\gamma}}|x' - \tilde{x}'|. \tag{2.30}$$

Now we prove (2.21). Take the case when $\psi \equiv 0$ for instance. Firstly, for $\alpha = n$, since $\varphi(x) \in C^{1,\gamma}(\Gamma_1^+)$, (2.7), (2.25), and (2.30), we have

$$\begin{aligned}|\partial_n \widetilde{\mathrm{u}}_\ell(x) - \partial_n \widetilde{\mathrm{u}}_\ell(\tilde{x})| \le& \left| \varphi^{(\ell)}(x', \frac{\varepsilon}{2} + h_1(x')) \right| |\partial_n \bar{u}(x) - \partial_n \bar{u}(\tilde{x})| \\ &+ |\partial_n \bar{u}(\tilde{x})| \left| \varphi^{(\ell)}(x', \frac{\varepsilon}{2} + h_1(x')) - \varphi^{(\ell)}(\tilde{x}', \frac{\varepsilon}{2} + h_1(\tilde{x}') \right| \\ \le& C \left| \varphi^{(\ell)}(z', \frac{\varepsilon}{2} + h_1(z')) \right| \delta(z')^{-1-\frac{1}{1+\gamma}}|x' - \tilde{x}'| \\ &+ C\|\nabla \varphi^{(\ell)}\|_{L^\infty(\Gamma_1^+)}\delta(z')^{-1}|x' - \tilde{x}'| \left(\delta(z')^{-\frac{1}{1+\gamma}}s + 1 \right), \end{aligned}$$

where we used the fact that $|x' - z'| < s$. Similarly, by using $|x' - \tilde{x}'| \le s$ we obtain

$$\begin{aligned}[\partial_n \widetilde{\mathrm{u}}_\ell]_{\gamma, \widehat{\Omega}_s(z')} \le& C \left| \varphi^{(\ell)}(z', \frac{\varepsilon}{2} + h_1(z')) \right| \delta(z')^{-1-\frac{1}{1+\gamma}}s^{1-\gamma} \\ &+ C\|\nabla \varphi^{(\ell)}\|_{L^\infty(\Gamma_1^+)}(\delta(z')^{-1-\frac{1}{1+\gamma}}s^{2-\gamma} + \delta(z')^{-1}s^{1-\gamma}). \end{aligned} \tag{2.31}$$

For $\alpha = 1, 2, \cdots, n-1$,

$$\begin{aligned}|\partial_\alpha \widetilde{\mathrm{u}}_\ell(x) - \partial_\alpha \widetilde{\mathrm{u}}_\ell(\tilde{x})| \leq & \left|\partial_\alpha \varphi^{(\ell)}(x', \frac{\varepsilon}{2} + h_1(x'))\bar{u}(x) - \partial_\alpha \varphi^{(\ell)}(\tilde{x}', \frac{\varepsilon}{2} + h_1(\tilde{x}'))\bar{u}(\tilde{x})\right| \\ & + \left|\varphi^{(\ell)}(x', \frac{\varepsilon}{2} + h_1(x'))\partial_\alpha \bar{u}(x) - \varphi^{(\ell)}(x', \frac{\varepsilon}{2} + h_1(x'))\partial_\alpha \bar{u}(\tilde{x})\right| \\ =: & \ \mathrm{I} + \mathrm{II}. \end{aligned} \tag{2.32}$$

Using (2.25) and (2.30), we obtain

$$\begin{aligned}\mathrm{I} \leq & C\|\nabla\varphi^{(\ell)}\|_{L^\infty(\Gamma_1^+)}\left(\frac{|x_n - \tilde{x}_n|}{\delta(x')} + |\tilde{x}_n|\left|\frac{1}{\delta(x')} - \frac{1}{\delta(\tilde{x}')}\right|\right) + C\|\nabla\varphi^{(\ell)}\|_{C^{0,\gamma}(\Gamma_1^+)}|x' - \tilde{x}'|^\gamma \\ \leq & C\|\nabla\varphi^{(\ell)}\|_{C^{0,\gamma}(\Gamma_1^+)}\left(\delta(z')^{-1}|x_n - \tilde{x}_n| + \delta(z')^{-\frac{1}{1+\gamma}}|x' - \tilde{x}'| + |x' - \tilde{x}'|^\gamma\right).\end{aligned}$$

In view of $|x' - \tilde{x}'| < s$ and $|x_n - \tilde{x}_n| \leq 2\delta(z')$, we obtain

$$\sup_{x,\tilde{x}\in\widehat{\Omega}_s(z'), x\neq\tilde{x}} \frac{\mathrm{I}}{|x - \tilde{x}|^\gamma} \leq C\left\|\nabla\varphi^{(\ell)}\right\|_{C^{0,\gamma}(\Gamma_1^+)}\left(\delta(z')^{-\gamma} + \delta(z')^{-\frac{1}{1+\gamma}}s^{1-\gamma}\right). \tag{2.33}$$

It follows from (2.7) and (2.29) that

$$\begin{aligned}\mathrm{II} \leq & C\left(|\varphi^{(\ell)}(z', \frac{\varepsilon}{2} + h_1(z'))| + s\|\nabla\varphi^{(\ell)}\|_{L^\infty(\Gamma_1^+)}\right) \\ & \left(\delta(z')^{-\frac{2}{1+\gamma}}|x' - \tilde{x}'| + \delta(z')^{-1}|x' - \tilde{x}'|^\gamma + \delta(z')^{-1-\frac{1}{1+\gamma}}|x_n - \tilde{x}_n|\right) \\ & + C\|\nabla\varphi^{(\ell)}\|_{L^\infty(\Gamma_1^+)}\delta(z')^{-1+\frac{\gamma}{1+\gamma}}|x' - \tilde{x}'|.\end{aligned}$$

Obviously,

$$\begin{aligned}\sup_{x,\tilde{x}\in\widehat{\Omega}_s(z'), x\neq\tilde{x}} \frac{\mathrm{II}}{|x - \tilde{x}|^\gamma} \leq & C\left|\varphi^{(\ell)}(z', \frac{\varepsilon}{2} + h_1(z'))\right|\left(\delta(z')^{-\frac{2}{1+\gamma}}s^{1-\gamma} + \delta(z')^{-\gamma-\frac{1}{1+\gamma}}\right) \\ & + C\|\nabla\varphi^{(\ell)}\|_{L^\infty(\Gamma_1^+)}\left(\delta(z')^{-\frac{2}{1+\gamma}}s^{2-\gamma} + \delta(z')^{-\frac{1}{1+\gamma}}s^{1-\gamma} + \delta(z')^{-\gamma-\frac{1}{1+\gamma}}s\right).\end{aligned}$$

This, combining (2.32) and (2.33), we have for $\alpha = 1, 2, \cdots, n-1$,

$$\begin{aligned}[\partial_\alpha \widetilde{\mathrm{u}}_\ell]_{\gamma,\widehat{\Omega}_s(z')} \leq & C\left|\varphi^{(\ell)}(z', \frac{\varepsilon}{2} + h_1(z'))\right|\left(\delta(z')^{-\frac{2}{1+\gamma}}s^{1-\gamma} + \delta(z')^{-\gamma-\frac{1}{1+\gamma}}\right) \\ & + C\|\varphi^{(\ell)}\|_{C^{1,\gamma}(\Gamma_1^+)}\left(\delta(z')^{-\frac{2}{1+\gamma}}s^{2-\gamma} + \delta(z')^{-\frac{1}{1+\gamma}}s^{1-\gamma} + \delta(z')^{-\gamma-\frac{1}{1+\gamma}}s + \delta(z')^{-\gamma}\right).\end{aligned}$$

From the above formula and (2.31), we have proved Proposition 2.1. $\square$

Let $z = (z', z_n)$ be as in (2.20), $\delta(z')$ be as in (2.8), and $\widehat{\Omega}_s(z)$ be as in (2.19) for $0 < s \leq C\delta(z')$. Let $A_{ij}^{\alpha\beta}$ be as in Definition 1.4, $\widetilde{\mathrm{u}}_\ell$ be as in (2.9) for $\ell = 1, \cdots, m$, and $|\widehat{\Omega}_s(z)|$ be the volume of region $\widehat{\Omega}_s(z)$. For $i = 1, 2, \cdots, m$ and $\alpha = 1, \cdots, n$, we define

$$\mathcal{M}_i^{(\alpha)} := (\mathfrak{a}_i^\alpha)_{j\beta} = \left(\frac{1}{|\widehat{\Omega}_s(z)|}\int_{\widehat{\Omega}_s(z)} \sum_{\beta,j}\left(A_{ij}^{\alpha\beta}(y)\partial_\beta \tilde{u}_\ell^{(j)}(y)\right)dy\right)_{j\beta}. \tag{2.34}$$

Clearly, by using (2.13) we have w_ℓ, $\ell = 1, \cdots, m$ as in (2.12) satisfies

$$\sum_{\alpha,\beta,i,j} \partial_\alpha\big(A_{ij}^{\alpha\beta}\partial_\beta w_\ell^{(j)} + B_{ij}^{\alpha} w_\ell^{(j)}\big) + C_{ij}^{\beta}\partial_\beta w_\ell^{(j)} + D_{ij} w_\ell^{(j)} = H^{(i)} - \sum_\alpha \partial_\alpha(F_i^{(\alpha)} - \mathcal{M}_i^{(\alpha)}), \quad \text{in} \quad \Omega_1. \tag{2.35}$$

Lemma 2.2. *Let $z = (z', z_n) \in \Omega_{1/2}$ be as in (1.7), $\delta(z')$ be as in (2.8) and $\widehat{\Omega}_{\delta(z')}(z)$ be as in (2.19). Let h_1, h_2 satisfy (1.4)–(1.19), Γ_1^+, Γ_1^- be as in (1.9) and φ, ψ be as in (1.11). Let ε be as in (1.3) and w_ℓ be as in (2.35), $\ell = 1, \cdots, m$. Then there exists a positive constant C independent of ε, ℓ, such that, for $0 \le |z'| \le \varepsilon^{\frac{1}{1+\gamma}}$,*

$$\begin{aligned}\int_{\widehat{\Omega}_{\delta(z')}(z)} |\nabla \mathsf{w}_l|^2 \, dx \le & C\varepsilon^{n-\frac{2}{1+\gamma}} \left(\left| \varphi^{(\ell)}(z', \frac{\varepsilon}{2} + h_1(z')) - \psi^{(\ell)}(z', -\frac{\varepsilon}{2} + h_2(z')) \right|^2 \right) \\ & + C\varepsilon^{n-\frac{2}{1+\gamma}+2} \left(\|\mathsf{w}_\ell\|^2_{L^2(\Omega_1)} + \left\|\varphi^{(\ell)}\right\|^2_{C^{1,\gamma}(\Gamma_1^+)} + \left\|\psi^{(\ell)}\right\|^2_{C^{1,\gamma}(\Gamma_1^-)} \right),\end{aligned} \tag{2.36}$$

and for $\varepsilon^{\frac{1}{1+\gamma}} < |z'| \le 1/2$,

$$\begin{aligned}\int_{\widehat{\Omega}_{\delta(z')}(z)} |\nabla \mathsf{w}_\ell|^2 \, dx \le & C|z'|^{(1+\gamma)(n-\frac{2}{1+\gamma})} \left(\left| \varphi^{(\ell)}(z', \frac{\varepsilon}{2} + h_1(z')) - \psi^{(\ell)}(z', -\frac{\varepsilon}{2} + h_2(z')) \right|^2 \right) \\ & + C|z'|^{(1+\gamma)(n-\frac{2}{1+\gamma}+2)} \left(\|\mathsf{w}_\ell\|^2_{L^2(\Omega_1)} + \left\|\varphi^{(\ell)}\right\|^2_{C^{1,\gamma}(\Gamma_1^+)} + \left\|\psi^{(\ell)}\right\|^2_{C^{1,\gamma}(\Gamma_1^-)} \right).\end{aligned} \tag{2.37}$$

Proof. For simplicity, we assume that $\psi \equiv 0$. Indeed, for $0 < t < s < 1/2$, let η be a cut-off function satisfying $0 \le \eta(x') \le 1$,

$$\eta(x') = \begin{cases} 1, \text{ if } |x' - z'| < t \\ 0, \text{ if } |x' - z'| > s \end{cases}, \quad |\nabla \eta(x')| \le \frac{2}{s-t}.$$

Multiplying (2.35) by $\eta^2 \mathsf{w}_\ell$ and using the integration by parts, one has

$$\begin{aligned}&\sum_{\alpha,\beta,i,j} \int_{\widehat{\Omega}_s(z)} \eta^2 A_{ij}^{\alpha\beta}(x)\partial_\beta w_\ell^{(j)} \partial_\alpha w_\ell^{(i)} \, dx \\ = & -\sum_{\alpha,\beta,i,j} \int_{\widehat{\Omega}_s(z)} A_{ij}^{\alpha\beta}(x)\partial_\beta w_\ell^{(j)} w_\ell^{(i)} 2\eta\partial_\alpha \eta \, dx - \sum_{\alpha,i,j} \int_{\widehat{\Omega}_s(z)} B_{ij}^{\alpha}(x) w_\ell^{(j)} \partial_\alpha(\eta^2 w_\ell^{(i)}) \\ & - \sum_{\beta,i,j} \int_{\widehat{\Omega}_s(z)} C_{ij}^{\beta}(x)\partial_\beta w_\ell^{(j)} (\eta^2 w_\ell^{(i)}) - \sum_{i,j} \int_{\widehat{\Omega}_s(z)} D_{ij}(x) w_\ell^{(j)} (\eta^2 w_\ell^{(i)}) \, dx \\ & + \sum_i \int_{\widehat{\Omega}_s(z)} H^{(i)}(\eta^2 w_\ell^{(i)}) \, dx + \sum_{\alpha,i} \int_{\widehat{\Omega}_s(z)} (F_i^{(\alpha)} - \mathcal{M}_i^{(\alpha)})\partial_\alpha(\eta^2 w_\ell^{(i)}) \, dx.\end{aligned} \tag{2.38}$$

Now we can bound with (1.18), Young's inequality $2ab \le \zeta a^2 + \frac{b^2}{\zeta}$, and the properties of η,

$$\sum_{\alpha,\beta,i,j} \left| \int_{\widehat{\Omega}_s(z)} A_{ij}^{\alpha\beta}(x)\partial_\beta w_\ell^{(j)} w_\ell^{(i)} 2\eta\partial_\alpha \eta \, dx \right|$$

$$\leq \zeta_1 \Lambda \int\limits_{\widehat{\Omega}_s(z)} \eta^2 |\nabla \mathsf{w}_\ell|^2 \, \mathrm{d}x + \frac{4\Lambda}{\zeta_1 (s-t)^2} \int\limits_{\widehat{\Omega}_s(z)} |\mathsf{w}_\ell|^2 \, \mathrm{d}x, \tag{2.39}$$

combining with (1.20), we deduce

$$\begin{aligned}\sum_{\alpha,i,j} \left| \int\limits_{\widehat{\Omega}_s(z)} B_{ij}^{\alpha}(x) w_\ell^{(j)} \partial_\alpha (\eta^2 w_\ell^{(i)}) \, dx \right| &= \sum_{\alpha,i,j} \left| \int\limits_{\widehat{\Omega}_s(z)} B_{ij}^{\alpha}(x) w_\ell^{(j)} (2\eta \partial_\alpha \eta w_\ell^{(i)} + \eta^2 \partial_\alpha w_\ell^{(i)}) \, dx \right| \\ &\leq \frac{\zeta_2 \kappa_3}{2} \int\limits_{\widehat{\Omega}_s(z)} \eta^2 |\nabla \mathsf{w}_\ell|^2 \, dx + \frac{C}{\zeta_2 (s-t)^2} \int\limits_{\widehat{\Omega}_s(z)} |\mathsf{w}_\ell|^2 \, dx. \end{aligned} \tag{2.40}$$

Similarly, using Young's inequality again, we have

$$\sum_{\beta,i,j} \left| \int\limits_{\widehat{\Omega}_s(z)} C_{ij}^{\beta}(x) \partial_\beta w_\ell^{(j)} \eta^2 w_\ell^{(i)} \, dx \right| \leq \frac{\zeta_3 \kappa_3}{2} \int\limits_{\widehat{\Omega}_s(z)} \eta^2 |\nabla \mathsf{w}_\ell|^2 \, dx + \frac{C}{\zeta_3 (s-t)^2} \int\limits_{\widehat{\Omega}_s(z)} |\mathsf{w}_\ell|^2 \, dx. \tag{2.41}$$

Noticing $0 < s - t < 1$, we obtain

$$\sum_{i,j} \left| \int\limits_{\widehat{\Omega}_s(z)} D_{ij}(x) w_\ell^{(j)} \eta^2 w_\ell^{(i)} \, dx \right| \leq \frac{C}{(s-t)^2} \int\limits_{\widehat{\Omega}_s(z)} |\mathsf{w}_\ell|^2 \, dx. \tag{2.42}$$

Recalling (2.9) and (2.14), we also have

$$\sum_{i} \left| \int\limits_{\widehat{\Omega}_s(z)} H^{(i)} \eta^2 w_\ell^{(i)} \, dx \right| \leq \frac{C}{(s-t)^2} \int\limits_{\widehat{\Omega}_s(z)} |\mathsf{w}_\ell|^2 \, dx + (s-t)^2 \sum_i \int\limits_{\widehat{\Omega}_s(z)} |H^{(i)}|^2 \, dx. \tag{2.43}$$

For the last term in the right-hand side of (2.38), using the Young's inequality again we obtain

$$\begin{aligned} &\sum_{\alpha,i} \left| \int\limits_{\widehat{\Omega}_s(z)} (F_i^{(\alpha)} - \mathcal{M}_i^{(\alpha)}) \partial_\alpha (\eta^2 w_\ell^{(i)}) \, dx \right| \\ \leq &\frac{\zeta_4}{2} \int\limits_{\widehat{\Omega}_s(z)} \eta^2 |\nabla \mathsf{w}_\ell|^2 \, dx + \frac{C}{\zeta_4} \sum_{\alpha,i} \int\limits_{\widehat{\Omega}_s(z)} |F_i^{(\alpha)} - \mathcal{M}_i^{(\alpha)}|^2 \, dx + \frac{C}{\zeta_4 (s-t)^2} \int\limits_{\widehat{\Omega}_s(z)} |\mathsf{w}_\ell|^2 \, dx. \end{aligned} \tag{2.44}$$

Choosing $0 < \zeta_i < 1, i = 1, 2, 3, 4$ satisfy

$$\Lambda \zeta_1 + \frac{\kappa_3}{2} \zeta_2 + \frac{\kappa_3}{2} \zeta_3 + \frac{1}{2} \zeta_4 < \frac{\lambda}{2}.$$

In view of the strong ellipticity condition (1.10), we obtain

$$\lambda \int\limits_{\widehat{\Omega}_s(z)} \eta^2 |\nabla \mathsf{w}_\ell|^2 \, dx \leq \sum_{\alpha,\beta,i,j} \int\limits_{\widehat{\Omega}_s(z)} \eta^2 A_{ij}^{\alpha\beta}(x) \partial_\beta w_\ell^{(j)} \partial_\alpha w_\ell^{(i)} \, dx,$$

this, combining with (2.38)–(2.44), yields

$$\int\limits_{\widehat{\Omega}_t(z)} |\nabla \mathsf{w}_\ell|^2\,dx \leq \frac{C}{(s-t)^2}\int\limits_{\widehat{\Omega}_s(z)} |\mathsf{w}_\ell|^2\,dx + C(s-t)^2\sum_i \int\limits_{\widehat{\Omega}_s(z)} |H^{(i)}|^2\,dx + C\sum_{\alpha,i}\int\limits_{\widehat{\Omega}_s(z)} |F_i^{(\alpha)} - \mathcal{M}_i^{(\alpha)}|^2\,dx. \tag{2.45}$$

It follows from (2.10) and (2.11) that

$$\sum_i \int\limits_{\widehat{\Omega}_s(z)} |H^{(i)}|^2\,dx \leq C\left|\varphi^{(\ell)}(z',\varepsilon/2+h_1(z'))\right|^2 \int\limits_{|x'-z'|<s}\left(\frac{1}{\varepsilon+|x'|^{1+\gamma}}\right)dx' + \|\nabla\varphi^{(\ell)}\|^2_{L^\infty(\Gamma_1^+)}\int\limits_{|x'-z'|<s}\left(\frac{|x'-z'|^2}{\varepsilon+|x'|^{1+\gamma}}\right)dx'. \tag{2.46}$$

Case 1. For $|z'| \leq \varepsilon^{\frac{1}{1+\gamma}}$ and $0 < s < \varepsilon^{\frac{1}{1+\gamma}}$, we have $\varepsilon \leq \delta(z') \leq C\varepsilon$. By a direct calculation, we have

$$\int\limits_{\widehat{\Omega}_s(z)} |\mathsf{w}_\ell|^2\,dx = \int\limits_{\widehat{\Omega}_s(z)}\left|\int\limits_{-\frac{\varepsilon}{2}+h_2(x')}^{x_n} \partial_n w_\ell(x',x_n)\,dx_n\right|^2 dx \leq C\varepsilon^2 \int\limits_{\widehat{\Omega}_s(z)} |\nabla \mathsf{w}_\ell|^2\,dx, \tag{2.47}$$

and from (2.46), we have

$$\sum_i \int\limits_{\widehat{\Omega}_s(z)} |H^{(i)}|^2\,dx \leq C\left|\varphi^{(\ell)}(z',\frac{\varepsilon}{2}+h_1(z'))\right|^2\frac{s^{n-1}}{\varepsilon} + C\|\nabla\varphi^{(\ell)}\|^2_{L^\infty(\Gamma_1^+)}\frac{s^{n+1}}{\varepsilon} := G_{11}(s).$$

By using (1.20), (2.14), and (2.34) for any $\alpha = 1,\cdots,n$, $i = 1,\cdots,m$, we have

$$\begin{aligned}\left|F_i^{(\alpha)} - \mathcal{M}_i^{(\alpha)}\right|^2 \leq& \frac{C\kappa_3}{|\widehat{\Omega}_s(z)|^2}\left([\nabla\tilde{\mathsf{u}}_\ell]_{\gamma;\widehat{\Omega}_s(z)}\int\limits_{\widehat{\Omega}_s(z)}|x-y|^\gamma\,dy + \delta(z')^{-1}\int\limits_{\widehat{\Omega}_s(z)}|x-y|^\gamma\,dy\right)^2 + C\kappa_3\|\varphi^{(\ell)}\|_{L^\infty(\Gamma_1^+)}\\ \leq& C\left([\nabla\tilde{\mathsf{u}}_\ell]^2_{\gamma;\widehat{\Omega}_s(z)} + \delta(z')^{-2}\right)\left(\delta(z')^{2\gamma}+s^{2\gamma}\right),\end{aligned}$$

thus, from Proposition 2.1 we have

$$\sum_{\alpha,i}\int\limits_{\widehat{\Omega}_s(z)} |F_i^\alpha - \mathcal{M}_i^\alpha|^2\,dx \leq C\left(\left|\varphi^{(\ell)}(z',\frac{\varepsilon}{2}+h_1(z'))\right|^2 + s^2\|\varphi^{(\ell)}\|^2_{C^{1,\gamma}(\Gamma_1^+)}\right)\left(\frac{s^{n+1}}{\varepsilon^{1+\frac{2}{1+\gamma}}} + \frac{s^{n-1}}{\varepsilon^{\frac{2}{1+\gamma}-1}} + \frac{s^{n+1-2\gamma}}{\varepsilon^{1+\frac{2}{1+\gamma}-2\gamma}} + \frac{s^{n-1+2\gamma}}{\varepsilon^{1+\frac{2\gamma^2}{1+\gamma}}}\right) =: G_{12}(s). \tag{2.48}$$

Denote $F(t) := \int_{\widehat{\Omega}_t(z_1)} |\nabla \mathsf{w}_\ell|^2\,dx$. It follows from (2.45), (2.47), and (2.48) that

$$F(t) \leq \left(\frac{c_1\varepsilon}{s-t}\right)^2 F(s) + C(s-t)^2 G_{11}(s) + CG_{12}(s). \tag{2.49}$$

Similarly as in [15], let $k=(4c_1\varepsilon^{\frac{\gamma}{1+\gamma}})^{-1}$ and $t_\tau=\delta(z')+2c_1\tau\varepsilon$, $\tau=0,1,2,\cdots,k$. It is easy to see from the definition of $G_{11}(s)$ and $G_{12}(s)$ that

$$G_{11}(t_{\tau+1})\le C\varepsilon^{n-2}\left(\left|\varphi^{(\ell)}(z',\varepsilon/2+h_1(z'))\right|^2+\varepsilon^2\|\nabla\varphi^{(\ell)}\|^2_{L^\infty(\Gamma_1^+)}\right)(\tau+1)^{n+1},$$

and

$$G_{12}(t_{\tau+1})\le C\varepsilon^{n-\frac{2}{1+\gamma}}\left(\left|\varphi^{(\ell)}(z',\frac{\varepsilon}{2}+h_1(z'))\right|^2+\varepsilon^2\left\|\varphi^{(\ell)}\right\|^2_{C^{1,\gamma}(\Gamma_1^+)}\right)(\tau+1)^{n+3}.$$

Taking $s=t_{i+1}$ and $t=t_i$ in (2.49), we have the following iteration formula

$$\begin{aligned}F(t_\tau)\le&\frac{1}{4}F(t_{\tau+1})+C\varepsilon^n\left(\left|\varphi^{(\ell)}(z',\varepsilon/2+h_1(z'))\right|^2+\varepsilon^2\|\nabla\varphi^{(\ell)}\|^2_{L^\infty(\Gamma_1^+)}\right)(\tau+1)^{n+1}\\&+C\varepsilon^{n-\frac{2}{1+\gamma}}\left(\left|\varphi^{(\ell)}(z',\frac{\varepsilon}{2}+h_1(z'))\right|^2+\varepsilon^2\left\|\varphi^{(\ell)}\right\|^2_{C^{1,\gamma}(\Gamma_1^+)}\right)(\tau+1)^{n+3}\\ \le&C\varepsilon^{n-\frac{2}{1+\gamma}}\left(|\varphi^{(\ell)}(z',\frac{\varepsilon}{2}+h_1(z'))|^2+\varepsilon^2\left\|\varphi^{(\ell)}\right\|^2_{C^{1,\gamma}(\Gamma_1^+)}\right)(\tau+1)^{n+3},\end{aligned}$$

after k iterations, and by virtue of Lemma 2.1, we have

$$\begin{aligned}F(t_0)\le(\frac{1}{4})^kF(t_k)+C\varepsilon^{n-\frac{2}{1+\gamma}}&\left(|\varphi^{(\ell)}(z',\frac{\varepsilon}{2}+h_1(z'))|^2+\varepsilon^2\left\|\varphi^{(\ell)}\right\|^2_{C^{1,\gamma}(\Gamma_1^+)}\right)\sum_{i=0}^{k-1}(\frac{1}{4})^i(i+1)^{n+3}\\ \le&C\varepsilon^{n-\frac{2}{1+\gamma}}\left(|\varphi^{(\ell)}(z',\frac{\varepsilon}{2}+h_1(z'))|^2+\varepsilon^2(\left\|\varphi^{(\ell)}\right\|^2_{C^{1,\gamma}(\Gamma_1^+)}+\|\mathsf{w}_l\|^2_{L^2(\Omega_1)})\right).\end{aligned}$$

This implies that Lemma 2.2 holds when $|z'|\le\varepsilon^{\frac{1}{1+\gamma}}$.

Case 2. For $\varepsilon^{\frac{1}{1+\gamma}}\le|z'|\le\frac{1}{2}$ and $0<s<|z'|$, we have $\frac{1}{C}|z'|^{1+\gamma}\le\delta(z')\le C|z'|^{1+\gamma}$. Estimates (2.47) and (2.48) become, respectively,

$$\begin{aligned}\int\limits_{\widehat{\Omega}_s(z)}|\mathsf{w}_\ell|^2\,dx\le&C|z'|^{2(1+\gamma)}\int\limits_{\widehat{\Omega}_s(z')}|\nabla\mathsf{w}_\ell|^2\,dx,\quad\text{if }0<s<\frac{2}{3}|z'|,\\ \sum_i\int\limits_{\widehat{\Omega}_s(z)}|H^{(i)}|^2\,dx\le&C\left|\varphi^{(\ell)}(z',\frac{\varepsilon}{2}+h_1(z'))\right|^2\frac{s^{n-1}}{|z'|^2}+C\|\nabla\varphi^{(\ell)}\|^2_{L^\infty(\Gamma_1^+)}\frac{s^{n+1}}{|z'|^2}:=G_{21}(s),\end{aligned}$$

and

$$\begin{aligned}\sum_{\alpha,i}\int\limits_{\widehat{\Omega}_s(z)}\left|F_i^{(\alpha)}-\mathcal{M}_i^{(\alpha)}\right|^2\,dx\le&C\left(\left|\varphi^{(\ell)}(z',\frac{\varepsilon}{2}+h_1(z'))\right|^2+s^2\left\|\varphi^{(\ell)}\right\|^2_{C^{1,\gamma}(\Gamma_1^+)}\right)\\&\left(\frac{s^{n+1}}{|z'|^{3+\gamma}}+\frac{s^{n-1}}{|z'|^{-1-\gamma}}+\frac{s^{n+1-2\gamma}}{|z'|^{1-\gamma-2\gamma^2}}+\frac{s^{n-1+2\gamma}}{|z'|^{\gamma-1+2\gamma^2}}\right)\\=&:G_{22}(s).\end{aligned}$$

Let $k=(4c_2|z'|^\gamma)^{-1}$ and $t_\tau=\delta(z')+2c_2\iota\,|z'|^{1+\gamma}$, $\tau=0,1,2,\cdots,k$, one has

$$G_{21}(s) \leq C|z'|^{(1+\gamma)(n-2)} \left(\left| \varphi^{(\ell)}(z', \varepsilon/2 + h_1(z')) \right|^2 + C|z'|^{2(1+\gamma)} \|\nabla \varphi^{(\ell)}\|^2_{L^\infty(\Gamma_1^+)} \right) (\tau+1)^{n+1},$$

and

$$G_{22}(t_{\tau+1}) \leq C|z'|^{(1+\gamma)(n-\frac{2}{1+\gamma})} \left(\left|\varphi^{(\ell)}(z', \frac{\varepsilon}{2} + h_1(z'))\right|^2 + C|z'|^{2(1+\gamma)} \left\|\varphi^{(\ell)}\right\|^2_{C^{1,\gamma}(\Gamma_1^+)} \right) (\tau+1)^{n+3}.$$

Then, we obtain that, for $0 < t < s < \frac{2}{3}|z'|$,

$$F(t_\tau) \leq \frac{1}{4} F(t_{\tau+1}) + C|z'|^{(1+\gamma)(n-\frac{2}{1+\gamma})} \left(\left|\varphi^{(\ell)}(z', \frac{\varepsilon}{2} + h_1(z'))\right|^2 + C|z'|^{2(1+\gamma)} \left\|\varphi^{(\ell)}\right\|^2_{C^{1,\gamma}(\Gamma_1^+)} \right) (\tau+1)^{n+3},$$

after k iterations, and using Lemma 2.1 again, we have

$$F(t_0) \leq C|z'|^{(1+\gamma)(n-\frac{2}{1+\gamma})} \left(\left|\varphi^{(\ell)}(z', \frac{\varepsilon}{2} + h_1(z'))\right|^2 + C|z'|^{2(1+\gamma)} (\left\|\varphi^{(\ell)}\right\|^2_{C^{1,\gamma}(\Gamma_1^+)} + \|\mathsf{w}_\ell\|^2_{L^2(\Omega_1)}) \right).$$

This implies that Lemma 2.2 holds when $|z'| \geq \varepsilon^{\frac{1}{1+\gamma}}$. $\square$

Lemma 2.3. *Let $z = (z', z_n) \in \Omega_{1/2}$ be as in (1.7), $\delta(z')$ be as in (2.8) and $\widehat{\Omega}_{\delta(z')}(z)$ be as in (2.19). Let h_1, h_2 satisfy (1.4)–(1.19), Γ_1^+, Γ_1^- be as in (1.9) and φ, ψ be as in (1.11). Let ε be as in (1.3) and w_ℓ be as in (2.35), $\ell = 1, \cdots, m$. Then there exists a positive constant C independent of ε, such that, for $|z'| \leq \varepsilon^{\frac{1}{1+\gamma}}$,*

$$|\nabla \mathsf{w}_\ell(z', z_n)| \leq C\varepsilon^{-\frac{1}{1+\gamma}} \left| \varphi^{(\ell)}(z', \frac{\varepsilon}{2} + h_1(z')) - \psi^l(z', -\frac{\varepsilon}{2} + h_2(z')) \right| + C\varepsilon^{-\frac{\gamma}{1+\gamma}} \left(\|\mathsf{w}_\ell\|_{L^2(\Omega_1)} + \|\varphi^{(\ell)}\|_{C^{1,\gamma}(\Gamma_1^+)} + \|\psi^{(\ell)}\|_{C^{1,\gamma}(\Gamma_1^-)} \right),$$

and for $\varepsilon^{\frac{1}{1+\gamma}} < |z'| < \frac{1}{2}$,

$$|\nabla \mathsf{w}_\ell(z', z_n)| \leq C|z'|^{-1} \left| \varphi^{(\ell)}(z', \frac{\varepsilon}{2} + h_1(z')) - \psi^{(\ell)}(z', -\frac{\varepsilon}{2} + h_2(z')) \right| + C|z'|^{-\gamma} \left(\|\mathsf{w}_\ell\|_{L^2(\Omega_1)} + \|\varphi^{(\ell)}\|_{C^{1,\gamma}(\Gamma_1^+)} + \|\psi^{(\ell)}\|_{C^{1,\gamma}(\Gamma_1^-)} \right).$$

Consequently, by (2.10) and (2.11), we have, for sufficiently small ε and $z \in \Omega_{1/2}$,

$$|\nabla \mathsf{v}_\ell(z)| \leq \frac{C\left|\varphi^{(\ell)}(z', \frac{\varepsilon}{2} + h_1(z')) - \psi^{(\ell)}(z', -\frac{\varepsilon}{2} + h_2(z'))\right|}{\varepsilon + |z'|^{1+\gamma}} + C\left(\|\mathsf{v}_\ell\|_{L^2(\Omega_1)} + \|\varphi^{(\ell)}\|_{C^{1,\gamma}(\Gamma_1^+)} + \|\psi^{(\ell)}\|_{C^{1,\gamma}(\Gamma_1^-)} \right). \tag{2.50}$$

Moreover, if $\varphi^{(\ell)}(\vec{0}_{n-1}, \frac{\varepsilon}{2}) \neq \psi^{(\ell)}(\vec{0}_{n-1}, -\frac{\varepsilon}{2})$, then there exists a positive constant C independent of ε, such that, for any $z_n \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$,

$$\left|\nabla \mathsf{v}_\ell(\vec{0}_{n-1}, z_n)\right| \geq \frac{|\varphi^{(\ell)}(\vec{0}_{n-1}, z_n) - \psi^{(\ell)}(\vec{0}_{n-1}, z_n)|}{C\varepsilon}.$$

Proof. For simplicity, we assume that $\psi \equiv 0$. Given $z = (z', z_n) \in \Omega_{1/2}$, making the following change of variables on $\widehat{\Omega}_{\delta(z')}(z)$, as in [15], we have

$$\begin{cases} x' - z' = \delta(z')y', \\ x_n = \delta(z')y_n, \end{cases}$$

then $\widehat{\Omega}_{\delta(z')}(z)$ becomes $\boldsymbol{Q}_1$ of nearly unit size, where

$$\begin{aligned} \boldsymbol{Q}_r = \Big\{ y \in \mathbb{R}^n : &-\frac{\varepsilon}{2\delta(z')} + \frac{1}{\delta(z')} h_2(\delta(z')y' + z') \\ &< y_n < \frac{\varepsilon}{2\delta(z')} + \frac{1}{\delta(z')} h_1(\delta(z')y' + z'),\ |y'| < r \Big\}, \end{aligned} \tag{2.51}$$

for $r \le 1$, and the top and bottom boundaries become

$$\widehat{\Gamma}_r^+ := \left\{ y \in \mathbb{R}^n \,:\, y_n = \frac{\varepsilon}{2\delta(z')} + \frac{1}{\delta(z')} h_1(\delta(z')y' + z'), \quad |y'| < r \right\},$$

and

$$\widehat{\Gamma}_r^- := \left\{ y \in \mathbb{R}^n \,:\, y_n = -\frac{\varepsilon}{2\delta(z')} + \frac{1}{\delta(z')} h_2(\delta(z')y' + z'), \quad |y'| < r \right\},$$

respectively. Let

$$\widehat{\mathsf{w}}_\ell(y', y_n) := \mathsf{w}_\ell(\delta(z')y' + z', \delta(z')y_n),\ \widehat{\mathsf{u}}_\ell(y', y_n) := \widetilde{\mathsf{u}}_\ell(\delta(z')y' + z', \delta(z')y_n), \quad \text{for } (y', y_n) \in \boldsymbol{Q}_1.$$

It follows from (2.13) that $\widehat{\mathsf{w}}_\ell(y)$ satisfies

$$\begin{cases} \partial_\alpha \left(\widehat{A}_{ij}^{\alpha\beta} \partial_\beta \widehat{w}_\ell^{(j)} + \widehat{B}_{ij}^{\alpha} \widehat{w}_\ell^{(j)} \right) + \widehat{C}_{ij}^{\beta} \partial_\beta \widehat{w}_\ell^{(j)} + \widehat{D}_{ij} \widehat{w}_\ell^{(j)} = \widehat{H}^{(i)} - \partial_\alpha \widehat{F}_i^\alpha, & \text{in } \boldsymbol{Q}_1, \\ \widehat{\mathsf{w}}_\ell = 0, & \text{on } \widehat{\Gamma}_1^{\pm}, \end{cases} \tag{2.52}$$

where

$$\begin{aligned} \widehat{A}_{ij}^{\alpha\beta}(y) &:= A_{ij}^{\alpha\beta}(\delta y' + z', \delta y_n), \quad \widehat{B}_{ij}^{\alpha}(y) := \delta B_{ij}^{\alpha}(\delta y' + z', \delta y_n), \\ \widehat{C}_{ij}^{\beta}(y) &:= \delta C_{ij}^{\beta}(\delta y' + z', \delta y_n), \quad \widehat{D}_{ij}(y) := \delta^2 D_{ij}(\delta y' + z', \delta y_n), \\ \widehat{F}_i^{\alpha}(y) &:= \widehat{A}_{ij}^{\alpha\beta}(y) \partial_\beta \widehat{u}_\ell^{(j)} + \widehat{B}_{ij}^{\alpha}(y) \widehat{u}_\ell^{(j)}, \quad \widehat{H}^{(i)}(y) := \widehat{C}_{ij}^{\beta}(y) \partial_\beta \widehat{u}_\ell^{(j)} - \widehat{D}_{ij}(y) \widehat{u}_\ell^{(j)}. \end{aligned} \tag{2.53}$$

It follows from Theorem 2.2 that

$$\|\widehat{\mathsf{w}}_l\|_{L^\infty(\boldsymbol{Q}_{1/2})} \le C \left(\|\widehat{\mathsf{w}}_l\|_{L^2(\boldsymbol{Q}_1)} + [\widehat{\mathsf{F}}]_{\gamma, \boldsymbol{Q}_1} + \|\widehat{\mathsf{H}}\|_{L^\infty(\boldsymbol{Q}_1)} \right). \tag{2.54}$$

Applying Theorem 2.1 for (2.52) with (2.53) on $\boldsymbol{Q}_{1/2}$, we have

$$\|\widehat{\mathsf{w}}_\ell\|_{C^{1,\gamma}(\boldsymbol{Q}_{1/4})} \le C \left(\|\widehat{\mathsf{w}}_\ell\|_{L^\infty(\boldsymbol{Q}_{1/2})} + [\widehat{\mathsf{F}}]_{\gamma, \boldsymbol{Q}_1} + \|\widehat{\mathsf{H}}\|_{L^\infty(\boldsymbol{Q}_1)} \right).$$

This, combining with (2.54) and using the Poincaré inequality, yields

$$\|\nabla \widehat{\mathsf{w}}_\ell\|_{L^\infty(\boldsymbol{Q}_{1/4})} \le C \left(\|\nabla \widehat{\mathsf{w}}_{l\ell}\|_{L^2(\boldsymbol{Q}_1)} + [\widehat{\mathsf{F}}]_{\gamma, \boldsymbol{Q}_1} + \|\widehat{\mathsf{H}}\|_{L^\infty(\boldsymbol{Q}_1)} \right).$$

In the following proofs, we will briefly refer to $\delta(z')$ as δ. Recalling back to the original region $\widehat{\Omega}_\delta(z)$, we have

$$\|\nabla\widehat{\mathsf{w}}_\ell\|_{L^\infty(Q_{1/4})} = \delta\|\nabla\mathsf{w}_\ell\|_{L^\infty(\widehat{\Omega}_{\delta/4}(z))}, \quad \|\nabla\widehat{\mathsf{w}}_\ell\|_{L^2(Q_1)} = \delta^{1-\frac{n}{2}}\|\nabla\mathsf{w}_\ell\|_{L^2(\widehat{\Omega}_\delta(z))},$$
$$\|\nabla\widehat{\mathsf{u}}_\ell\|_{L^\infty(Q_1)} = \delta\|\nabla\widetilde{\mathsf{u}}_\ell\|_{L^\infty(\widehat{\Omega}_\delta(z))}, \quad [\nabla\widehat{\mathsf{u}}_\ell]_{\gamma,Q_1} = \delta^{1+\gamma}[\nabla\widetilde{\mathsf{u}}_\ell]_{\gamma,\widehat{\Omega}_\delta(z)},$$

thus,

$$[\widehat{\mathsf{F}}]_{\gamma,Q_1} \le C\delta^{1+\gamma}\left(\|\nabla\widetilde{\mathsf{u}}_\ell\|_{L^\infty(\widehat{\Omega}_\delta(z))} + [\nabla\widetilde{\mathsf{u}}_\ell]_{\gamma,\widehat{\Omega}_\delta(z)} + \|\widetilde{\mathsf{u}}_\ell\|_{L^\infty(\widehat{\Omega}_\delta(z))}\right),$$

and

$$\|\widehat{\mathsf{H}}\|_{L^\infty(Q)} \le C\delta^2\left(\|\nabla\widetilde{\mathsf{u}}_\ell\|_{L^\infty(\widehat{\Omega}_\delta(z))} + \|\widetilde{\mathsf{u}}_\ell\|_{L^\infty(\widehat{\Omega}_\delta(z))}\right).$$

It follows that

$$\begin{aligned}&\|\nabla\mathsf{w}_\ell\|_{L^\infty(\widehat{\Omega}_{\delta/4}(z))}\\ \le& C\delta^{-\frac{n}{2}}\|\nabla\mathsf{w}_\ell\|_{L^2(\widehat{\Omega}_\delta(z))} + C\delta^\gamma\left([\nabla\widetilde{\mathsf{u}}_\ell]_{\gamma,\widehat{\Omega}_\delta(z)} + \|\nabla\widetilde{\mathsf{u}}_\ell\|_{L^\infty(\widehat{\Omega}_\delta(z))} + \|\widetilde{\mathsf{u}}_\ell\|_{L^\infty(\widehat{\Omega}_\delta(z))}\right).\end{aligned}$$

Case 1. For $0 \le |z'| \le \varepsilon^{\frac{1}{1+\gamma}}$.

By (2.36) and Proposition 2.1, we have

$$\delta^{-\frac{n}{2}}\|\nabla\mathsf{w}\|_{L^2(\widehat{\Omega}_{\delta(z')}(z))} \le \frac{C}{\varepsilon^{\frac{1}{1+\gamma}}}\left|\varphi^{(\ell)}(z', \frac{\varepsilon}{2} + h_1(z'))\right| + C(\left\|\varphi^{(\ell)}\right\|_{C^{1,\gamma}(\Gamma_1^+)} + \|\mathsf{w}_\ell\|_{L^2(\Omega_1)}),$$

and

$$\delta^\gamma[\nabla\widetilde{\mathsf{u}}_\ell]_{\gamma,\widehat{\Omega}_\delta(z)} \le \frac{C}{\varepsilon^{\frac{1}{1+\gamma}}}\left|\varphi^{(\ell)}(z', \varepsilon/2 + h_1(z'))\right| + C\left\|\varphi^{(\ell)}\right\|_{C^{1,\gamma}(\Gamma_1^+)}.$$

By using (2.9), (2.10), and (2.11), we can obtain

$$\delta^\gamma\|\widetilde{\mathsf{u}}_\ell\|_{L^\infty(\widehat{\Omega}_\delta(z))} \le \|\varphi^{(\ell)}\|_{L^\infty(\Gamma_{1+})},$$

and

$$\delta^\gamma\|\nabla\widetilde{\mathsf{u}}_\ell\|_{L^\infty(\widehat{\Omega}_\delta(z))} \le \frac{C}{\varepsilon^{1-\gamma}}\left|\varphi^{(\ell)}(z', \varepsilon/2 + h_1(z'))\right| + C\|\nabla\varphi^{(\ell)}\|_{L^\infty(\Gamma_1^+)}.$$

Therefore,

$$\|\nabla\mathsf{w}_\ell\|_{L^\infty\widehat{\Omega}_{\delta(z')/4}(z)} \le \frac{C}{\varepsilon^{\frac{1}{1+\gamma}}}\left|\varphi^{(\ell)}(z', \varepsilon/2 + h_1(z'))\right| + C(\left\|\varphi^{(\ell)}\right\|_{C^{1,\gamma}(\Gamma_1^+)} + \|\mathsf{w}_\ell\|_{L^2(\Omega_1)}).$$

Case 2. For $\varepsilon^{\frac{1}{1+\gamma}} < |z'| < 1/2$. By (2.37) and Proposition 2.1, we have

$$\delta^{-\frac{n}{2}}\|\nabla\mathsf{w}_\ell\|_{L^2(\widehat{\Omega}_{\delta(z')}(z))} \le \frac{C}{|z'|}\left|\varphi^{(\ell)}(z', \frac{\varepsilon}{2} + h_1(z'))\right| + C(\left\|\varphi^{(\ell)}\right\|_{C^{1,\gamma}(\Gamma_1^+)} + \|\mathsf{w}_\ell\|_{L^2(\Omega_1)}),$$

and

$$\delta^\gamma[\nabla\widetilde{\mathsf{u}}_\ell]_{\gamma,\widehat{\Omega}_\delta(z)} \le \frac{C}{|z'|}\left|\varphi^{(\ell)}(z',\varepsilon/2+h_1(z'))\right| + C\left\|\varphi^{(\ell)}\right\|_{C^{1,\gamma}(\Gamma_1^+)}.$$

By using (2.9), (2.10), and (2.11), we can obtain

$$\delta^\gamma\|\widetilde{\mathsf{u}}_\ell\|_{L^\infty(\widehat{\Omega}_\delta(z))} \le \|\varphi^{(\ell)}\|_{L^\infty(\Gamma_{1+})},$$

and

$$\delta^\gamma\|\nabla\widetilde{\mathsf{u}}_\ell\|_{L^\infty(\widehat{\Omega}_\delta(z))} \le \frac{C}{|z'|^{1-\gamma^2}}\left|\varphi^{(\ell)}(z',\varepsilon/2+h_1(z'))\right| + C\|\nabla\varphi^{(\ell)}\|_{L^\infty(\Gamma_1^+)}.$$

It follows that

$$\|\nabla\mathsf{w}_\ell\|_{L^\infty(\widehat{\Omega}_{\delta(z')/4}(z))} \le \frac{C}{|z'|}\left|\varphi^{(\ell)}(z',\varepsilon/2+h_1(z'))\right| + C\left(\left\|\varphi^{(\ell)}\right\|_{C^{1,\gamma}(\Gamma_1^+)} + \|\mathsf{w}_\ell\|_{L^2(\Omega_1)}\right).$$

Noticing that $|\nabla\mathsf{v}_\ell| \le |\nabla\mathsf{w}_\ell| + |\nabla\widetilde{\mathsf{u}}_\ell|$, and by (2.10), (2.11), (2.36), and (2.37), we obtain (2.50).

It is clear that if $\varphi^{(\ell)} \ne 0$, then

$$|\nabla\mathsf{v}_\ell(\vec{0}_{n-1},x_n)| \ge |\nabla\widetilde{\mathsf{u}}_\ell(\vec{0}_{n-1},x_n)| - |\nabla\mathsf{w}_\ell(\vec{0}_{n-1},x_n)| \ge \frac{|\varphi^{(\ell)}(\vec{0}_{n-1},x_n)|}{C\varepsilon}.$$

The Lemma 2.3 is proved when $\psi \equiv 0$. □

Proof of Theorem 1.1. By Lemma 2.1, Lemma 2.2, and Lemma 2.3, we have for $x \in \Omega_{1/2}$,

$$\begin{aligned}|\nabla\mathsf{u}(x)| \le \sum_{\ell=1}^m |\nabla\mathsf{v}_\ell(x)| \le & \frac{C\,|\varphi(x',\varepsilon/2+h_1(x')) - \psi(x',-\varepsilon/2+h_2(x'))|}{\varepsilon+|x'|^{1+\gamma}} \\ & + C\left(\|\varphi\|_{C^{1,\gamma}(\Gamma_1^+)} + \|\psi\|_{C^{1,\gamma}(\Gamma_1^-)} + \|\mathsf{u}\|_{L^2(\Omega_1)}\right).\end{aligned}$$

If $\varphi^{(\ell)}(\vec{0}_{n-1},\varepsilon/2) \ne \psi^{(\ell)}(\vec{0}_{n-1},-\varepsilon/2)$ for some integer ℓ, then by Lemma 2.3, we can obtain

$$|\nabla\mathsf{u}(\vec{0}_{n-1},x_n)| \ge \frac{\left|\varphi^{(\ell)}(\vec{0}_{n-1},\varepsilon/2) - \psi^{(\ell)}(\vec{0}_{n-1},-\varepsilon/2)\right|}{C\varepsilon} \quad \forall\, x_n \in (-\frac{\varepsilon}{2},\frac{\varepsilon}{2}).$$

The proof of Theorem 1.1 is completed. □

3. Proof of Corollary 1.1

Since the Lamé system as in (1.15) has more applications in practice, such as shear modulus in high contrast linear elastic composites. This section establishes the gradient estimates of Lamé system under the assumptions in Definition 1.4. We give a sketched proof of Corollary 1.1 and only list its main ingredients. In order to make the proof more clear and concise, we will prove the case when $n = 2$.

Let $\mathsf{v}_1 = (v_1^{(1)}, v_1^{(2)}) = (u^{(1)}, 0)$ be a weak solution to

$$\begin{cases}\mathcal{L}_{\lambda_1,\mu_1}\mathsf{v}_1 = \nabla\cdot(\mathbb{C}^0 e(\mathsf{v}_1)) = 0, & \text{in}\,\Omega_1,\\ \mathsf{v}_1 = (\varphi^{(1)},0), & \text{on}\,\Gamma_1^+,\\ \mathsf{v}_1 = (\psi^{(1)},0), & \text{on}\,\Gamma_1^-,\end{cases} \tag{3.1}$$

and $\mathsf{v}_2 = (v_2^{(1)}, v_2^{(2)}) = (0, u^{(2)})$ be a weak solution to

$$\begin{cases} \mathcal{L}_{\lambda_1,\mu_1}\mathsf{v}_2 = \nabla\cdot(\mathbb{C}^0 e(\mathsf{v}_2)) = 0, & \text{in}\,\Omega_1, \\ \mathsf{v}_2 = (0,\varphi^{(2)}), & \text{on}\,\Gamma_1^+, \\ \mathsf{v}_2 = (0,\psi^{(1)}), & \text{on}\,\Gamma_1^-. \end{cases} \tag{3.2}$$

It is obvious that

$$\mathsf{u} = \mathsf{v}_1 + \mathsf{v}_2 \quad \text{and} \quad \nabla\mathsf{u} = \nabla\mathsf{v}_1 + \nabla\mathsf{v}_2. \tag{3.3}$$

Then we still construct the auxiliary function $\widetilde{\mathsf{u}}_\ell$ for $\ell = 1, 2$ as shown in (2.9) in Section 2:

$$\begin{aligned} \widetilde{\mathsf{u}}_1 &:= \Big(\varphi^{(1)}(x_1, \frac{\varepsilon}{2} + h_1(x_1))\bar{u}(x) + \psi^{(1)}(x_1, -\frac{\varepsilon}{2} + h_2(x_1))(1-\bar{u}(x)),\ 0\Big), \\ \widetilde{\mathsf{u}}_2 &:= \Big(0,\ \varphi^{(2)}(x_1, \frac{\varepsilon}{2} + h_1(x_1))\bar{u}(x) + \psi^{(2)}(x_1, -\frac{\varepsilon}{2} + h_2(x_1))(1-\bar{u}(x))\Big). \end{aligned} \tag{3.4}$$

So $|\nabla\widetilde{\mathsf{u}}_\ell|$ also has the gradient estimate as in (2.10)–(2.11), and the Hölder semi-norm estimate of $\nabla\widetilde{\mathsf{u}}_\ell$ as in Proposition 2.1.

Denote $\mathsf{w}_\ell = \mathsf{v}_\ell - \widetilde{\mathsf{u}}_\ell$ for any $\ell = 1, 2$, which satisfies the following boundary value problem:

$$\begin{cases} \mathcal{L}_{\lambda_1,\mu_1}\mathsf{w}_\ell = -\mathcal{L}_{\lambda_1,\mu_1}\widetilde{\mathsf{u}}_\ell, & \text{in } \Omega_1, \\ \mathsf{w}_\ell = 0, & \text{on } \Gamma_1^+, \\ \mathsf{w}_\ell = 0, & \text{on } \Gamma_1^-. \end{cases} \tag{3.5}$$

Because the result in Corollary 1.1 independent of ℓ, we might as well consider only the case of $\ell = 1$.

Lemma 3.1. *Under the hypotheses of Lemma 2.1, and in addition that w_1 is the weak solution to (3.5), then there exists a positive constant C independent of ε, such that,*

$$\int\limits_{\Omega_{1/2}} |\nabla w_1|^2\,dx \le C\left(\|\mathsf{w}_1\|^2_{L^2(\Omega_1)} + \|\varphi^{(1)}\|^2_{C^{1,\gamma}(\Gamma_1^+)} + \|\psi^{(1)}\|^2_{C^{1,\gamma}(\Gamma_1^-)}\right). \tag{3.6}$$

Proof. Multiplying (3.5) by w_1 and making use of the integration by parts in $\Omega_{1/2}$, in view of $\mathsf{w}_1 = 0$ on $\Gamma_1^\pm$, we have

$$\int\limits_{\Omega_{1/2}} \Big(\mathbb{C}^0 e(\mathsf{w}_1), e(\mathsf{w}_1)\Big)dx = \int\limits_{\Omega_{1/2}} \big(\nabla\cdot(\mathbb{C}^0 e(\widetilde{\mathsf{u}}_1))\big)\cdot\mathsf{w}_1\,dx. \tag{3.7}$$

For the right-hand side of (3.7), noticing that $\partial_{22}\widetilde{u}_1^{(1)} = 0$ in $\Omega_{1/2}$, and using integration by parts, (2.10), (2.17), we have

$$\begin{aligned} &\left|\int\limits_{\Omega_{1/2}} \big(\nabla\cdot(\mathbb{C}^0 e(\widetilde{\mathsf{u}}_1))\big)\cdot\mathsf{w}_1\,dx\right| \\ \le& C\left|\int\limits_{\Omega_{1/2}} \partial_1(\partial_1\widetilde{u}_1^{(1)})w_1^{(1)}\,dx + \int\limits_{\Omega_{1/2}} \partial_2(\partial_1\widetilde{u}_1^{(1)})w_1^{(2)}\,dx\right| \\ \le& C\int\limits_{\Omega_{1/2}} \left|\partial_1\widetilde{u}_1^{(1)}\right| |\nabla w_1|\,dx + \int\limits_{\substack{|x_1|=1/2 \\ -\frac{\varepsilon}{2}+h_2(x_1)<x_2<\frac{\varepsilon}{2}+h_1(x_1)}} |w_1|\left|\partial_1\widetilde{u}_1^{(1)}\right|dx_2 \end{aligned}$$

$$\leq \frac{\lambda}{2}\int\limits_{\Omega_{1/2}} |\nabla w_1|^2\,dx + C\left(\|\mathsf{w}_1\|^2_{L^2(\Omega_1)} + \|\varphi^{(1)}\|^2_{C^{1,\gamma}(\Gamma_1^+)} + \|\psi^{(1)}\|^2_{C^{1,\gamma}(\Gamma_1^-)}\right). \tag{3.8}$$

For the left-hand side of (3.7), it follows from strong ellipticity condition as in (1.10) there exists a positive constant λ, such that,

$$\lambda \int\limits_{\Omega_{1/2}} |\nabla \mathsf{w}_1|^2\,dx \leq \int\limits_{\Omega_{1/2}} \Big(\mathbb{C}^0 e(\mathsf{w}_1), e(\mathsf{w}_1)\Big)dx. \tag{3.9}$$

By (3.7)–(3.9), we obtain (3.6). $\square$

From the definition of $\widetilde{\mathsf{u}}_1$, we can get

$$\mathbb{C}^0 e(\widetilde{\mathsf{u}}_1) = \begin{pmatrix} (\lambda_1+2\mu_1)\partial_1\widetilde{\mathsf{u}}_1^{(1)} & \mu_1\partial_2\widetilde{\mathsf{u}}_1^{(1)} \\ \mu_1\partial_2\widetilde{\mathsf{u}}_1^{(1)} & \lambda_1\partial_1\widetilde{\mathsf{u}}_1^{(1)} \end{pmatrix}.$$

Let

$$\mathcal{M} := \fint\limits_{\widehat{\Omega}_s(z_1)} \mathbb{C}^0 e(\bar{u}_1^1(y))\,dy := \frac{1}{|\widehat{\Omega}_s(z_1)|}\int\limits_{\widehat{\Omega}_s(z_1)} \mathbb{C}^0 e(\bar{u}_1^1(y))\;dy. \tag{3.10}$$

It follows from (3.5) that w_1 satisfies

$$\mathcal{L}_{\lambda_1,\mu_1}\mathsf{w}_1 = -\mathcal{L}_{\lambda_1,\mu_1}(\widetilde{\mathsf{u}}_1 - \mathcal{M}). \tag{3.11}$$

Lemma 3.2. *Under the hypotheses of Lemma 2.2 and in addition that w_1 is the weak solution to (3.5), then there exists a positive constant C independent of ε, such that, for $0 \leq |z_1| \leq \varepsilon^{\frac{1}{1+\gamma}}$,*

$$\begin{aligned}\int\limits_{\widehat{\Omega}_{\delta(z_1)}(z)} |\nabla \mathsf{w}_1|^2\;\,dx \leq & C\varepsilon^{\frac{2\gamma}{1+\gamma}}\left(\left|\varphi^{(1)}(z_1,\frac{\varepsilon}{2}+h_1(z_1)) - \psi^{(1)}(z_1,-\frac{\varepsilon}{2}+h_2(z_1))\right|^2\right) \\ & + C\varepsilon^{\frac{2\gamma}{1+\gamma}+2}\left(\|\mathsf{w}_1\|^2_{L^2(\Omega_1)} + \left\|\varphi^{(1)}\right\|^2_{C^{1,\gamma}(\Gamma_1^+)} + \left\|\psi^{(1)}\right\|^2_{C^{1,\gamma}(\Gamma_1^-)}\right),\end{aligned} \tag{3.12}$$

and for $\varepsilon^{\frac{1}{1+\gamma}} < |z_1| \leq 1/2$,

$$\begin{aligned}\int\limits_{\widehat{\Omega}_{\delta(z_1)}(z)} |\nabla \mathsf{w}_1|^2\;\,dx \leq & C|z_1|^{2\gamma}\left(\left|\varphi^{(1)}(z_1,\frac{\varepsilon}{2}+h_1(z_1)) - \psi^{(1)}(z_1,-\frac{\varepsilon}{2}+h_2(z_1))\right|^2\right) \\ & + C|z_1|^{2\gamma+2(1+\gamma)}\left(\|\mathsf{w}_1\|^2_{L^2(\Omega_1)} + \left\|\varphi^{(1)}\right\|^2_{C^{1,\gamma}(\Gamma_1^+)} + \left\|\psi^{(1)}\right\|^2_{C^{1,\gamma}(\Gamma_1^-)}\right).\end{aligned} \tag{3.13}$$

Proof. For simplicity, we assume that $\psi \equiv 0$. Similar to the proof of Lemma 2.2, for a fixed point $z = (z_1, z_2)$, we consider the following cut-off function $\eta(x_1)$: for $0 < t < s < 1/2$, let η be a cut-off function satisfying

$$\eta(x_1) = \begin{cases} 1 & \text{if } |x_1 - z_1| < t, \\ 0 & \text{if } |x_1 - z_1| > s, \end{cases} \quad \text{and} \quad |\eta'(x_1)| \leq \frac{2}{s-t}.$$

Multiplying (3.11) by $\eta^2 \mathsf{w}_1$ and using the integration by parts, one has

$$\int\limits_{\widehat{\Omega}_s(z)} \left(\mathbb{C}^0 e(\mathsf{w}_1), e(\eta^2 \mathsf{w}_1)\right) dx = -\int\limits_{\widehat{\Omega}_s(z)} \left(\mathbb{C}^0 e(\widetilde{\mathsf{u}}_1) - \mathcal{M}, \nabla(\eta^2 \mathsf{w}_1)\right) dx. \tag{3.14}$$

Using the same proof method as Proposition 2.1 in [15] to deal with (3.14), we get

$$\int\limits_{\widehat{\Omega}_t(z)} |\nabla \mathsf{w}_1|^2 \, dx \leq \frac{C}{(s-t)^2} \int\limits_{\widehat{\Omega}_s(z)} |\mathsf{w}_1|^2 \, dx + C \int\limits_{\widehat{\Omega}_s(z)} |\mathbb{C}^0 e(\widetilde{\mathsf{u}}_1) - \mathcal{M}|^2 \, dx. \tag{3.15}$$

Case 1. For $|z_1| \leq \varepsilon^{\frac{1}{1+\gamma}}$ and $0 < s < \varepsilon^{\frac{1}{1+\gamma}}$, we have $\varepsilon \leq \delta(z_1) \leq C\varepsilon$. By a direct calculation, we have

$$\int\limits_{\widehat{\Omega}_s(z)} |\mathsf{w}_1|^2 \, dx = \int\limits_{\widehat{\Omega}_s(z)} \left| \int\limits_{-\frac{\varepsilon}{2}+h_2(x_1)}^{x_2} \partial_2 \mathsf{w}_1(x_1, x_2) \, dx_2 \right|^2 dx \leq C\varepsilon^2 \int\limits_{\widehat{\Omega}_s(z)} |\nabla \mathsf{w}_1|^2 \, dx. \tag{3.16}$$

In view of Proposition 2.1 and (3.10), one has

$$\begin{aligned} \int\limits_{\widehat{\Omega}_s(z)} |\mathbb{C}^0 e(\widetilde{\mathsf{u}}_1) - \mathcal{M}|^2 \, dx \leq & C[\nabla \widetilde{\mathsf{u}}_1^{(1)}]^2_{\gamma, \widehat{\Omega}_s(z_1)} \int\limits_{\widehat{\Omega}_s(z_1)} (s^{2\gamma} + \delta(z_1)^{2\gamma}) \, dx \\ \leq & C \left(\left| \varphi^{(1)}(z_1, \frac{\varepsilon}{2} + h_1(z_1)) \right|^2 + s^2 \|\varphi^{(1)}\|^2_{C^{1,\gamma}(\Gamma_1^+)} \right) \\ & \left(\frac{s^3}{\varepsilon^{1+\frac{2}{1+\gamma}}} + \frac{s}{\varepsilon^{\frac{2}{1+\gamma}-1}} + \frac{s^{3-2\gamma}}{\varepsilon^{1+\frac{2}{1+\gamma}-2\gamma}} + \frac{s^{1+2\gamma}}{\varepsilon^{1+\frac{2\gamma^2}{1+\gamma}}} \right) =: G_1(s). \end{aligned} \tag{3.17}$$

Case 2. For $\varepsilon^{\frac{1}{1+\gamma}} \leq |z_1| \leq \frac{1}{2}$ and $0 < s < |z_1|$, we have $\frac{1}{C}|z_1|^{1+\gamma} \leq \delta(z_1) \leq C|z_1|^{1+\gamma}$. Estimates (3.16) and (3.17) become, respectively,

$$\int\limits_{\widehat{\Omega}_s(z)} |\mathsf{w}_1|^2 \, dx \leq C|z_1|^{2(1+\gamma)} \int\limits_{\widehat{\Omega}_s(z_1)} |\nabla \mathsf{w}_1|^2 \, dx, \quad \text{if } 0 < s < \frac{2}{3}|z'|, \tag{3.18}$$

and

$$\begin{aligned} \int\limits_{\widehat{\Omega}_s(z)} |\mathbb{C}^0 e(\widetilde{\mathsf{u}}_1) - \mathcal{M}|^2 \, dx \leq & C \left(\left|\varphi^{(\ell)}(z', \frac{\varepsilon}{2} + h_1(z'))\right|^2 + s^2 \left\|\varphi^{(\ell)}\right\|^2_{C^{1,\gamma}(\Gamma_1^+)} \right) \\ & \left(\frac{s^3}{|z_1|^{3+\gamma}} + \frac{s}{|z_1|^{-1-\gamma}} + \frac{s^{3-2\gamma}}{|z_1|^{1-\gamma-2\gamma^2}} + \frac{s^{2+2\gamma}}{|z_1|^{\gamma-1+2\gamma^2}} \right). \end{aligned} \tag{3.19}$$

Next, similar to the proof of Lemma 2.2, we complete the proofs of (3.12) and (3.13). □

Lemma 3.3. *Under the hypotheses of Lemma 2.3 and in addition that w_1 is the weak solution to (3.5), then there exists a positive constant C independent of ε such that, for $|z_1| \leq \varepsilon^{\frac{1}{1+\gamma}}$,*

$$\begin{aligned} |\nabla \mathsf{w}_1(z_1, z_2)| \leq & C\varepsilon^{-\frac{1}{1+\gamma}} \left| \varphi^{(1)}(z', \frac{\varepsilon}{2} + h_1(z_1)) - \psi^{(1)}(z_1, -\frac{\varepsilon}{2} + h_2(z_1)) \right| \\ & + C\varepsilon^{-\frac{\gamma}{1+\gamma}} \left(\|\mathsf{w}_1\|_{L^2(\Omega_1)} + \|\varphi^{(1)}\|_{C^{1,\gamma}(\Gamma_1^+)} + \|\psi^{(1)}\|_{C^{1,\gamma}(\Gamma_1^-)} \right), \end{aligned} \tag{3.20}$$

and for $\varepsilon^{\frac{1}{1+\gamma}} < |z_1| < \frac{1}{2}$,

$$
\begin{aligned}
|\nabla \mathsf{w}_1(z_1,z_2)| \leq & C|z_1|^{-1}\left|\varphi^{(1)}(z_1,\frac{\varepsilon}{2}+h_1(z')) - \psi^{(1)}(z_1,-\frac{\varepsilon}{2}+h_2(z_1))\right| \\
&+ C|z_1|^{-\gamma}\left(\|\mathsf{w}_1\|_{L^2(\Omega_1)} + \|\varphi^{(1)}\|_{C^{1,\gamma}(\Gamma_1^+)} + \|\psi^{(1)}\|_{C^{1,\gamma}(\Gamma_1^-)}\right).
\end{aligned} \tag{3.21}
$$

Proof. Let $\mathcal{Q}_1$ be as in (2.51) and C_{ijkl} be as in (1.16). For any $(y_1,y_2) \in \mathcal{Q}_1$, we denote

$$
\widetilde{\mathsf{w}}_1(y_1,y_2) := \mathsf{w}_1(\delta(z_1)y_1 + z_1, \delta(z_1)y_2), \quad \widehat{\mathsf{u}}_1(y_1,y_2) := \widetilde{\mathsf{u}}_1(\delta(z_1)y_1 + z_1, \delta(z_1)y_2),
$$

then, after the same coordinate transformation as in Lemma 2.3, we can obtain that $\widetilde{\mathsf{w}}_1$ satisfies

$$
\begin{cases}
-\displaystyle\sum_{j,k,l}\partial_j\big(C_{ijkl}\partial_l\widetilde{w}_1^{(k)}\big) = \sum_{j,k,l}\partial_j\big(C_{ijkl}\partial_l\widehat{u}_1^{(k)}\big) & \text{in } \mathcal{Q}_1, \\
\widetilde{\mathsf{w}}_1 = 0 & \text{on } \widehat{\Gamma}_1^{\pm}.
\end{cases} \tag{3.22}
$$

Similar to the proof in Lemma 2.3, recalling back to the original region $\widehat{\Omega}_{\delta(z_1)}(z)$, one has

$$
\|\nabla \mathsf{w}_1\|_{L^\infty(\widehat{\Omega}_{\delta(z_1)/4}(z))} \leq C\delta(z_1)^{-1}\|\nabla \mathsf{w}_1\|_{L^2(\widehat{\Omega}_{\delta(z_1)}(z_1))} + C\delta(z_1)^{\gamma}[\nabla\widetilde{\mathsf{u}}_1]_{\gamma,\widehat{\Omega}_{\delta(z_1)}(z)}). \tag{3.23}
$$

Therefore, by using (3.12), (3.13), and Proposition 2.1, we prove that (3.20) and (3.23) hold. □

Proof of Corollary 1.1. Consequently, by (2.10) and (2.11), we have for sufficiently small ε and $z \in \Omega_{1/2}$,

$$
\begin{aligned}
|\nabla \mathsf{v}_1(z)| \leq & \frac{C\left|\varphi^{(1)}(z_1,\frac{\varepsilon}{2}+h_1(z_1)) - \psi^{(1)}(z_1,-\frac{\varepsilon}{2}+h_2(z_1))\right|}{\varepsilon+|z_1|^{1+\gamma}} \\
&+ C\left(\|\mathsf{v}_1\|_{L^2(\Omega_1)} + \|\varphi^{(1)}\|_{C^{1,\gamma}(\Gamma_1^+)} + \|\psi^{(1)}\|_{C^{1,\gamma}(\Gamma_1^-)}\right).
\end{aligned} \tag{3.24}
$$

By (3.3), we have for $x \in \Omega_{1/2}$,

$$
\begin{aligned}
|\nabla \mathsf{u}(x)| \leq & |\nabla \mathsf{v}_1(x)| + |\nabla \mathsf{v}_2(x)| \\
\leq & \frac{C\left|\varphi(x_1,\varepsilon/2+h_1(x_1)) - \psi(x_1,-\varepsilon/2+h_2(x_1))\right|}{\varepsilon+|x_1|^{1+\gamma}} \\
&+ C\left(\|\varphi\|_{C^{1,\gamma}(\Gamma_1^+)} + \|\psi\|_{C^{1,\gamma}(\Gamma_1^-)} + \|\mathsf{u}\|_{L^2(\Omega_1)}\right).
\end{aligned}
$$

If $\varphi^{(\ell)}(0,\varepsilon/2) \neq \psi^{(\ell)}(0,-\varepsilon/2)$ for some integer ℓ, then by Lemma 2.3, we can obtain

$$
|\nabla \mathsf{u}(0,x_2)| \geq \frac{\left|\varphi^{(\ell)}(0,\varepsilon/2) - \psi^{(\ell)}(0,-\varepsilon/2)\right|}{C\varepsilon} \quad \forall\, x_2 \in (-\frac{\varepsilon}{2},\frac{\varepsilon}{2}).
$$

The proof of Corollary 1.1 is completed. □

4. Appendix. Proof of $C^{1,\gamma}$ estimates and $W^{1,p}$ estimates

In this section, we show the proofs of the Theorem 2.1 and Theorem 2.2, which play a key role in the proof of Theorem 1.1, with the help of the Campanato's approach, Schauder estimates and L^p estimates for elliptic systems in [19].

4.1. Proof of Theorem 2.1

To prove Theorem 2.1, we first introduce the definition of the spaces of Morrey and Campanato (see [19, Chapter 5]).

Let $Q \subset \mathbb{R}^n$ be any domain and $\rho > 0$, for any $x_0 \in Q$ we use the *symbol* $Q(x_0;\rho)$ to denote the set $Q \cap B_\rho(x_0)$ and the *symbol* $\dim Q$ to denote the diameter of Q. The domain Ω is said to be a *Lipschitz domain* if $\partial\Omega$ is Lipschitz defined as in Definition 1.2.

Definition 4.1. Let Q be a Lipschitz domain in $\mathbb{R}^n$. For every $1 \le p \le +\infty$, $\lambda > 0$, define the Morrey space

$$L^{p,\lambda}(Q) := \Big\{u \in L^p(Q) : \sup_{x_0\in Q,\ \rho>0} \rho^{-\lambda} \int\limits_{Q(x_0,\rho)} |u|^p\,dx < +\infty\Big\},$$

endowed with the norm defined by

$$\|u\|_{L^{p,\lambda}(Q)} := \Big(\sup_{x_0\in Q,\ \rho>0} \rho^{-\lambda} \int\limits_{Q(x_0,\rho)} |u|^p\,dx\Big)^{\frac{1}{p}}.$$

Definition 4.2. Let Q be a Lipschitz domain in $\mathbb{R}^n$. For every $1 \le p \le +\infty$, $\lambda > 0$, define the Campanato space

$$\mathcal{L}^{p,\lambda}(Q) := \{u \in L^p(Q) : \sup_{x_0\in Q,\rho>0} \rho^{-\lambda} \int\limits_{Q(x_0,\rho)} |u - u_{x_0,\rho}|^p\,dx < +\infty\},$$

endowed with the norm defined by

$$\begin{aligned}\|u\|_{\mathcal{L}^{p,\lambda}(Q)} &: = [u]_{p,\lambda} + \|u\|_{L^p}\\ &:=\Big(\sup_{x_0\in Q,\rho>0} \rho^{-\lambda} \int\limits_{Q(x_0,\rho)} |u - u_{x_0,\rho}|^p\,dx\Big)^{\frac{1}{p}} + \Big(\int\limits_{Q} |u|^p\,dx\Big)^{\frac{1}{p}} < +\infty, \end{aligned} \tag{4.1}$$

where $u_{x_0,\rho} := \frac{1}{|Q(x_0,\rho)|}\int_{Q(x_0,\rho)} u\,dx$.

The following lemma is just [19, Theorem 5.5].

Lemma 4.1. *For $n < \lambda \le n + p$ and $\gamma = \frac{\lambda-n}{p}$ we have $\mathcal{L}^{p,\lambda}(Q) = C^{0,\gamma}(\overline{Q})$. Moreover the Hölder semi-norm $[u]_{0,\gamma}$ as in (1.2) is equivalent to $[u]_{p,\lambda}$ as in (4.1). If $\lambda > n + p$ and $u \in \mathcal{L}^{p,\lambda}(\Omega)$, then u is a constant.*

Referring to [19, Theorem 5.14], we can obtain the following interior estimates. In what follows, for any domain $Q \subset \mathbb{R}^n$ we denote by the *symbol* $\mathcal{L}^{p,\lambda}_{loc}(Q)$ the set of all functions u which satisfies for any $Q' \subset\subset Q$, $\|u\|_{\mathcal{L}^{p,\lambda}(Q')} < \infty$.

Lemma 4.2. *Let Q be a Lipschitz domain in $\mathbb{R}^n$. Let $A^{\alpha\beta}_{ij}$ be constant and satisfy (1.10), (1.18). Let $0 < \gamma < 1$, $\mu := n + 2\gamma - 2$ and for any $\alpha = 1, \cdots, n$, $i = 1, \cdots, m$, $F^{(\alpha)}_i \in \mathcal{L}^{2,\mu+2}(Q)$ and $H^{(i)} \in L^{2,\mu}(Q)$. Let $\mathsf{w} = (w^{(1)}, \cdots, w^{(m)}) \in W^{1,2}(Q \subset \mathbb{R}^n; \mathbb{R}^m)$ be a weak solution to*

$$\sum_{\alpha,\beta,j} \partial_\alpha(A^{\alpha\beta}_{ij}\partial_\beta w^{(j)}) = H^{(i)} - \sum_\alpha \partial_\alpha F^{(\alpha)}_i, \quad \text{in } Q. \tag{4.2}$$

Then $\partial_\alpha w^{(i)} \in \mathcal{L}^{2,\mu}_{loc}(Q)$ *for any* $\alpha = 1, \cdots, n$ *and* $i = 1, \cdots, m$*, and there exists a positive constant* C *depending on* $n, m, \gamma, R, \lambda, \Lambda$ *such that, for* $B_R(x_0) \subset Q$*,*

$$[\nabla \mathsf{w}]_{\gamma, B_{R/2}} := \max_{\alpha,i} [\partial_\alpha w^{(i)}]_{\gamma, B_{R/2}} \leq C\Big(\frac{1}{R^{1+\gamma}} \|\mathsf{w}\|_{L^\infty(B_R)} + [\mathsf{F}]_{\gamma, B_R} + \|\mathsf{H}\|_{L^{2,\mu}(B_R)}\Big), \tag{4.3}$$

where $\|\mathsf{H}\|_{L^{2,\mu}(B_R)} := \max_i \|H^{(i)}\|_{L^{2,\mu}(B_R)}$.

Proof. By Proposition 4.1 we have $F_i^{(\alpha)} \in \mathcal{L}^{2,n+2\gamma}(Q)$. For a given ball $B_R := B_R(x_0) \subset Q$, the decomposition of w is as follows

$$\mathsf{w} = \mathsf{w}_1 + \mathsf{w}_2, \quad \text{in } B_R, \tag{4.4}$$

where w_1 and w_2 satisfy, respectively,

$$\begin{cases} \sum\limits_{\alpha,\beta,j} \partial_\alpha(A_{ij}^{\alpha\beta} \partial_\beta w_1^{(j)}) = 0, \text{ in } B_R, \\ \mathsf{w}_1 = \mathsf{w}, \qquad\qquad\qquad\quad \text{on } \partial B_R, \end{cases} \tag{4.5}$$

and

$$\begin{cases} \sum_{\alpha,\beta,j} \partial_\alpha(A_{ij}^{\alpha\beta} \partial_\beta w_2^{(j)}) = H^{(i)} - \sum_\alpha \partial_\alpha(F_i^{(\alpha)} - (F_i^{(\alpha)})_R), \text{ in } B_R, \\ \mathsf{w}_2 = 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{on } \partial B_R, \end{cases} \tag{4.6}$$

where $(F_i^{(\alpha)})_R = \frac{1}{|B_R|} \int_{B_R} F_i^\alpha \, dx$.

By [19, Proposition 5.8], for $0 < \rho < \frac{3R}{4}$ we have

$$\int\limits_{B_\rho} |\nabla \mathsf{w}_1 - (\nabla \mathsf{w}_1)_\rho|^2 \, dx \leq C \left(\frac{\rho}{R}\right)^{n+2} \int\limits_{B_{3R/4}} |\nabla \mathsf{w}_1 - (\nabla \mathsf{w}_1)_R|^2 \, dx, \tag{4.7}$$

and for w_2, multiplying (4.6) by w_2 and using the integration by parts, one has

$$\int\limits_{B_{3R/4}} |\nabla \mathsf{w}_2|^2 \, dx \leq C R^{\mu+2} \left([\mathsf{F}]_{\mathcal{L}^{2,\mu+2}(B_R)} + \|\mathsf{H}\|_{L^{2,\mu}(B_R)}\right). \tag{4.8}$$

Consequently,

$$\begin{aligned} \int\limits_{B_\rho} |\nabla \mathsf{w} - (\nabla \mathsf{w})_\rho|^2 \, dx \leq & \int\limits_{B_\rho} |\nabla \mathsf{w}_1 - (\nabla \mathsf{w}_1)_\rho + \nabla \mathsf{w}_2 - (\nabla \mathsf{w}_2)_\rho|^2 \, dx \\ \leq & C_1 \left(\frac{\rho}{R}\right)^n \int\limits_{B_{3R/4}} |\nabla \mathsf{w}_1 - (\nabla \mathsf{w}_1)_R|^2 \, dx + C_2 \int\limits_{B_{3R/4}} |\nabla \mathsf{w}_2 - (\nabla \mathsf{w}_2)_{3R/4}|^2 \, dx \\ \leq & C_1 \left(\frac{\rho}{R}\right)^n \int\limits_{B_{3R/4}} |\nabla \mathsf{w} - (\nabla \mathsf{w})_{3R/4}|^2 \, dx + C_2 \int\limits_{B_{3R/4}} |\nabla \mathsf{w}_2|^2 \, dx. \end{aligned} \tag{4.9}$$

Inserting (4.8) in (4.9) and using [19, Lemma 5.13], we obtain

$$\int_{B_\rho} |\nabla \mathsf{w} - (\nabla \mathsf{w})_\rho|^2 \, dx \le C\Big[\Big(\frac{\rho}{R}\Big)^{\mu+2} \int_{B_{3R/4}} |\nabla \mathsf{w}|^2 \, dx + \rho^{\mu+2} ([\mathsf{F}]^2_{\mathcal{L}^{2,\mu+2}(B_R)} + \|\mathsf{H}\|^2_{L^{2,\mu}(B_R)})\Big]. \tag{4.10}$$

We assert that the following inequality holds:

$$\int_{B_{3R/4}} |\nabla \mathsf{w}|^2 \, dx \le C\Big(\frac{1}{R^2} \int_{B_R} |\mathsf{w}|^2 \, dx + R^{\mu+2}\Big([\mathsf{F}]^2_{\mathcal{L}^{2,\mu+2}(B_R)} + \|\mathsf{H}\|^2_{L^{2,\mu}(B_R)}\Big)\Big), \tag{4.11}$$

where C depends on λ and boundedness of coefficients of (4.2).

Actually, define a cut-off function $\zeta \in C_c^\infty(Q)$ as follows, $0 \le \zeta(x) \le 1$,

$$\zeta(x) = \begin{cases} 1, & \text{on } B_{3R/4}, \\ 0, & \text{on } B_R \backslash B_{3R/4} \end{cases}, \quad |\nabla \zeta(x)| \le \frac{8}{R},$$

and substitute $\mathsf{w}\zeta^2$ as test function into (2.1). From strong ellipticity condition (1.10), we obtain

$$\begin{aligned} \lambda \int_{B_{3R/4}} \zeta^2 |\nabla \mathsf{w}|^2 \, dx \le & \sum_{\alpha,\beta,i,j} \int_{B_{3R/4}} \zeta^2 A^{\alpha\beta}_{ij} \partial_\beta w^{(j)} \partial_\alpha w^{(i)} \, dx \\ = & - \sum_{\alpha,\beta,i,j} \int_{B_{3R/4}} 2\zeta w^{(i)} A^{\alpha\beta}_{ij} \partial_\beta w^{(j)} \partial_\alpha \zeta \, dx - \sum_{\alpha,i,j} \int_{B_{3R/4}} B^{\alpha}_{ij} w^{(j)} \partial_\alpha \left(w^{(i)}\zeta^2\right) dx \\ & - \sum_{\beta,i,j} \int_{B_{3R/4}} w^{(i)} \zeta^2 (C^{\beta}_{ij} \partial_\beta w^{(j)} + D_{ij} w^{(j)}) \, dx + \sum_i \int_{B_{3R/4}} w^{(i)} \zeta^2 H^{(i)} \, dx \\ & + \sum_{\alpha,i} \int_{B_{3R/4}} (F^{(\alpha)}_i - (F^{(\alpha)}_i)_{3R/4}) \partial_\alpha \left(w^{(i)}\zeta^2\right) dx. \end{aligned}$$

Then by using Cauchy's inequality and the properties of ζ, we proved (4.11).

Therefore, using (4.10) and (4.11), we have

$$\begin{aligned} \frac{1}{\rho^{\mu+2}} \int_{B_\rho} |\nabla \mathsf{w} - (\nabla \mathsf{w})_\rho|^2 \, dx \le & C\Big(\frac{1}{R^{\mu+4}} \int_{B_R} |\mathsf{w}|^2 \, dx + [\mathsf{F}]^2_{\mathcal{L}^{2,\mu+2}(B_R)} + \|\mathsf{H}\|^2_{L^{2,\mu}(B_R)}\Big) \\ \le & C\Big(\frac{1}{R^{2+2\gamma}} \int_{B_R} |\mathsf{w}|^2 \, dx + [\mathsf{F}]^2_{\mathcal{L}^{2,\mu+2}(B_R)} + \|\mathsf{H}\|^2_{L^{2,\mu}(B_R)}\Big), \end{aligned}$$

where C depends on n, γ. For any $x = (x', x_n) \in B_{R/2}$ and $0 < \rho \le R/4$, we have

$$\begin{aligned} \frac{1}{\rho^{\mu+2}} \int_{B_\rho(x)\cap B_{R/2}} |\nabla \mathsf{w} - (\nabla \mathsf{w})_{B_\rho(x)\cap B_{R/2}}|^2 \, dy \le & \frac{1}{\rho^{\mu+2}} \int_{B_\rho} |\nabla \mathsf{w} - (\nabla \mathsf{w})_\rho|^2 \, dy \\ \le & C\Big(\frac{1}{R^{2+2\gamma}} \int_{B_R} |\mathsf{w}|^2 \, dx + [\mathsf{F}]^2_{\mathcal{L}^{2,\mu+2}(B_R)} + \|\mathsf{H}\|^2_{L^{2,\mu}(B_R)}\Big). \end{aligned}$$

By the equivalence between the Hölder space and the Campanato space (see Lemma 4.1), this implies that (4.3) holds. □

Next, we give the boundary estimate on half space $\partial\mathbb{R}^n_+$. Consider

$$\begin{cases} \sum\limits_{\alpha,\beta,j} \partial_\alpha(A^{\alpha\beta}_{ij}\partial_\beta w^{(j)}) = H^{(i)} - \sum\limits_{\alpha}\partial_\alpha F^{(\alpha)}_i, & \text{in } \mathbb{R}^n_+, \\ \mathsf{w} = 0, & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{4.12}$$

Corollary 4.1. *Under the hypothesis of Lemma 4.2, let* $\mathsf{w} \in W^{1,2}(\mathbb{R}^n_+;\mathbb{R}^m)$ *be the solution to* (4.12), *for any* $x_0 \in \partial\mathbb{R}^n_+$ *and* $\mathcal{B}^+_R(x_0) := B_R(x_0) \cap \partial\mathbb{R}^n_+$, *then there is a constant* C *only depending on* $n, \gamma, \lambda, \Lambda$ *such that,*

$$[\nabla \mathsf{w}]_{\gamma,\mathcal{B}^+_{R/2}(x_0)} \le C\Big(\frac{1}{R^{1+\gamma}}\|\mathsf{w}\|_{L^\infty(\mathcal{B}^+_R(x_0))} + [\mathsf{F}]_{\gamma,\mathcal{B}^+_R(x_0)} + \|\mathsf{H}\|_{L^{2,\mu}(\mathcal{B}^+_R(x_0))}\Big).$$

Proof. Firstly, we decompose $\mathsf{w} = \mathsf{w}_1 + \mathsf{w}_2$ as shown in (4.4), where w_1, w_2 satisfy (4.5) and (4.6) in $\mathcal{B}^+_R(x_0)$ respectively. It follows from [19, (5.38) in Theorem 5.21] that (4.7) holds for w_1. The proof of the corollary can be obtained by the method in the proof of the Lemma 4.2 and some elementary arguments. We omit the details. □

Proof of Theorem 2.1. Since Γ is $C^{1,\gamma}$, then for any $x_0 \in \Gamma$, there exist a neighborhood U of x_0 and a homeomorphism $\Psi \in C^{1,\gamma}(U)$ such that

$$\Psi(U \cap Q) = \mathcal{B}^+_1 := \{y \in B_1(0), y_n > 0\}, \qquad \Psi(U \cap \Gamma) = \partial\mathcal{B}^+_1 \cap \{y \in \mathbb{R}^n : y_n = 0\}.$$

Through the transformation $y = \Psi(x) = (\Psi^{(1)}(x), \cdots, \Psi^{(n)}(x))$, we denote

$$\mathcal{W}(y) := \mathsf{w}(\Psi^{-1}(y)), \quad \mathcal{J}(y) := \frac{\partial\left((\Psi^{-1})^{(1)}, \cdots, (\Psi^{-1})^{(n)}\right)}{\partial(y_1, \cdots, y_n)}, \quad |\mathcal{J}(y)| := \det \mathcal{J}(y),$$

and

$$\begin{aligned}
\mathcal{A}^{\alpha\beta}_{ij}(y) :&= \sum_{\hat\alpha,\hat\beta} |\mathcal{J}(y)| A^{\hat\alpha\hat\beta}_{ij}\left(\Psi^{-1}(y)\right)(\partial_{\hat\beta}(\Psi^{-1})^{(\beta)}(y))^{-1}\partial_{\hat\alpha}(\Psi)^{(\alpha)}(\Psi^{-1}(y)),\\
\mathcal{B}^{\alpha}_{ij}(y) :&= \sum_{\hat\alpha} |\mathcal{J}(y)| B^{\hat\alpha}_{ij}\left(\Psi^{-1}(y)\right)\partial_{\hat\alpha}(\Psi)^{(\alpha)}(\Psi^{-1}(y)),\\
\mathcal{C}^{\beta}_{ij}(y) :&= \sum_{\hat\beta} |\mathcal{J}(y)| C^{\hat\beta}_{ij}\left(\Psi^{-1}(y)\right)\partial_{\hat\beta}(\Psi)^{(\beta)}\left(\Psi^{-1}(y)\right), \quad \mathcal{D}_{ij}(y) := |\mathcal{J}(y)| D_{ij}\left(\Psi^{-1}(y)\right),\\
\mathcal{F}^{(\alpha)}_{i}(y) :&= \sum_{\hat\alpha} |\mathcal{J}(y)| F^{\hat\alpha}_{i}\left(\Psi^{-1}(y)\right)\partial_{\hat\alpha}(\Psi)^{(\alpha)}\left(\Psi^{-1}(y)\right), \quad \mathcal{H}^{(i)}(y) := |\mathcal{J}(y)| H^{(i)}\left(\Psi^{-1}(y)\right),
\end{aligned}$$

where $\hat\alpha, \hat\beta = 1, \cdots, n$. Then (2.1) becomes

$$\sum_{\alpha,\beta,i,j} \partial_\alpha(\mathcal{A}^{\alpha\beta}_{ij}\partial_\beta \mathcal{W}^{(j)} + \mathcal{B}^{\alpha}_{ij}\mathcal{W}^{(j)}) + \mathcal{C}^{\beta}_{ij}\partial_\beta\mathcal{W}^{(j)} + \mathcal{D}_{ij}\mathcal{W}^{(j)} = \mathcal{H}^{(i)} - \sum_{\alpha}\partial_\alpha \mathcal{F}^{(\alpha)}_i \quad \text{in } \mathcal{B}^+_R,$$

and $\mathcal{W} = 0$ on $\partial\mathcal{B}^+_R \cap \partial\mathbb{R}^n_+$. Let $y_0 = \Psi(x_0)$. Freezing coefficients and rewriting the above formula into the following form,

$$\begin{aligned}
\sum_{\alpha,\beta,j} \partial_\alpha(\mathcal{A}^{\alpha\beta}_{ij}(y_0)\partial_\beta\mathcal{W}^{(j)}(y)) = &\sum_{\alpha,\beta,j} -\partial_\alpha((\mathcal{A}^{\alpha\beta}_{ij}(y) - \mathcal{A}^{\alpha\beta}_{ij}(y_0))\partial_\beta\mathcal{W}^{(j)}(y) + \mathcal{B}^{\alpha}_{ij}(y)\mathcal{W}^{(j)}(y))\\
&+ \sum_{\alpha,\beta,j}(\mathcal{C}^{\beta}_{ij}(y)\partial_\beta\mathcal{W}^{(j)}(y) - \mathcal{D}_{ij}(y)\mathcal{W}^{(j)}(y) + \mathcal{H}^{(i)}(y) - \partial_\alpha\mathcal{F}^{(\alpha)}_i(y)).
\end{aligned}$$

Then, by Corollary 4.1 we have that for $0 < R < 1$,

$$\begin{aligned}\left[\nabla \mathcal{W}\right]_{\gamma,\mathcal{B}^+_{R/2}} \leq & C\Big(\frac{1}{R^{1+\gamma}}\|\mathcal{W}\|_{L^\infty(\mathcal{B}^+_R)} + [\mathcal{F}]_{\gamma,\mathcal{B}^+_R}\Big)\\ & + C\sum_{\beta,j}\Big(\left[(\mathcal{A}^{\alpha\beta}_{ij}(y) - \mathcal{A}^{\alpha\beta}_{ij}(y_0))\partial_\beta \mathcal{W}^{(j)}\right]_{\gamma,\mathcal{B}^+_R} + \left[\mathcal{B}^\alpha_{ij}(y)\mathcal{W}^{(j)}\right]_{\gamma,\mathcal{B}^+_R}\Big)\\ & + C\Big(\sum_{\beta,j}\left\|\mathcal{C}^\beta_{ij}(y)\partial_\beta \mathcal{W}^{(j)} - \mathcal{D}_{ij}(y)\mathcal{W}^{(j)}\right\|_{L^{2,\mu}(B^+_R)} + \|\mathcal{H}\|_{L^{2,\mu}(B^+_R)}\Big).\end{aligned}$$

Since $\mathcal{A}^{\alpha\beta}_{ij}(y), \mathcal{B}^\alpha_{ij}(y), \mathcal{C}^\beta_{ij}(y), \mathcal{D}_{ij}(y) \in C^\gamma(\mathcal{B}^+_R)$, we have

$$\sum_{\beta,j}\left[(\mathcal{A}^{\alpha\beta}_{ij}(y) - \mathcal{A}^{\alpha\beta}_{ij}(y_0))\partial_\beta \mathcal{W}^{(j)}\right]_{\gamma,\mathcal{B}^+_R} \leq C\left(R^\gamma[\nabla \mathcal{W}]_{\gamma,\mathcal{B}^+_R} + \|\nabla \mathcal{W}\|_{L^\infty(\mathcal{B}^+_R)}\right),$$

$$\sum_j \left[\mathcal{B}^\alpha_{ij}(y)\mathcal{W}^{(j)}\right]_{\gamma,\mathcal{B}^+_R} \leq C R^\gamma \|\mathcal{W}\|_{L^\infty(\mathcal{B}^+_R)},$$

and

$$\sum_{\beta,j}\left\|\mathcal{C}^\beta_{ij}(y)\partial_\beta \mathcal{W}^{(j)} - \mathcal{D}_{ij}(y)\mathcal{W}^{(j)}\right\|_{L^{2,\mu}(\mathcal{B}^+_R)} \leq C\|\nabla \mathcal{W}\|_{L^\infty(\mathcal{B}^+_R)} + \|\mathcal{W}\|_{L^\infty(\mathcal{B}^+_R)}.$$

In view of the interpolation inequality ([20, Lemma 6.35]), we can obtain

$$\|\nabla \mathcal{W}\|_{L^\infty(\mathcal{B}^+_R)} \leq R^\gamma[\nabla \mathcal{W}]_{\gamma,\mathcal{B}^+_R} + \frac{C}{R}\|\mathcal{W}\|_{L^\infty(\mathcal{B}^+_R)},$$

where $C = C(n)$. Hence,

$$\left[\nabla \mathcal{W}\right]_{\gamma,\mathcal{B}^+_{R/2}} \leq C(\frac{1}{R^{1+\gamma}}\|\mathcal{W}\|_{L^\infty(\mathcal{B}^+_R)} + R^\gamma[\nabla \mathcal{W}]_{\gamma,\mathcal{B}^+_R} + [\mathcal{F}]_{\gamma,\mathcal{B}^+_R} + \|\mathcal{H}\|_{L^{2,\mu}(B^+_R)}).$$

Since Ψ is a homeomorphism, thus, changing back to the variable x, we obtain

$$\left[\nabla \mathsf{w}\right]_{\gamma,\mathcal{N}'} \leq C\Big(\frac{1}{R^{1+\gamma}}\|\widetilde{\mathsf{w}}\|_{L^\infty(\mathcal{N})} + R^\gamma[\nabla \mathsf{w}]_{\gamma,\mathcal{N}} + [\mathsf{F}]_{\gamma,\mathcal{N}} + \|\widetilde{\mathsf{H}}\|_{L^{2,\mu}(\mathcal{N})}\Big),$$

where $\mathcal{N} = \Psi^{-1}(\mathcal{B}^+_R)$, $\mathcal{N}' = \Psi^{-1}(\mathcal{B}^+_{R/2})$ and $C = C(n,\gamma,\Psi)$. Furthermore, there exists a constant $0 < \sigma < 1$, independent on R, such that $B_{\sigma R}(x_0) \cap Q \subset \mathcal{N}'$.

Therefore, recalling that $\Gamma \subset \partial Q$ is a boundary portion, for any domain $Q' \subset\subset Q \cup \Gamma$ and for each $x_0 \in Q' \cap \Gamma$, there exist $\mathcal{R}_0 := \mathcal{R}_0(x_0)$ and $C_0 = C_0(n,\gamma,x_0)$ such that,

$$\left[\nabla \mathsf{w}\right]_{\gamma,B_{\mathcal{R}_0}(x_0)\cap Q'} \leq C_0\Big(\mathcal{R}_0^\gamma[\nabla \mathsf{w}]_{\gamma,Q'} + \frac{1}{\mathcal{R}_0^{1+\gamma}}\|\mathsf{w}\|_{L^\infty(Q)} + [\mathsf{F}]_{\gamma,Q} + \|\mathsf{H}\|_{L^{2,\mu}(Q)}\Big). \tag{4.13}$$

By using the boundary estimates (4.13) near Γ, the finite covering theorem, and Lemma 4.2, we can obtain

$$\left[\nabla \mathsf{w}\right]_{\gamma,Q'} \leq C\left(\|\mathsf{w}\|_{L^\infty(Q)} + [\mathsf{F}]_{\gamma,Q} + \|\mathsf{H}\|_{L^{2,\mu}(Q)}\right),$$

where $C = C(n,\gamma,Q',Q)$. The proof details can be referred to the proof of [15, Theorem 2.3]. By using the interpolation inequality ([20, Lemma 6.35]), we obtain (2.2). $\square$

4.2. Proof of Theorem 2.2

In this subsection, we give the proof of $W^{1,p}$ estimates of the weak solution to the system as in Definition 2.1.

Proof of Theorem 2.2. First, we give the $W^{1,p}$ interior estimates. For any ball $B_{3R/4} := B_{3R/4}(x_0) \subset Q$ with $R \leq 1$, since $\mathsf{w} \neq 0$ on $\partial B_{3R/4}$, we can choose a cut-off function $\eta \in C_0^\infty(B_{3R/4})$ such that for $0 < \rho < 3R/4$,

$$0 \leq \eta \leq 1 \quad \eta = 1 \quad \text{in} \quad B_\rho, \quad |\nabla \eta| \leq \frac{C}{R-\rho}.$$

It is easy to see that

$$\begin{aligned}
&\sum_{\alpha,\beta,i,j} \int_{B_{3R/4}} A_{ij}^{\alpha\beta}(x_0)\partial_\beta(\eta w^{(j)})\partial_\alpha \phi^{(i)}\, dx \\
&= \sum_{\alpha,\beta,i,j} \int_{B_{3R/4}} \left(A_{ij}^{\alpha\beta}(x_0) - A_{ij}^{\alpha\beta}(x)\right)\partial_\beta(\eta w^{(j)})\partial_\alpha \phi^{(i)}\, dx \\
&\quad + \sum_{\alpha,i}\Big(\int_{B_{3R/4}} T^{(i)}\phi^{(i)}\, dx + \int_{B_{3R/4}} K_i^{(\alpha)}\partial_\alpha \phi^{(i)}\Big), \quad \forall\, \phi \in C_0^\infty(B_{3R/4};\mathbb{R}^m),
\end{aligned}$$

where

$$\begin{aligned}
T^{(i)} &= -\sum_{\alpha,\beta,j}\Big(\big(A_{ij}^{\alpha\beta}(x)\partial_\beta w^{(j)} + B_{ij}^{\alpha}(x)w^{(j)}\big)\partial_\alpha\eta - C_{ij}^{\beta}(x)\big(2w^{(j)}\partial_\beta\eta + \eta\partial_\beta w^{(j)}\big)\Big) \\
&\qquad - \sum_j D_{ij}(x)(\eta w^{(j)}) + H^{(i)}\eta + \sum_\alpha (F_i^{(\alpha)} - (F_i^{(\alpha)})_{3R/4})\partial_\alpha\eta, \\
K_i^{(\alpha)} &= \sum_{\beta,j}\left(A_{ij}^{\alpha\beta}(x)w^{(j)}\partial_\beta\eta - B_{ij}^{\alpha}(x)w^{(j)}\eta + (F_i^{(\alpha)} - (F_i^{(\alpha)})_{3R/4})\eta\right).
\end{aligned}$$

Let $\mathsf{v} = (v^{(1)}, \cdots, v^{(m)}) \in H_0^1(B_{3R/4};\mathbb{R}^m)$ be the weak solution to

$$-\Delta v^{(i)} = T^{(i)}. \tag{4.14}$$

Thus, we can obtain that $\eta\mathsf{w}$ satisfies

$$\begin{aligned}
&\sum_{\alpha,\beta,i,j} \int_{B_{3R/4}} A_{ij}^{\alpha\beta}(x_0)\partial_\beta(\eta w^{(j)})\partial_\alpha \phi^{(i)}\, dx \\
&\qquad = \sum_{\alpha,\beta,i,j} \int_{B_{3R/4}} \left((A_{ij}^{\alpha\beta}(x_0) - A_{ij}^{\alpha\beta}(x))\partial_\beta(\eta w^{(j)}) + \mathcal{K}_i^{(\alpha)}\right)\partial_\alpha \phi^{(i)}\, dx,
\end{aligned}$$

where $\mathcal{K}_i^{(\alpha)} := K_i^{(\alpha)} + \partial_\alpha v^{(i)}$.

Since $F_i^{(\alpha)} \in C^\gamma(B_{3R/4})$, then $(F_i^{(\alpha)} - (F_i^{(\alpha)})_{3R/4})) \in L^p(B_{3R/4})$ for any $n \leq p < \infty$. We firstly assume that $\mathsf{w} \in W^{1,q}(B_{3R/4};\mathbb{R}^n)$, $q \geq 2$. Then, combining with Sobolev embedding theorem and the boundedness of coefficients, $H^{(i)} \in L^p(B_{3R/4})$, we can get

$$K_i^{(\alpha)} \in L^{\min(p,q^*)}(B_{3R/4}) \quad \text{and} \quad T^{(i)} \in L^{\min(p,q)}(B_{3R/4}),$$

where $q^* := \frac{nq}{n-q}$ is the Sobolev conjugate of q. To write simply, we use $a \wedge b$ to represent $\min(a, b)$. By (4.14),

$$-\Delta(\partial_\alpha v^{(i)}) = \partial_\alpha T^{(i)}, \quad \text{for} \quad \alpha = 1, 2, \cdots, n.$$

The [19, Theorem 7.1] guarantees that $\nabla(\partial_\alpha v^{(i)}) \in L^{p\wedge q}(B_{3R/4})$ and

$$\|\nabla(\partial_\alpha v^{(i)})\|_{L^{p\wedge q}(B_{3R/4})} \le C\|T^{(i)}\|_{L^{p\wedge q}(B_{3R/4})},$$

where C depends on n, λ, p, q. Combining with the Sobolev embedding theorem, we have

$$\partial_\alpha v^{(i)} \in L^{(p\wedge q)^*}(B_{3R/4}).$$

It follows from $p \wedge q^* \le (p \wedge q)^*$ that $K_i^{(\alpha)} \in L^{p\wedge q^*}(B_{3R/4})$. Let $s := p \wedge q^*$ and define $T : W_0^{1,s}(B_{3R/4}) \to W_0^{1,s}(B_{3R/4})$ as follows,

$$T(\mathsf{V}) = \mathsf{v}, \quad \text{for any } \mathsf{V} \in W_0^{1,s}(B_{3R/4}),$$

where $v \in W_0^{1,s}(B_{3R/4})$ is the solution to the following elliptic system:

$$\sum_{\alpha,\beta,i,j} \int_{B_{3R/4}} A_{ij}^{\alpha\beta}(x_0)\partial_\beta v^{(j)}\partial_\alpha \phi^{(i)}\, dx = \sum_{\alpha,\beta,i,j} \int_{B_{3R/4}} \left((A_{ij}^{\alpha\beta}(x_0) - A_{ij}^{\alpha\beta}(x))\partial_\beta V^{(j)} + K_i^{(\alpha)}\right)\partial_\alpha \phi^{(i)}\, dx.$$

By using [19, Theorem 7.1] again, we have

$$\|\nabla \mathsf{v}\|_{L^s(B_{3R/4})} \le C\|[\mathsf{A}(x_0) - \mathsf{A}(x)]\nabla \mathsf{V}\|_{L^s(B_{3R/4})} + C\|\mathsf{K}\|_{L^s(B_{3R/4})}, \tag{4.15}$$

where $\mathsf{A}(x)\nabla\mathsf{V}$ represent matrix $(A_{ij}^{\alpha\beta}(x)\partial_\beta V^{(j)})$, and C depends on $n, \lambda, \|\mathsf{A}\|_{C^\gamma(Q)}, p, q$.

When R is sufficiently small, it is proved by Poincaré inequality and (4.15) that T is a contractive mapping and $\eta\mathsf{w}$ is the only fixed point. See [19, Theorem 7.2] for details. By (4.15), we have

$$\|\nabla(\eta\mathsf{w})\|_{L^s(B_R)} \le C\|[\mathsf{A}(x_0) - \mathsf{A}(x)]\nabla(\eta\mathsf{w})\|_{L^s(B_R)} + C\|\mathsf{K}\|_{L^s(B_R)}.$$

Therefore, for sufficiently small R, we can obtain

$$\begin{aligned}
\|\nabla(\eta\mathsf{w})\|_{L^{p\wedge q^*}(B_{3R/4})} &\le C\|\mathsf{K}\|_{L^{p\wedge q^*}(B_{3R/4})} \\
&\le \|\mathsf{T}\|_{L^{p\wedge q}(B_{3R/4})} + \|\mathsf{K}\|_{L^{p\wedge q^*}(B_{3R/4})} \\
&\le \frac{C}{R-\rho}\left([\mathsf{F}]_{\gamma,B_{3R/4}} + \|\mathsf{H}\|_{L^\infty(B_{3R/4})} + \|\mathsf{w}\|_{W^{1,q}(B_{3R/4})}\right),
\end{aligned}$$

where C depends on $n, \lambda, \|\mathsf{A}\|_{C^\gamma(Q)}, p, q$. Thus

$$\|\nabla\mathsf{u}\|_{L^{p\wedge q^*}(B_\rho)} \le \frac{C}{R-\rho}\left([\mathsf{F}]_{\gamma,B_{3R/4}} + \|\mathsf{H}\|_{L^\infty(B_{3R/4})} + \|\mathsf{w}\|_{W^{1,q}(B_{3R/4})}\right). \tag{4.16}$$

Next, we prove that $\nabla\mathsf{w} \in L^p(B_{R/2})$. Similar to the proof of [15, Theorem 2.4], choose a series of balls with radii

$$\frac{R}{2} < \cdots < R_k < \cdots < R_2 < R_1 < \frac{3R}{4}.$$

First, let $\rho = R_1, q = 2$ in (4.16), then

$$\|\nabla \mathsf{u}\|_{L^{p\wedge 2^*}(B_{R_1})} \le \frac{C}{R-R_1}\left([\mathsf{F}]_{\gamma,B_{3R/4}} + \|\mathsf{H}\|_{L^\infty(B_{3R/4})} + \|\mathsf{w}\|_{W^{1,2}(B_{3R/4})}\right).$$

If $p \le 2^*$, it can be obtained by interpolation inequality ([1, Theorem 5.8]) that

$$\|\mathsf{w}\|_{L^p(B_{R_1})} \le C\|\mathsf{w}\|^{\theta}_{W^{1,2}(B_{R_1})}\|\mathsf{w}\|^{1-\theta}_{L^2(B_{R_1})} \le C\|\mathsf{w}\|_{W^{1,2}(B_{R_1})},$$

where $\theta = n/2 - n/p$ with $2 \le p \le 2^*$. Combining with (4.16), the proof is completed. If $p > 2^*$, then $\nabla \mathsf{w} \in L^{2^*}(B_{R_1})$ and

$$\|\nabla \mathsf{w}\|_{L^{2^*}(B_{R_1})} \le \frac{C}{R-R_1}\left([\mathsf{F}]_{\gamma,B_{3R/4}} + \|\mathsf{H}\|_{L^\infty(B_{3R/4})} + \|\mathsf{w}\|_{W^{1,2}(B_{3R/4})}\right). \tag{4.17}$$

By taking $R = R_1$, $\rho = R_2$, $q = 2^*$ in (4.16), and combining with (4.17), one has

$$\|\nabla \mathsf{w}\|_{L^{p\wedge 2^{**}}(B_{R_2})} \le \frac{C}{(R-R_1)(R_1-R_2)}\left([\mathsf{F}]_{\gamma,B_{3R/4}} + \|\mathsf{H}\|_{L^\infty(B_{3R/4})} + +\|\mathsf{w}\|_{W^{1,2}(B_{3R/4})}\right).$$

Similarly, if $p \le 2^{**}$, using the above formula and interpolation inequality ([1, Theorem 5.8]), we have completed the proof of the theorem.

If $p > 2^{**}$, continuing the above argument within finite steps, with the help of interpolation inequality, we obtain

$$\|\mathsf{w}\|_{W^{1,p}(B_{R/2})} \le C\left([\mathsf{F}]_{\gamma,B_{3R/4}} + \|\mathsf{H}\|_{L^\infty(B_{3R/4})} + \|\mathsf{w}\|_{W^{1,2}(B_{3R/4})}\right), \tag{4.18}$$

where C depends on $n, \lambda, p, \|\mathsf{A}\|_{C^\gamma(Q)}, \mathrm{dist}(B_R, \partial Q))$. Similar to the proof of (4.11), we can obtain

$$\int_{B_{3R/4}} |\nabla \mathsf{w}|^2\, dx \le C\left(\|\mathsf{w}\|^2_{L^2(B_R)} + [\mathsf{F}]^2_{\gamma,B_R} + \|\mathsf{H}\|^2_{L^{2,\mu}(B_R)}\right).$$

This, combining with (4.18), we have

$$\|\mathsf{w}\|_{W^{1,p}(B_{R/2})} \le C\left(\|\mathsf{w}\|_{L^2(B_R)} + [\mathsf{F}]_{\gamma,B_R} + \|\mathsf{H}\|_{L^\infty(B_R)}\right). \tag{4.19}$$

Now, we prove the $W^{1,p}$ estimates near boundary Γ by using the technology of locally flattening the boundary, which is the same to the proof in Theorem 2.1. For simplicity, we use the same notation. Hence, we have that $\mathcal{W}(y) := \mathsf{w}(\Psi^{-1}(y)) \in W^{1,2}(\mathcal{B}^+_R \subset \mathbb{R}^n, \mathbb{R}^m)$ satisfies

$$\sum_{\alpha,\beta,i,j}\int_{\mathcal{B}^+_R}\left(\mathcal{A}^{\alpha\beta}_{ij}\partial_\beta \mathcal{W}^{(j)} + \mathcal{B}^{\alpha}_{ij}\mathcal{W}^{(j)}\right)\partial_\alpha\phi^{(i)} + \mathcal{C}^{\beta}_{ij}\partial_\beta\mathcal{W}^{(j)}\phi^{(i)} + \mathcal{D}_{ij}\mathcal{W}^{(j)}\phi^{(i)}\, dy$$

$$= \sum_{\alpha,i}\int_{\mathcal{B}^+_R}\mathcal{H}^{(i)}\phi^{(i)} + \mathcal{F}^{(\alpha)}_i\partial_\alpha\phi^{(i)}\, dy,$$

for any $\phi \in W^{1,2}_0(\mathcal{B}^+_R, \mathbb{R}^m)$. In this special case, we can obtain the boundary estimate of the upper half space by using the above method of proving the interior estimate (4.19), thus, for $n \le p < \infty$ we have

$$\|\mathcal{W}\|_{W^{1,p}(\mathcal{B}^+_{R/2})} \le C\left(\|\mathcal{W}\|_{L^2(\mathcal{B}^+_R)} + [\mathcal{F}]_{\gamma,\mathcal{B}^+_R} + \|\mathcal{H}\|_{L^\infty(\mathcal{B}^+_R)}\right),$$

where C depends on $\lambda, \Lambda, p, R, \Psi$. Then, changing back to the original variable x, we obtain

$$\|\mathsf{w}\|_{W^{1,p}(\mathcal{B}^+_{R/2})} \leq C\left(\|\mathsf{w}\|_{L^2(\mathcal{B}^+_R)} + [\mathsf{F}]_{\gamma,\mathcal{B}^+_R} + \|\mathsf{H}\|_{L^\infty(\mathcal{B}^+_R)}\right),$$

where $\mathcal{N}' = \Psi^{-1}(\mathcal{B}^+_{R/2})$, $\mathcal{N} = \Psi^{-1}(\mathcal{B}^+_R)$ and $C = C(\lambda, \mu, p, R, \Psi)$. Furthermore, there exists a constant $0 < \sigma < 1$, independent on R, such that $B_{\sigma R}(x_0) \cap Q \subset \mathcal{N}'$. Therefore, for any $x_0 \in Q' \cap \Gamma$, there exists $R_0 := R_0(x_0) > 0$ such that,

$$\|\nabla\mathsf{w}\|_{W^{1,p}(B_{\sigma R_0}(x_0)\cap Q')} \leq C(\|\mathsf{w}\|_{L^2(Q)} + [\mathsf{F}]_{\gamma,Q} + \|\mathsf{H}\|_{L^\infty(Q)}), \tag{4.20}$$

where C depends on $\lambda, \Lambda, p, x_0, R$. Combining (4.19) and (4.20), and making use of the finite covering theorem, we have completed the proof of Theorem 2.2. Refer to the proof of [15, Theorem 2.4] for more details. □

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