中国石油大学（北京）

2025 年申报硕士研究生指导教师评审材料目录

申报人姓名	胡张楠	申报学科专业	数学

序号	送审材料名称
1	代表性论文全文
2	论文检索证明

Received: 9 May 2024 | Revised: 8 May 2025 | Accepted: 27 August 2025

DOI: 10.1112/jlms.70287

RESEARCH ARTICLE

Journal of the London Mathematical Society

On shrinking targets for linear expanding and hyperbolic toral endomorphisms

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Funding information
Science Foundation of China University of Petroleum, Beijing, Grant/Award Number: 2462023SZBH013; China Postdoctoral Science Foundation, Grant/Award Number: 2023M743878; Postdoctoral Fellowship Program of CPSF, Grant/Award Number: GZB20240848; NSFC, Grant/Award Numbers: 12001245, 12161141002, 12271432; Natural Science Foundation of Jiangsu Province, Grant/Award Number: BK20201025; Guangdong Basic and Applied Basic Research Foundation, Grant/Award Number: 2024A1515010974

Abstract

Let A be an invertible $d \times d$ matrix with integer elements. Then A determines a self-map T of the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Given a real number $\tau > 0$, and a sequence $\{z_n\}$ of points in $\mathbb{T}^d$, let W_τ be the set of points $x \in \mathbb{T}^d$ such that $T^n(x) \in B(z_n, e^{-n\tau})$ for infinitely many $n \in \mathbb{N}$. The Hausdorff dimension of W_τ has previously been studied by Hill–Velani and Li–Liao–Velani–Zorin. We provide a lower bound on the Hausdorff dimension of W_τ for any expanding matrix. For hyperbolic matrices, we compute the dimension of W_τ only when A is a 2×2 matrix. We give counterexamples to a natural candidate for a dimension formula for general dimension d.

MSC 2020
37C45, 37E05 (Primary)

1 INTRODUCTION

Let $T : X \to X$ be a measure-preserving transformation on a metric space X which is equipped with an ergodic Borel probability measure m. For any fixed subset $B \subset X$ of positive measure, Birkhoff's ergodic theorem implies that

$$\{x \in X : T^n x \in B \text{ for infinitely many } n \in \mathbb{N}\}$$

has m-measure 1. Hill and Velani [13] considered this set when $B = B(n)$ is a ball that shrinks with time n. They called the points in the set the *well-approximable* points in analogy with the classical theory of metric Diophantine approximation [6, 22], in particular the Jarník–Besicovitch theorem [3, 16], and introduced the so-called *shrinking target problems*: if at time n, one has a ball $B(n) = B(x_0, r_n)$ centred at x_0 with radius $r_n \to 0$, then what kind of properties does the set of points x have, whose images $T^n(x)$ are in $B(n)$ for infinitely many n?

There are plenty of related works such as the so-called quantitative recurrence properties [4], dynamical Borel–Cantelli lemma [7], shrinking target problems [11, 23], uniform Diophantine approximation [5], recurrence time [1], waiting time [12] and so on. We refer to the survey article by Wang and Wu [26] for more information.

Let A be a $d \times d$ matrix with integral coefficients, let X be the d dimensional torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ and let $T : \mathbb{T}^d \to \mathbb{T}^d$ be a transformation of $\mathbb{T}^d$ defined by

$$T(x) = Ax \mod 1.$$

Given $\tau > 0$ and a sequence $\{z_n : n \in \mathbb{N}\}$ of points in the d dimensional torus $\mathbb{T}^d$, we consider the set

$$\begin{aligned} W_\tau &= \{x \in \mathbb{T}^d : T^n(x) \in B(z_n, e^{-n\tau}) \text{ for infinitely many } n \in \mathbb{N}\} \\ &= \limsup_{n \in \mathbb{N}} T^{-n} B(z_n, e^{-n\tau}). \end{aligned}$$

Let μ denote the d-dimensional normalised Lebesgue measure on $\mathbb{T}^d$. In this paper, the radii of the balls decay exponentially as $e^{-\tau n}$. Li, Liao, Velani and Zorin [17] also studied other decay rates and they gave conditions when the Lebesgue measure of the corresponding set is 0 or 1. We will only study the exponential decay rate, and we note that by the Borel–Cantelli lemma, it follows immediately that for any $\tau > 0$, we have $\mu(W_\tau) = 0$.

Since $\mu(W_\tau) = 0$, it is natural to calculate the Hausdorff dimension of the set W_τ. This is the problem that we will deal with in this paper. It has previously been studied by Hill and Velani [14] and by Li, Liao, Velani and Zorin [17, Theorem 6 and 8]. Our results are similar to some of those in the mentioned papers, but our assumptions are somewhat different, and we also prove a large intersection property of the set W_τ. The main difficulty in the problem of determining the Hausdorff dimension of W_τ is that W_τ is a limsup set of increasingly eccentric ellipsoids.

Li, Liao, Velani and Zorin used mass transference to obtain their results. We use the geometry of the involved sets to obtain estimates on measures which leads to our results through results on Riesz energies.

Our results about hyperbolic endomorphisms are only for $d = 2$, but for expanding endomorphism, our results hold for any $d \geqslant 2$. We also point out an error in Theorem 1 of Hill and Velani [14], see Example 2.6 below.

2 | RESULTS

Our results will involve the eigenvalues of the matrix A. Throughout this text, we denote by $\lambda_1, \dots, \lambda_d$ the eigenvalues of A counted with multiplicity and ordered so that

$$0 < |\lambda_1| \leqslant |\lambda_2| \leqslant \dots \leqslant |\lambda_d|.$$

Hence, we will always assume that A is invertible. Put $l_j = \log|\lambda_j|$, $j = 1, 2, \dots, d$.

With this notation, we have the following theorems.

Theorem 2.1. *Let A be a 2×2 integer matrix with eigenvalues $0 < |\lambda_1| < 1 < |\lambda_2|$. Then*

$$\dim_{\mathrm{H}} W_\tau = s_\tau = \begin{cases} \dfrac{2l_2}{\tau + l_2}, & 0 < \tau < -l_1, \\ \min\left\{ \dfrac{l_1 + l_2}{\tau + l_1}, \dfrac{2l_2}{\tau + l_2} \right\}, & -l_1 < \tau. \end{cases}$$

Moreover, W_τ has large intersection property in the sense that $W_\tau \in \mathscr{G}^{s_\tau}$, provided $s_\tau > 0$. (See Section 3 for the definition of $\mathscr{G}^{s_\tau}$).

We remark that the dimension formula above differs from the one for recurrence in Hu and Persson [15]. In one-dimensional cases, the dimension formulae are often the same for hitting and recurrence, see [17, 24].

Remark 2.2. In Theorem 2.1, for the critical case $\tau = -l_1$, the dimension of W_τ is simpler to describe when $|\det A| > 1$. When $|\det A| > 1$, we have $\dim_{\mathrm{H}} W_{-l_1} = \frac{2l_2}{l_2 - l_1}$, and the dimension of W_τ is continuous as a function of τ. We refer to the proof of Theorem 2.1.

When $|\det A| = 1$ and λ is an eigenvalue with $|\lambda| > 1$, we have by Theorem 2.1 that

$$\dim_{\mathrm{H}} W_\tau = \begin{cases} \dfrac{2\log|\lambda|}{\tau + \log|\lambda|}, & 0 < \tau < \log|\lambda|, \\ 0, & \log|\lambda| < \tau. \end{cases}$$

It is interesting (but natural) that the dimension formula above is not continuous as a function of τ. Figure 1 shows the graph of $\dim_{\mathrm{H}} W_\tau$ as a function of τ when $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

The dimension of W_τ when $\tau = \log|\lambda|$ depends on the choice of z_n. It is possible to choose z_n so that W_τ is a line segment, and hence $\dim_{\mathrm{H}} W_\tau = 1$, and it is possible to choose z_n so that $W_\tau = \emptyset$ and $\dim_{\mathrm{H}} W_\tau = 0$.

Remark 2.3. With the following adjustments, our results are valid for general non-increasing sequences $\{r_n\}_{n \geqslant 1}$ of positive real numbers instead of $\{e^{-n\tau}\}_{n \geqslant 1}$. Put

$$\tau = \liminf_{n \to \infty} \frac{-\log r_n}{n}.$$

With this definition of τ in mind, Theorems 2.1, 2.4 and 2.5 hold for such sequence $\{r_n\}_{n \geqslant 1}$.

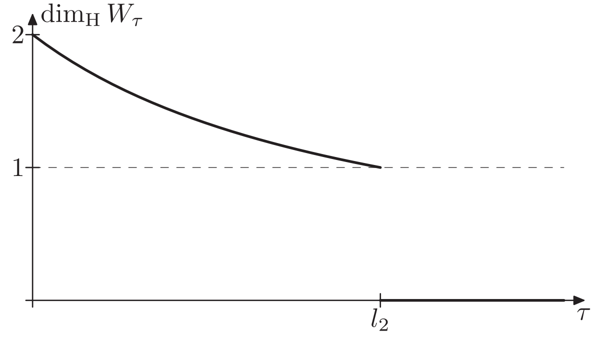

FIGURE 1 The graph of $\tau \mapsto \dim_{\mathrm{H}} W_\tau$ for the cat map, that is, when $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

We now turn to the case of a torus of general dimension d. We prove an upper bound on the dimension of W_τ.

Theorem 2.4. *If A is an invertible $d \times d$ integer matrix, then for any $\tau \geqslant 0$, we have*

$$\dim_{\mathrm{H}} W_\tau \leqslant \min_k \left\{ \frac{k l_k + \sum_{j>k} l_j}{\tau + l_k} \right\}, \tag{1}$$

where the minimum is over those k for which $\tau + l_k > 0$, and the minimum of the empty set should be interpreted as d.

Hill and Velani [14, Theorem 1 and 2] stated that under some assumptions, there is equality in (1). As we will show in Examples 2.6 and 7.2, those statements are incorrect for hyperbolic matrices. These examples also show that the lower bound mentioned immediately after Hill's and Velani's Theorem 2 is not always correct. However, the following theorem shows that their statements hold for expanding toral endomorphisms.

We further state a lower bound on the dimension for expanding endomorphisms. Without the assumption that all eigenvalues are outside the unit circle, the result may fail, as we will explain after the theorem.

Theorem 2.5. *Let A be a $d \times d$ integer matrix. Assume that $l_i > 0$ for $1 \leqslant i \leqslant d$. Then,*

$$\dim_{\mathrm{H}} W_\tau \geqslant \tilde{s}_\tau := \min_{1 \leqslant k \leqslant d} \left\{ \frac{k l_k + \sum_{j>k} l_j - \sum_{j=1}^{d} (l_j - l_k - \tau)_+}{\tau + l_k} \right\},$$

and $(x)_+ = \max\{0, x\}$. Moreover, $W_\tau \in \mathscr{G}^{\tilde{s}_\tau}$.

Li, Liao, Velani and Zorin [17, Theorem 6 and 8] obtained dimension formulae under some conditions on the matrix A. They assumed that either A is an integer matrix, diagonalisable over $\mathbb{Z}$, with all eigenvalues outside the unit circle, or that A is diagonal, not necessarily an integer matrix, and that all eigenvalues are outside the unit circle. The novelty in Theorem 2.5 is that we do not need any assumption on diagonalisability over $\mathbb{Z}$, but on the other hand, we only treat integer matrices, and we only obtain estimates on the dimension. Note that for most values of τ, the lower bound provided by Theorem 2.5 differs from the upper bound provided by Theorem 2.4.

One could possibly expect that the lower bound of Theorem 2.5 holds also without any assumptions on the eigenvalues. Although this might be the case in certain cases, it is not always so. To explain what can go wrong, we define a probability measure μ_n supported on $E_n = T^{-n}B(z_n, e^{-n\tau})$ by

$$\mu_n = c_d e^{nd\tau} \mu|_{E_n},$$

where c_d is a constant that depends only on d. In words, μ_n is the normalised restriction of the Lebesgue measure to the nth inverse image of $B(z_n, e^{-\tau n})$.

It turns out that when μ_n does not converge weakly to the Lebesgue measure, then the lower bound in Theorem 2.5 can fail. In the proof of Theorem 2.1, we will see (and use) that μ_n converges weakly to the Lebesgue measure, unless in the case when $\dim_{\mathrm{H}} W_\tau = 0$. Hence, in the case of Theorem 2.1, we do not need the assumption that μ_n converges weakly to the Lebesgue measure, but it seems that such an assumption is needed in the general case.

If A has an eigenvalue on the unit circle, then it is sometimes possible to, given any $\tau > 0$, choose z_n such that $W_\tau = \emptyset$, and μ_n will not converge to the Lebesgue measure. The following example shows a particular instance of this. In Section 7, there are further examples of things that can go wrong if the assumptions are not fulfilled. In particular, the examples in Section 7 show that it is not enough to assume that there are no eigenvalues on the unit circle, in order for the bound in Theorem 2.5 to hold.

Example 2.6. Assume that b is an integer, and define the 2×2 matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}.$$

If $z_n = (x_n, y_n)$ with $x_n \in \mathbb{T}$ and $y_n \in \mathbb{T}$, then

$$T^{-n}B(z_n, e^{-n\tau}) \subset [x_n - e^{-n\tau}, x_n + e^{-n\tau}] \times \mathbb{T}.$$

From this, it is clear that we can choose x_n such that

$$\limsup_{n\in\mathbb{N}} [x_n - e^{-n\tau}, x_n + e^{-n\tau}] = \emptyset.$$

We then also have

$$W_\tau \subset \limsup_{n\in\mathbb{N}} ([x_n - e^{-n\tau}, x_n + e^{-n\tau}] \times \mathbb{T}) = \emptyset.$$

The dimension formula in [14] is not quite correct in this and similar cases, such as when

$$A = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix},$$

and B is any $(d-1) \times (d-1)$ integer matrix. In particular, the main result of Hill and Velani [14, Theorem 1] is not correct as stated, see also Example 7.2.

The paper is organised as follows. Section 3 is devoted to preliminaries, where we recall the definition of the Hausdorff dimension and give a critical lemma. In this paper, we deal with two kinds of endomorphisms, hyperbolic ones with both expansion and contraction and purely expanding ones. Theorem 2.1 is for hyperbolic endomorphisms, Theorem 2.4 is for both hyperbolic and expanding endomorphisms and Theorem 2.5 is for expanding endomorphisms. In Section 4, we give the proofs of Theorem 2.4 and the upper bound in Theorem 2.5. The strategy of dealing with the lower bound for these two kinds of endomorphisms is different, so the proofs are divided into two sections. Section 5 contains the proof of Theorem 2.1, and the proof of Theorem 2.5 is given in Section 6. In the last section, we give counterexamples to a natural candidate for a dimension formula for general dimension d.

Our method of proof is the following. We give upper bounds on the Hausdorff dimension by direct covering arguments. The lower bounds are obtained by estimates of the type $\mu_n(B(x,r)) \leqslant Cr^s$, where, as above, μ_n is the normalised restriction of the Lebesgue measure to the set $T^{-n}(B(z_n, e^{-\tau n}))$. According to Lemma 3.2 below, such estimates lead to a lower bound on the Hausdorff dimension, provided that μ_n converges weakly to the Lebesgue measure on $\mathbb{T}^d$.

3 | HAUSDORFF MEASURE AND DIMENSION

Limsup sets often possess a large intersection property, see [9, 10, 19]. This means that the set belongs to a particular class $\mathscr{G}^s$ of G_δ sets that, among other properties, is closed under countable intersections and consists of sets of Hausdorff dimension at least s.

To get an upper bound on the Hausdorff dimension of a set is frequently easier than obtaining a lower bound, at least if the set is a limsup set. The mass transference principle [2] is often used to get a lower bound of the Hausdorff dimension of a set. In this paper, we will use the following special case of a lemma from Persson and Reeve [21]. See also Persson [19]. It is a slight variation of a lemma in [19].

Let μ denote the d-dimensional Lebesgue measure on $\mathbb{T}^d$. We recall that the s-dimensional Riesz energy of the measure μ is defined as

$$I_s(\mu) = \iint |x-y|^{-s}\, \mathrm{d}\mu(x)\mathrm{d}\mu(y).$$

Lemma 3.1 ([20, Lemma 2.1]). *Let E_n be open sets in $\mathbb{T}^d$ and let μ_n be probability measures with $\mu_n(\mathbb{T}^d \setminus E_n) = 0$. If there is a constant $C > 1$ such that*

$$C^{-1} \leqslant \liminf_{n\to\infty} \frac{\mu_n(B)}{\mu(B)} \leqslant \limsup_{n\to\infty} \frac{\mu_n(B)}{\mu(B)} \leqslant C \tag{2}$$

for any ball B, and

$$\iint |x-y|^{-s} \mathrm{d}\mu_n(x)\mathrm{d}\mu_n(y) < C$$

for all n, then $\limsup_{n\to\infty} E_n \in \mathscr{G}^s$, *and, in particular, we have*

$$\dim_{\mathrm{H}}\left(\limsup_{n\to\infty} E_n\right) \geqslant s.$$

Sometimes, the assumption that the s-dimensional Riesz energy of μ_n is uniformly bounded is not easy to check. We observe that this assumption can be replaced by a stronger but sometimes more manageable one as the following lemma shows, which is a slight variation of Lemma 3.1.

Lemma 3.2. *Let E_n be open sets in $\mathbb{T}^d$ and let μ_n be probability measures with $\mu_n(\mathbb{T}^d \setminus E_n) = 0$. Suppose that there are constants C and s such that*

$$C^{-1} \leqslant \liminf_{n\to\infty} \frac{\mu_n(B)}{\mu(B)} \leqslant \limsup_{n\to\infty} \frac{\mu_n(B)}{\mu(B)} \leqslant C \tag{3}$$

for any ball B and $\mu_n(B) \leqslant Cr^s$ for all n and any ball B of radius r. Then $\limsup_{n\to\infty} E_n \in \mathscr{G}^s$.

Proof. Let $t < s$. Using the estimate $\mu_n(B) \leqslant Cr^s$, we can write

$$\begin{aligned}\int |x-y|^{-t}\mathrm{d}\mu_n(y) &= 1 + \int_1^\infty \mu_n\big(B(x, u^{-1/t})\big)\mathrm{d}u \leqslant 1 + 2C\int_1^\infty u^{-s/t}\mathrm{d}u \\ &= 1 + 2C\frac{t}{s-t}.\end{aligned}$$

Therefore, $\iint |x-y|^{-t}\mathrm{d}\mu_n(x)\mathrm{d}\mu_n(y) \leqslant 1 + 2Ct/(s-t)$, and by Lemma 3.1, we can conclude that $\limsup_{n\to\infty} E_n$ belongs to the intersection class with dimension t. Since t can be taken as close to s as we like, it follows that $\limsup_{n\to\infty} E_n \in \mathscr{G}^s$. $\square$

4 | THE UPPER BOUND

We start this section with several elementary observations concerning $T^{-n}(B(z_n, e^{-\tau n}))$.

In the proofs, the singular values of A^n will be of importance. The following easy lemma tells us that the singular values and eigenvalues of A^n are comparable.

Lemma 4.1. *Let $\sigma_{n,1} \leqslant \ldots \leqslant \sigma_{n,d}$ be the singular values of A^n. For any $\varepsilon > 0$, there exists a constant $c_0 > 0$ such that*

$$c_0^{-1}e^{-\varepsilon n} \leqslant \frac{e^{l_k n}}{\sigma_{n,k}} \leqslant c_0 e^{\varepsilon n}.$$

for all k and n.

We shall also need the following lemma that we compile from the book by Everest and Ward [8].

Lemma 4.2 ([8, Lemma 2.2 and 2.3]). *If A is an invertible $d \times d$ integer matrix with no eigenvalue being a root of unity, and $T(x) = Ax \pmod 1$, then*

$$\#\{\, x \in \mathbb{T}^d \,:\, A^n x \pmod 1 = 0 \,\} = |\det A|^n.$$

Put $L := \sum_{i=1}^d l_i$, then $e^{nL} = |\det A|^n$, which is an integer. Let $\tau > 0$. We let k be the smallest number such that $\tau + l_k > 0$. By Lemma 4.2, the set $T^{-n}(B(z_n, e^{-\tau n}))$ consists of e^{nL} ellipsoids with semi-axes $r_1 \geqslant r_2 \geqslant \ldots \geqslant r_d > 0$, where $r_i = e^{-(\tau n + \sigma_{n,i})}$, $1 \leqslant i \leqslant d$. It follows from Lemma 4.1 that for $n \geqslant 1$ and $1 \leqslant j \leqslant d$,

$$c_0^{-1} e^{-(l_k+\tau+\varepsilon)n} < r_j \leqslant c_0 e^{-(l_k+\tau-\varepsilon)n}.$$

By the choice of k, we have that $r_k, \ldots, r_d$ are exponentially small (in n), whereas r_j for $j \leqslant k-1$ are exponentially large. From now on, we shall simplify notation by assuming that $c_0 = 1$.

We are now ready to prove Theorem 2.4.

Proof of Theorem 2.4. To obtain an upper bound on the Hausdorff dimension of W_τ, we look for covers of the set W_τ. Recall that

$$W_\tau = \limsup_{n \in \mathbb{N}} T^{-n} B(z_n, e^{-n\tau}).$$

Let $\varepsilon > 0$. For all large n, the set $T^{-n}B(z_n, e^{-n\tau})$ consists of e^{Ln} ellipsoids with semi-axes not more than $e^{-(l_j+\tau-\varepsilon)n}$. Fix an integer k such that $1 \leqslant k \leqslant d$, we may cover the set $T^{-n}B(z_n, e^{-n\tau})$ by balls of radius $r = e^{-(l_k+\tau-\varepsilon)n}$. We assume that k is such that $l_k + \tau - \varepsilon > 0$, and hence also that the radius goes to zero as n goes to infinity.

To cover each ellipsoid, we need about

$$\prod_{j<k} \frac{e^{-(l_j+\tau-\varepsilon)n}}{r} = \prod_{j<k} \frac{e^{-(l_j+\tau-\varepsilon)n}}{e^{-(l_k+\tau-\varepsilon)n}} = e^{n((k-1)l_k - \sum_{j<k} l_j)}$$

such balls. Hence, we need in total not more than about

$$e^{n(L+(k-1)l_k - \sum_{j<k} l_j)}$$

balls of radius $e^{-(l_k+\tau-\varepsilon)n}$ to cover the set $T^{-n}B(z_n, e^{-n\tau})$.

If $l_k + \tau > 0$, then the radius $e^{-(l_k+\tau)n}$ goes to 0 as $n \to \infty$ and the cover mentioned above can be used to get an upper bound on the Hausdorff dimension of W_τ. Hence, by letting $\varepsilon \to 0$, we get that

$$\dim_{\mathrm{H}} W_\tau \leqslant \frac{k l_k + \sum_{j>k} l_j}{\tau + l_k},$$

provided that $\tau + l_k > 0$. This proves the upper bound on the Hausdorff dimension of W_τ. □

5 | THE PROOF OF THEOREM 2.1

First, let us give some notation. We write $f_n \lesssim g_n$, $n \in \mathbb{N}$, if there is an absolute constant $0 < c < \infty$ such that $f_n \leqslant c g_n$ for large n. If $f_n \lesssim g_n$ and $g_n \lesssim f_n$, then we write $f_n \asymp g_n$.

For the lower bound in our theorems, we use Lemma 3.2. In this section, let μ denote the two-dimensional normalised Lebesgue measure on $\mathbb{T}^2$, and we define the probability measure

μ_n supported on $E_n = T^{-n}B(z_n, e^{-n\tau})$ by

$$\mu_n = \pi^{-1}e^{n2\tau}\mu|_{E_n}.$$

The following lemma is essential in estimating $\dim_{\mathrm{H}} W_\tau$ when A is a hyperbolic 2×2 integer matrix.

Lemma 5.1. *Let A be a 2×2 integer matrix with an eigenvalue $0 < |\lambda| < 1$. Then, both eigenvalues of A are irrational.*

Proof. Suppose $D = \det A$ and $\lambda = p/q$, where p/q is a reduced fraction. Then, the other eigenvalue must be qD/p since the product of the eigenvalues is D. The trace of the matrix is also an integer and is the sum of the eigenvalues. Hence,

$$p/q + Dq/p = (p^2 + Dq^2)/(pq)$$

is an integer. Therefore, $(p^2 + Dq^2)$ is divisible by q. But then q divides p^2 which is impossible, since p/q is a reduced fraction. □

Let θ be a real irrational number, and let $\theta - [\theta] = \{\theta\}$ be the fractional part of θ, where $[\theta]$ is the greatest integer not greater than θ. The distribution of $(\{\theta n\})_{n\geqslant 1}$ is a crucial ingredient in showing $(\mu_n)_n$ satisfy inequalities (3).

The set $\{\,\{\theta n\} : 1 \leqslant n \leqslant N\,\}$ partitions the interval $[0,1]$ into $N+1$ intervals. The three distance theorem states that these intervals have at most three distinct lengths, then we denote $d_\theta(N)$ and $d'_\theta(N)$, respectively, the maximum and minimum lengths. The following theorem describes the relationships between $d_\theta(N)$ and $d'_\theta(N)$.

Theorem 5.2 (Theorem 2 in [18]). *Under the setting given above, the sequence $(\frac{d_\theta(N)}{d'_\theta(N)})_{N=1}^{\infty}$ is bounded if and only if θ is of constant type (defined as irrationals with bounded partial quotients).*

5.1 | Proof of theorem 2.1

Without loss of generality, we assume that $\tau > 0$. Recall that $l_1 = \log|\lambda_1|$, and $l_2 = \log|\lambda_2|$, then $l_1 < 0 < l_2$. The assumptions in Theorem 2.1 imply that A is diagonalisable and we do not need to use the ε in Lemma 4.1 as in the proof of Theorem 2.4 above, and for simplicity, we assume that $c_0 = 1$ in Lemma 4.1. By Lemmata 4.1 and 4.2, the set $T^{-n}(B(z_n, e^{-\tau n}))$ consists of e^{Ln} ellipses with semi-axes of length about $e^{-(\tau+l_1)n}$ and $e^{-(\tau+l_2)n}$, where $L = l_1 + l_2 \geqslant 0$. These ellipses are denoted by $\{R_n^i\}_{i=1}^{e^{nL}}$.

The strategy of the proof is the following. We are going to use Lemma 3.2, and therefore, we are going to prove that $(\mu_n)_{n=1}^{\infty}$ satisfies the inequalities (3) and that there are constants C and s such that $\mu_n(B) \leqslant Cr^s$, with $s = s_\tau$, as defined in Theorem 2.1. In the case $s_\tau = 0$, there is nothing to prove and it is therefore enough to consider the case $s_\tau > 0$.

Lemma 5.3. *Assume that $s_\tau > 0$. If $\tau \neq -l_1$ or $|\det A| > 1$, then the measures μ_n satisfy inequalities (3).*

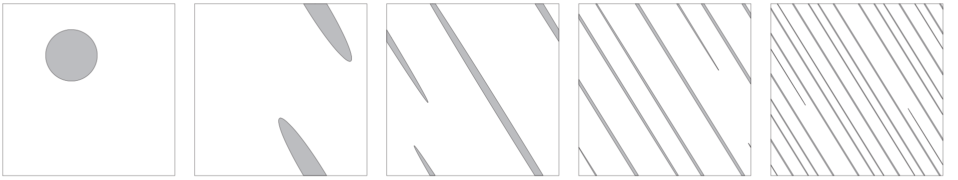

FIGURE 2 Artist's illustration of the structure of $T^{-n}(B(z_n, e^{-\tau n}))$, for $n = 0, 1, 2, 3, 4$ and $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

Proof. If $\tau < -l_1$, then $T^{-n}(B(z_n, e^{-\tau n}))$ consists of one or several disjoint long ellipses that wrap around the torus, see Figure 2. Because of Lemma 5.1, the longer semi-axis has an irrational direction, denoted by α and we will show that μ_n converges weakly to the Lebesgue measure on the torus.

Since α is a quadratic irrational number, α is of constant type. For $N \geqslant 1$, it follows from Theorem 5.2 that $\frac{d_\alpha(N)}{d'_\alpha(N)}$ is uniformly bounded, then

$$d_\alpha(N) \asymp d'_\alpha(N) \asymp N^{-1}.$$

Viewing $\mathbb{T}$ as $[0, 1)^d$, each ellipse R_n^i consists of many pieces as in Figure 2. We deduce from the above estimates that these pieces are separated by $\asymp e^{n(\tau+l_1)}$ for large enough n. Let $B = B(x, r)$. For large n, each ellipse R_n^i can intersect B and because of the separation, at most about $r/e^{n(\tau+l_1)}$ pieces can intersect B. Therefore,

$$\mu(B \cap R_n^i) \asymp r\frac{r}{e^{n(\tau+l_1)}}e^{-n(\tau+l_2)} = r^2 e^{-2n\tau} e^{-n(l_1+l_2)},$$

and we get that

$$\begin{aligned}\mu_n(B) &= \pi^{-1}e^{2n\tau}\sum_i \mu(B \cap R_n^i) = \pi^{-1}e^{2n\tau}e^{Ln}\mu(B \cap R_n^i) \\ &\asymp e^{2n\tau}e^{Ln}r^2 e^{-2n\tau}e^{-n(l_1+l_2)} = r^2\end{aligned}$$

hold for large enough n.

If $\tau > -l_1$, then there are two cases, either $|\det A| = 1$ or $|\det A| > 1$. If $|\det A| = 1$, then $A^{-n}B(z_n, e^{-n\tau})$ shrinks as $n \to \infty$. It implies that S_τ consists of a single point, or it is empty, which gives the Hausdorff dimension 0. This means that $s_\tau = 0$, but we assumed that $s_\tau > 0$, and hence, there is nothing to prove in this case.

If $|\det A| > 1$, then $T^{-n}(B(z_n, e^{-\tau n}))$ consists of e^{Ln} exponentially small ellipses $\{R_n^i\}_i$. Given $B = B(x, r)$, note that for large enough n,

$$\begin{aligned}\mu_n(B) &= \sum_{i\,:\, B\cap R_n^i \neq \varnothing} \pi^{-1}e^{2n\tau}\mu(B \cap R_n^i) \\ &\asymp e^{2n\tau}\mu(R_n^i)\#\{i : B \cap R_{n,i} \neq \varnothing\} \\ &\asymp e^{-n(l_1+l_2)}\#\{B \cap T^{-n}z_n\} \asymp e^{-n(l_1+l_2)}\#\{A^n B \cap \mathbb{Z}^2\}.\end{aligned} \tag{4}$$

Now we estimate $\#\{A^nB \cap \mathbb{Z}^2\}$. The set A^nB is an ellipsoid with centre (x_1, x_2), and semi-axes re^{nl_1} and $re^{l_2 n}$. The direction of the longer semi-axis of A^nB is irrational, denoted by θ. If $(z_1, z_2) \in \mathbb{Z}^2 \cap A^nB$, then

$$\begin{cases} |z_1 - x_1| < \frac{1}{\sqrt{1+\theta^2}} re^{l_2 n}, \\ |z_2 - x_2 - \theta(z_1 - x_1)| < \sqrt{1+\theta^2} re^{l_1 n}. \end{cases} \tag{5}$$

If $(z_1, z_2) \in \mathbb{Z}^2$ satisfies

$$\begin{cases} |z_1 - x_1| < \frac{1}{4\sqrt{1+\theta^2}} re^{l_2 n}, \\ |z_2 - x_2 - \theta(z_1 - x_1)| < \frac{3\sqrt{1+\theta^2}}{4} re^{l_1 n}, \end{cases} \tag{6}$$

then $(z_1, z_2) \in \mathbb{Z}^2 \cap A^nB$.

Notice that $|z_2 - x_2 - \theta(z_1 - x_1)| = \{\theta(z_1 - x_1) + x_2\}$. Therefore,

$$\begin{aligned} \#\{A^nB \cap \mathbb{Z}^2\} &\leqslant \#\{(z_1, z_2) \in \mathbb{Z}^2 : (z_1, z_2) \text{ satisfies (5)}\} \\ &\asymp \#\left\{ 1 \leqslant k \leqslant \frac{2}{\sqrt{1+\theta^2}} re^{l_2 n} : \{\theta k\} \in I_n \right\}, \end{aligned} \tag{7}$$

where I_n is an interval of length $2\sqrt{1+\theta^2} re^{l_1 n}$. Since θ is of constant type, then given $N \geqslant 1$, it follows from Theorem 5.2 that $\frac{d_\theta(N)}{d'_\theta(N)}$ is uniformly bounded, which implies that

$$d_\theta(N) \asymp d'_\theta(N) \asymp N^{-1}.$$

From what is written above, we see that

$$\begin{aligned} \#\{1 \leqslant k \leqslant \frac{2}{\sqrt{1+\theta^2}} re^{l_2 n} : \{\theta k\} \in I_n\} &\asymp \frac{|I_n|}{d_\theta\left(\frac{2}{\sqrt{1+\theta^2}} re^{l_2 n}\right)} \\ &\asymp re^{nl_1} re^{l_2 n}. \end{aligned} \tag{8}$$

By (7) and (8), we get

$$\#\{A^nB \cap \mathbb{Z}^2\} \lesssim re^{nl_1} re^{l_2 n}.$$

Similarly, we deduce from (6) that

$$\#\{A^nB \cap \mathbb{Z}^2\} \gtrsim re^{nl_1} re^{l_2 n}.$$

Recalling (4), we conclude that for any ball B,

$$\mu_n(B) \asymp r^2$$

holds for large enough n.

We now consider the case $\tau = -l_1$ and $|\det A| > 1$. In this case, $T^{-n}(B(z_n, e^{-\tau n}))$ consists of e^{nL} disjoint long ellipses transversal in B. Therefore,

$$\begin{aligned}\mu_n(B) &\asymp e^{2n\tau} \sum_{i\,:\, B\cap R_n^i \neq \emptyset} \mu(B\cap R_n^i) \\ &\asymp e^{2n\tau}\#\{i : B\cap R_n^i \neq \emptyset\} r e^{-n(\tau+l_2)}. \qquad (9)\end{aligned}$$

Let $R(p,q)$ be a rectangle with the same centre as B, the length in the stable direction is p, and q in the unstable direction. If R_n^i intersects B, then the centre of R_n^i is contained in a rectangle $R := R(p,q)$, where $p := 2(r+1)$ and $q := 2(r + e^{-n(\tau+l_2)})$. Recall that the centres of $\{R_n^i\}$ are $T^{-n}z_n$. Then, we obtain

$$\#\{i : B\cap R_n^i \neq \emptyset\} \leqslant \#\{T^{-n}z_n \cap R\} \lesssim \#\{\mathbb{Z}^2 \cap A^n R\}.$$

Using a similar argument in estimating $\#\{\mathbb{Z}^2 \cap A^n B\}$, we get

$$\#\{\mathbb{Z}^2 \cap A^n R\} \lesssim \{1 \leqslant k \leqslant 2e^{l_2 n} : \{\theta k\} \in \tilde{I}_n\},$$

where $\tilde{I}_n$ is a segment with $|\tilde{I}_n| = 2e^{l_1 n}(r+1)$. Then,

$$\#\{\mathbb{Z}^2 \cap A^n R\} \lesssim e^{l_2 n} e^{l_1 n} = e^{l_2 n} e^{-n\tau}.$$

On the contrary, if the centre of R_n^i is contained in rectangle $R(\frac{1}{4}, \frac{3}{4}r)$, then R_n^i intersects B. Then,

$$\#\{i : B\cap R_n^i \neq \emptyset\} \geqslant \#\left\{\mathbb{Z}^2 \cap A^n R\left(\tfrac{1}{4}, \tfrac{3}{4}r\right)\right\} \gtrsim e^{l_2 n} e^{l_1 n} = e^{l_2 n} e^{-n\tau}.$$

Combining this with (9), we have $\mu_n(B) \asymp e^{2n\tau} r e^{-n(\tau+l_2)} e^{l_2 n} e^{-n\tau} = r^2$. $\square$

Lemma 5.4. *There is a constant C such that*

$$\mu_n(B(x,r)) \leqslant \begin{cases} C r^{\frac{2l_2}{\tau+l_2}} & \textit{if } 0 < \tau < -l_1 \\ C r^{\min\{\frac{l_1+l_2}{\tau+l_1}, \frac{2l_2}{\tau+l_2}\}} & \textit{if } \tau \geqslant -l_1, \end{cases}$$

for all n, x and r.

Proof. Pick a point $x \in \mathbb{T}^2$ and $r > 0$. We want to give an estimate of the μ_n-measure of the ball $B(x,r)$ for $n \geqslant 1$. There are three cases to consider: $\tau < -l_1$, $-l_1 < \tau < \frac{l_2-l_1}{2}$ and $\frac{l_2-l_1}{2} \leqslant \tau$.

We consider three cases, depending on the size of τ.

- Case $\tau < -l_1$.

  In this case, we have $l_1 + \tau < 0 < \tau + l_2$. Then, on the torus, each ellipse in $T^{-n}(B(z_n, e^{-\tau n}))$ is very long, and hence wraps around the torus in a complicated way, see Figure 2. To get a good estimate on $\mu_n(B(x,r))$, we need to investigate how the strips are distributed on the torus. If they are too concentrated, then $\mu_n(B(x,r))$ can be very large.

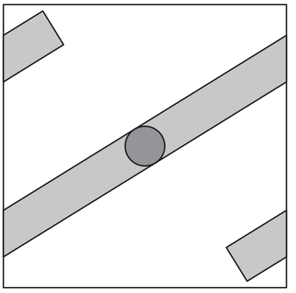

FIGURE 3 Illustration of the rectangle which does not overlap itself.

Firstly we estimate the distance between different parts of one ellipse. Consider a line in the unstable direction going out from the point $z_n = (z_{n,1}, z_{n,2})$. The equation of this line can be written as $y - z_{n,2} = \alpha(x - z_{n,1})$, and by Lemma 5.1, the number α is an algebraic number of degree 2. This line will wrap around the torus and come close to the point z_n.

Let p be an integer, and consider $(x, y) \in \mathbb{T}^2$ with $x = q + z_{n,1}$ and $y = z_{n,2} + \alpha(x - z_{n,1}) = z_{n,2} + \alpha q$. Suppose that the point (x, y) is very close to z_n, that is, there is an integer p such that $y = z_{n,2} + p \pm r$ with $0 < r < \frac{2}{\sqrt{1+\alpha^2}} e^{-\tau n}$. We then have $|\alpha q - p| = r$, or equivalently $|\alpha - \frac{p}{q}| = \frac{r}{q}$. Liouville's theorem on Diophantine approximation implies that there exists a constant $c_\alpha > 0$ which only depends on A such that $|\alpha - \frac{p}{q}| > \frac{c_\alpha}{q^2}$ and hence $\frac{r}{q} > \frac{c_\alpha}{q^2}$. We then get $q > c_\alpha r^{-1} > c_\alpha \frac{\sqrt{1+\alpha^2}}{2} e^{\tau n}$.

This implies that for $c < c_\alpha \frac{\sqrt{1+\alpha^2}}{2}$, the rectangle R around $B(z_n, e^{-\tau n})$ with side length $2e^{-\tau n}$ in the stable direction and $ce^{\tau n}$ in the unstable direction does not overlap itself. More precisely if, R is viewed as a subset of $\mathbb{R}^2$ and (a, b) is a non-zero integer vector, then $R \cap (R + (a, b)) = \emptyset$. Therefore, when R is projected to $\mathbb{T}^2$, the projection is injective on R, as illustrated in Figure 3. Hence, any ellipse in $T^{-n}B(z_n, e^{-\tau n})$ does not overlap itself and the ellipse has a side length $ce^{(\tau - l_2)n}$ in the unstable direction, which means that the separation of the strips in such ellipse is at least $ce^{(\tau - l_2)n}$, for some $c > 0$.

Consider again the rectangle R around $B(z_n, e^{-\tau n})$ with side lengths $2e^{-\tau n}$ in the stable direction and $ce^{\tau n}$ in the unstable direction. As we have seen above, the rectangle R does not overlap itself, which means that in $\mathbb{R}^2$, any two different translations of R by integer vectors are disjoint. The pre-image $T^{-n}(R)$ consists of e^{Ln} pieces, but they must all be disjoint, since if two of them intersect, then their images under T^n, which are $R + z_1$ and $R + z_2$, $z_1 \neq z_2 \in \mathbb{Z}^2$, must intersect. Since two different copies of R in $R + \mathbb{Z}^2$ do not intersect each other, all pieces in $T^{-n}(R)$ are disjoint.

This implies that we have the separation $ce^{(\tau - l_2)n}$ between the strips in $T^{-n}(B(z_n, e^{-\tau n}))$.

Now we estimate $\mu_n(B)$. There are two cases, depending on how large r is compared to the size of the ellipses that make up the set $T^{-n}(B(z_n, e^{-\tau n}))$.

(i) Assume that $r \leqslant ce^{(\tau - l_2)n}$. Then $B(x, r)$ intersects at most one strip. Since each ellipse is long, $B \cap T^{-n}B(z_n, e^{-n\tau})$ is contained in a rectangle with lengths of sides $2r$ and $2\min\{r, e^{-(\tau + l_2)n}\}$. We therefore have that

$$\begin{aligned} \mu_n(B(x, r)) &= \pi^{-2} e^{2n\tau} \mu\big(B \cap T^{-n}B(z_n, e^{-n\tau})\big) \\ &\leqslant 4\pi^{-2} e^{2n\tau} r \min\{r, e^{-(\tau + l_2)n}\}. \end{aligned}$$

If $r \leqslant e^{-(\tau+l_2)n}$, then

$$\mu_n(B(x,r)) \lesssim e^{2n\tau} r^2 < r^{2-\frac{2\tau}{\tau+l_2}} = r^{\frac{2l_2}{\tau+l_2}}.$$

If $e^{-(\tau+l_2)n} < r \leqslant ce^{(\tau-l_2)n}$, then

$$\begin{aligned}\mu_n(B(x,r)) &\lesssim e^{2n\tau} r e^{-(\tau+l_2)n} = r e^{n(\tau-l_2)} \\ &< r^{1-\frac{\tau-l_2}{\tau+l_2}} = r^{\frac{2l_2}{\tau+l_2}}.\end{aligned}$$

(ii) Suppose that $ce^{(\tau-l_2)n} < r < 1$. Then $B(x,r)$ intersects at most $c^{-1} r e^{(l_2-\tau)n}$ strips of $T^{-n}B(z_n, e^{-n\tau})$. The intersection of each strip and $B(x,r)$ is contained in a rectangle with lengths of sides $2r$ and $2e^{-(\tau+l_2)n}$. Hence,

$$\begin{aligned}\mu_n(B(x,r)) &= \pi^{-2} e^{2n\tau} \mu\big(B \cap T^{-n}B(z_n, e^{-n\tau})\big) \\ &\lesssim e^{2\tau n} r e^{(l_2-\tau)n} r e^{-(\tau+l_2)n} = r^2.\end{aligned}$$

Taken together, the two estimates imply that there is a constant C such that $\mu_n(B(x,r)) \leqslant Cr^{\frac{2l_2}{\tau+l_2}}$ holds for all n, x and r.

- Case $\tau \geqslant \frac{l_2-l_1}{2}$.

  In this case, we have $\tau + l_2 > \tau + l_1 \geqslant \frac{1}{2}(l_1 + l_2) \geqslant 0$. Since we do not consider the case $\tau = -l_1$ in this lemma, we actually have $\tau + l_2 > \tau + l_1 > 0$ in this case. Note that $B(z_n, r_n)$ is contained in a 'rectangle' R centred at z_n and with side lengths $ce^{\frac{1}{2}(l_2-l_1)n}$ in the unstable direction and $e^{\frac{1}{2}(l_1-l_2)n}$ in the stable direction. From what was written in the previous case, we conclude that such a rectangle does not intersect itself. The rectangles in the pre-image $T^{-n}R$ have a side length $ce^{-\frac{1}{2}(l_1+l_2)n}$ in the unstable direction and $e^{-\frac{1}{2}(l_1+l_2)n}$ in the stable direction and each such rectangle contains one of the ellipses in $T^{-n}(B(z_n, e^{-\tau n}))$. Therefore, the ellipses are separated by at least about $ce^{-\frac{1}{2}(l_1+l_2)n}$.

  We now bound the measure $\mu_n(B(x,r))$.

  (i) If $r < e^{-(\tau+l_2)n}$, then as before

$$\mu_n(B(x,r)) \leqslant r^{\frac{2l_2}{\tau+l_2}}.$$

  (ii) If $e^{-(\tau+l_2)n} < r < e^{-(\tau+l_1)n}$, the ball $B(x,r)$ intersects at most one ellipse, but cannot contain an ellipse. We then have

$$\mu_n(B(x,r)) \leqslant e^{2\tau n} r e^{-(\tau+l_2)n} = r e^{(\tau-l_2)n}.$$

  Since $\tau > \frac{l_2-l_1}{2}$, we have $\tau + l_1 > \frac{l_1+l_2}{2} \geqslant 0$. Therefore, $l_1 + l_2 \geqslant 0$ implies that $\tau - l_2 \leqslant \tau + l_1$ and hence that $\frac{\tau-l_2}{\tau+l_1} \leqslant 0$. Because of this, we can use $r < e^{-(\tau+l_1)n}$ to estimate that

$$\mu_n(B(x,r)) \leqslant re^{(\tau-l_2)n} = r\left(e^{-(\tau+l_1)n}\right)^{-\frac{\tau-l_2}{\tau+l_1}}$$
$$\leqslant r^{1-\frac{\tau-l_2}{\tau+l_1}} = r^{\frac{l_1+l_2}{\tau+l_1}}.$$

(iii) If $e^{-(\tau+l_1)n} < r < e^{-\frac{1}{2}(l_1+l_2)n}$, in this case, $B(x,r)$ intersect at most one ellipse, and can contain an ellipse. Hence,

$$\mu_n(B) \leqslant e^{-n(l_1+l_2)} < r^{\frac{l_1+l_2}{\tau+l_1}}.$$

(iv) When $e^{-\frac{1}{2}(l_1+l_2)n} < r$, we have that $B(x,r)$ intersects at most $cre^{\frac{1}{2}(l_1+l_2)n}$ ellipses. Then,

$$\mu_n(B) \leqslant re^{\frac{1}{2}(l_1+l_2)n}e^{2\tau n}e^{-(\tau+l_1)n}e^{-(\tau+l_2)n} = cre^{-\frac{1}{2}(l_1+l_2)n} = cr^2.$$

From above, we have $\frac{\log \mu_n(B(x,r))}{\log r} \geqslant \min\{\frac{l_1+l_2}{\tau+l_1}, \frac{2l_2}{\tau+l_2}, 2\} = \frac{l_1+l_2}{\tau+l_1}$. Hence, $\mu_n(B(x,r)) \leqslant Cr^{\frac{l_1+l_2}{\tau+l_1}}$.

- Case $-l_1 \leqslant \tau < \frac{l_2-l_1}{2}$.

Here, we can use the separation $e^{(\tau-l_2)n}$ between the ellipses in the unstable direction. We consider three cases.

(i) When $r \leqslant e^{-(\tau+l_2)n}$, the ball $B(x,r)$ intersects at most one ellipse, and we have

$$\mu(B(x,r)) \leqslant r^2e^{2\tau n} \leqslant r^2 r^{-\frac{2\tau}{\tau+l_2}} = r^{\frac{2l_2}{\tau+l_2}}.$$

(ii) When $e^{-(\tau+l_2)n} < r \leqslant e^{(\tau-l_2)n}$, here it is important to note that $\tau - l_2 < -\frac{l_1+l_2}{2} < 0$. Therefore,

$$e^{(\tau-l_2)n} = \left(e^{-(\tau+l_2)n}\right)^{-\frac{\tau-l_2}{\tau+l_2}} \leqslant r^{-\frac{\tau-l_2}{\tau+l_2}},$$

since $-\frac{\tau-l_2}{\tau+l_2} > 0$.

The ball $B(x,r)$ intersects at most one ellipse in the unstable direction. Hence,

$$\mu(B(x,r)) \leqslant e^{2\tau n}re^{-(\tau+l_2)n} = re^{(\tau-l_2)n}$$
$$\leqslant r^{1-\frac{\tau-l_2}{\tau+l_2}} = r^{\frac{2l_2}{\tau+l_2}}.$$

(iii) When $r > e^{(\tau-l_2)n}$, then the ball $B(x,r)$ intersects at most about $r/e^{(\tau-l_2)n}$ ellipses in the unstable direction, and intersects at most one ellipse in the stable direction, then we get

$$\mu(B(x,r)) \leqslant e^{2\tau n}\min\{r, e^{-(\tau+l_1)n}\}e^{-(\tau+l_2)n}\frac{r}{e^{(\tau-l_2)n}} \leqslant r^2.$$

Combining (i)–(iii) above, we have

$$\mu_n(B) \leqslant Cr^{\min\{\frac{2l_2}{\tau+l_2}, \frac{l_1+l_2}{\tau+l_1}\}}. \qquad \square$$

By Lemmata 5.3 and 5.4, the assumptions of Lemma 3.2 are satisfied, and therefore, the set W_τ has a large intersection property in the sense that $W_\tau \in \mathscr{G}^{s_\tau}$, where

$$s_\tau = \begin{cases} \dfrac{2l_2}{\tau + l_2} & 0 < \tau < -l_1 \\ \min\left\{ \dfrac{l_1 + l_2}{\tau + l_1}, \dfrac{2l_2}{\tau + l_2} \right\} & \tau > -l_1. \end{cases}$$

This finishes the proof of Theorem 2.1.

6 | PROOF OF THEOREM 2.5

The proof of Theorem 2.5 follows some ideas from Wang and Wu [25]. The following proposition is a simplification of the corresponding statement in their paper [25, Proposition 3.1].

Proposition 6.1. *Let $\tilde{s}_\tau$ be as in Theorem 2.5. For $\tau \geqslant 0$, we have*

$$\tilde{s}_\tau = \min_{t \in \{l_i + \tau : 1 \leqslant i \leqslant d\}} \left\{ \sum_{j \in \mathcal{K}_1(t) \cup \mathcal{K}_2(t)} 1 + \frac{1}{t}\left(\sum_{j \in \mathcal{K}_3(t)} l_j - \sum_{j \in \mathcal{K}_2(t)} \tau \right) \right\}$$
$$= \min_{t \in \mathcal{A}} \left\{ \sum_{j \in \mathcal{K}_1(t) \cup \mathcal{K}_2(t)} 1 + \frac{1}{t}\left(\sum_{j \in \mathcal{K}_3(t)} l_j - \sum_{j \in \mathcal{K}_2(t)} \tau \right) \right\},$$

where

$$\mathcal{A} = \{ l_i, l_i + \tau : 1 \leqslant i \leqslant d \},$$

and for each $t \in \mathcal{A}$, the sets $\mathcal{K}_1(t)$, $\mathcal{K}_2(t)$, $\mathcal{K}_3(t)$ give a partition of $\{1, 2, \dots, d\}$ defined by

$$\mathcal{K}_1(t) = \{ j : l_j \geqslant t \},$$
$$\mathcal{K}_2(t) = \{ j : l_j + \tau \leqslant t \} \setminus \mathcal{K}_1(t),$$
$$\mathcal{K}_3(t) = \{1, 2, \dots, d\} \setminus (\mathcal{K}_1(t) \cup \mathcal{K}_2(t)).$$

Proof. The second equality holds due to Proposition 3.1 in [25].

Since $l_1 \leqslant l_2 \leqslant \dots \leqslant l_d$, for any $1 \leqslant i \leqslant d$, assume that there is a $0 \leqslant k \leqslant d - i$ such that $l_i = \cdots = l_{i+k} < l_{i+k+1}$. Here, we adopt $l_{d+1} = \infty$. Then,

$$\mathcal{K}_1(l_i + \tau) = \{ j : l_j \geqslant l_i + \tau \},$$
$$\mathcal{K}_2(l_i + \tau) = \{1, 2, \dots, i, \dots, i + k\}.$$

Note that

$$\begin{aligned}
&\frac{1}{\tau+l_i}\left(\sum_{j\in\mathcal{K}_1(l_i+\tau)\cup\mathcal{K}_2(l_i+\tau)}\tau+l_i+\sum_{j\in\mathcal{K}_3(l_i+\tau)}l_j-\sum_{j\in\mathcal{K}_2(l_i+\tau)}\tau\right)\\
&=\frac{1}{\tau+l_i}\left(\sum_{j\in\mathcal{K}_1(l_i+\tau)}(l_i+\tau-l_j)+\sum_{j\in\mathcal{K}_1(l_i+\tau)\cup\mathcal{K}_3(l_i+\tau)}l_j+\sum_{j\in\mathcal{K}_2(l_i+\tau)}l_i\right)\\
&=\frac{1}{\tau+l_i}\left(\sum_{j\in\mathcal{K}_1(l_i+\tau)}(l_i+\tau-l_j)+(i+k)l_i+\sum_{j=i+k+1}^{d}l_j\right)\\
&=\frac{1}{\tau+l_i}\left(\sum_{j\in\mathcal{K}_1(l_i+\tau)}(l_i+\tau-l_j)+il_i+\sum_{j=i+1}^{d}l_j\right).
\end{aligned}$$

The last equality follows from the assumption on k. This finishes the proof. □

Proof of Theorem 2.5. Without loss of generality, we assume that $\tau>0$. Recall that $L=\sum_{j=1}^{d}l_j$, and put $W_n=T^{-n}B(z_n,e^{-\tau n})$, and W_n is the union of e^{Ln} ellipsoids, denoted by $\{R_n^k\}_{k=1}^{e^{Ln}}$. The ellipsoids in the set W_n have semi-axes $e^{-(\tau+l_j)n}$, $1\leqslant j\leqslant d$ which are all small. The separation of the ellipsoids is e^{-nl_j}, $1\leqslant j\leqslant d$ in the direction of the d semi-axes. Let

$$\mu_n=\frac{\mu|_{W_n}}{\mu(W_n)}=c_d e^{nd\tau}\mu|_{W_n},$$

where μ is the Lebesgue measure. Let $B:=B(x,r)$, where $x\in\mathbb{T}^d$ and $r>0$, then one has

$$\mu_n(B)=c_d e^{n\tau}\mu(B\cap W_n)=c_d e^{nd\tau}\sum_{\substack{1\leqslant k\leqslant e^{Ln}\\ B\cap R_n^k\neq\emptyset}}\mu(B\cap R_n^k).$$

Now we show that $\mu_n(B(x,r))\leqslant Cr^s$. We consider three cases, depending on the size of r.

(i) $r<e^{-n(\tau+l_d)}$.

In this case, a ball of radius r intersects only one ellipsoid of W_n. So,

$$\begin{aligned}
\mu_n(B)&\leqslant c_d e^{nd\tau}\sum_{\substack{1\leqslant k\leqslant e^{Ln}\\ B\cap R_n^k\neq\emptyset}}\mu(B)\\
&\lesssim e^{nd\tau}r^d<r^{d-\frac{d\tau}{\tau+l_d}}=r^{\frac{dl_d}{\tau+l_d}}.
\end{aligned}$$

(ii) $r\geqslant e^{-nl_1}$.

The ball B intersects at most

$$\prod_{k=1}^{d}re^{l_k n}=r^d e^{Ln}$$

ellipsoid of W_n. Thus,

$$\mu_n(B) \leqslant c_d e^{nd\tau} \sum_{\substack{1\leqslant k\leqslant e^{Ln} \\ B\cap R_n^k \neq \emptyset}} \mu(R_n^k) \lesssim e^{nd\tau} r^d e^{Ln} \frac{e^{-nd\tau}}{e^{Ln}} = r^d.$$

(iii) $e^{-n(\tau+l_d)} \leqslant r < e^{-nl_1}$.

In this case, the ball B intersects many ellipsoids, and each ellipsoid is in the worst case transversal in B. The ellipsoid in W_n are parallel and each ellipsoid has d semi-axes. In each of these d directions, we will estimate the number of ellipsoid segments which a ball of radius r intersects, aiming to get the total number of ellipsoid segments intersecting B.

Recall that

$$\mathcal{A} = \{ l_i, l_i + \tau \,:\, 1 \leqslant i \leqslant d \}.$$

Arrange the elements in $\mathcal{A}$ in non-descending order, and assume that there exists i such that

$$e^{-nt_{i+1}} \leqslant r < e^{-nt_i},$$

where $t_i,\ t_{i+1}$ are two consecutive and distinct terms in $\mathcal{A}$. For such i, put

$$N_1(i) := \{ j \,:\, l_j > t_i \} = \{ j \,:\, e^{-nl_j} < e^{-nt_i} \},$$

$$N_2(i) := \{ j \,:\, l_j + \tau \leqslant t_i \} = \{ j \,:\, e^{-n(l_j+\tau)} \geqslant e^{-nt_i} \},$$

and

$$N_3(i) := \{1, 2, \dots, d\} \setminus (N_1(i) \cup N_2(i)).$$

Since $t_i,\ t_{i+1}$ are consecutive, we have

$$N_1(i) = \{ j \,:\, l_j \geqslant t_{i+1} \}, \tag{10}$$

and

$$N_2(i) = \{ j \,:\, t_{i+1} > l_j + \tau \}. \tag{11}$$

(a) For $j \in N_1(i)$, by Equation (10), one has $e^{-nl_j} \leqslant r$. Hence, in each of these directions, the number of ellipsoids intersecting a ball with radius r is at most

$$\frac{r}{e^{-nl_j}}.$$

(b) For $j \in N_2(i)$, we have $r \leqslant e^{-n(l_j+\tau)}$. In the direction of this semi-axis, a ball with radius r intersects at most one ellipse of W_n.

(c) For $j \in N_3(i)$, it follows from the definition of $N_3(i)$ and Equation (11) that

$$e^{-n(l_j+\tau)} \leqslant e^{-nt_{i+1}} < e^{-nt_i} \leqslant e^{-nl_j},$$

which implies that

$$e^{-n(l_j+\tau)} \leqslant r < e^{-nl_j}.$$

In this direction, because of the assumption on r and the separation of the ellipsoids, the ball intersects at most one ellipsoid.

From above, the ball B intersects at most

$$\prod_{j\in N_1(i)} \frac{r}{e^{-nl_j}} \prod_{j\in N_2(i)\cup N_3(i)} 1 = \prod_{j\in N_1(i)} \frac{r}{e^{-nl_j}}$$

ellipsoids, and the intersection of B and each of these ellipsoids are contained in a rectangle of sides about

$$L_j = \begin{cases} 2e^{-n(\tau+l_j)} & \text{if } j \in N_1(i) \cup N_3(i), \\ 2r & \text{if } j \in N_2(i). \end{cases}$$

It follows that

$$\begin{aligned} \mu_n(B) &= c_d e^{nd\tau} \sum_{\substack{1\leqslant k\leqslant e^{Ln} \\ B\cap R_n^k \neq \emptyset}} \mu(R_n^k \cap B) \\ &\lesssim e^{nd\tau} \prod_{j\in N_1(i)} e^{-n(\tau+l_j)} \frac{r}{e^{-nl_j}} \prod_{j\in N_2(i)} r \prod_{j\in N_3(i)} e^{-n(\tau+l_j)} \\ &= e^{n(\sum_{j\in N_2(i)} \tau - \sum_{j\in N_3(i)} l_j)} \prod_{j\in N_1(i)\cup N_2(i)} r. \end{aligned} \tag{12}$$

The estimate obtained in case (iii) above will now be discussed in two cases.

(a) If $\sum_{j\in N_2(i)} \tau - \sum_{j\in N_3(i)} l_j \geqslant 0$, then using $r < e^{-nt_i}$, the inequalities (12) will be rewritten as

$$\mu_n(B) \lesssim r^{s_{i,1}},$$

where

$$s_{i,1} := \sum_{j\in N_1(i)\cup N_2(i)} 1 - \frac{1}{t_i}\left(\sum_{j\in N_2(i)} \tau - \sum_{j\in N_3(i)} l_j \right).$$

Notice that

$$\mathcal{K}_1(t_i) = N_1(i) \cup \{ j : l_j = t_i \},$$

$$\mathcal{K}_2(t_i) = N_2(i),$$
$$\mathcal{K}_3(t_i) = N_3(i) \setminus \{ j : l_j = t_i \}.$$

Hence,

$$\begin{aligned} s_{i,1} &= \sum_{j \in \mathcal{K}_1(t_i) \cup \mathcal{K}_2(t_i)} 1 - \sum_{j : l_j = t_i} 1 - \frac{1}{t_i}\left(\sum_{j \in \mathcal{K}_2(t_i)} \tau - \sum_{j \in \mathcal{K}_3(t_i)} l_j - \sum_{j : l_j = t_i} l_j \right) \\ &= \sum_{j \in \mathcal{K}_1(t_i) \cup \mathcal{K}_2(t_i)} 1 - \frac{1}{t_i}\left(\sum_{j \in \mathcal{K}_2(t_i)} \tau - \sum_{j \in \mathcal{K}_3(t_i)} l_j \right). \end{aligned} \tag{13}$$

This shows that $s_{i,1}$ is a term in

$$\left\{ \sum_{j \in \mathcal{K}_1(t) \cup \mathcal{K}_2(t)} 1 + \frac{1}{t}\left(\sum_{j \in \mathcal{K}_3(t)} l_j - \sum_{j \in \mathcal{K}_2(t)} \tau \right) \right\}_{t \in \mathcal{A}},$$

as given in Proposition 6.1.

(b) If $\sum_{j \in N_2(i)} \tau - \sum_{j \in N_3(i)} l_j < 0$, by $r \geqslant e^{-nt_{i+1}}$ and inequalities (12), then

$$\mu_n(B) \lesssim r^{s_{i,2}},$$

where

$$s_{i,2} := \sum_{j \in N_1(i) \cup N_2(i)} 1 - \frac{1}{t_{i+1}}\left(\sum_{j \in N_2(i)} \tau - \sum_{j \in N_3(i)} l_j \right).$$

Notice that

$$\mathcal{K}_1(t_{i+1}) = N_1(i),$$
$$\mathcal{K}_2(t_{i+1}) = N_2(i) \cup \{ j : l_j = t_{i+1} \},$$
$$\mathcal{K}_3(t_{i+1}) = N_3(i) \setminus \{ j : l_j = t_{i+1} \}.$$

Hence,

$$\begin{aligned} s_{i,2} &= \sum_{j \in \mathcal{K}_1(t_{i+1}) \cup \mathcal{K}_2(t_{i+1})} 1 - \sum_{j : l_j = t_{i+1}} 1 - \\ &\quad - \frac{\sum_{j \in \mathcal{K}_2(t_{i+1})} \tau - \sum_{j \in \mathcal{K}_3(t_{i+1})} l_j - \sum_{j : l_j = t_{i+1}} l_j}{t_{i+1}} \\ &= \sum_{j \in \mathcal{K}_1(t_{i+1}) \cup \mathcal{K}_2(t_{i+1})} 1 - \frac{1}{t_{i+1}}\left(\sum_{j \in \mathcal{K}_2(t_{i+1})} \tau - \sum_{j \in \mathcal{K}_3(t_{i+1})} l_j \right). \end{aligned} \tag{14}$$

Hence, $s_{i,2}$ is also a term among those in Proposition 6.1.

Let

$$\tilde{s}(i) = \sum_{j\in\mathcal{K}_1(t_i)\cup\mathcal{K}_2(t_i)} 1 - \frac{1}{t_i}\left(\sum_{j\in\mathcal{K}_2(t_i)} \tau - \sum_{j\in\mathcal{K}_3(t_i)} l_j\right).$$

Combining (13) and (14), we get that

$$\mu_n(B) \lesssim r^{\min_{1\leqslant i<d} \tilde{s}(i)}.$$

Note that $\tilde{s}(d) = \frac{dl_d}{\tau+l_d} < d$, together with Cases (i) and (ii) and Proposition 6.1, then we conclude that there is a constant C such that

$$\mu_n(B) \leqslant Cr^s,$$

where

$$s = \min_{1\leqslant j\leqslant d}\left\{\frac{jl_j + \sum_{i=j+1} l_i - (l_i - l_j - \tau)_+}{\tau + l_j}\right\}.$$

Lemma 3.2 now finishes the proof, since μ_n converges weakly to the Lebesgue measure on the torus if all eigenvalues are outside the unit circle. □

7 | EXAMPLES

Recall that by Theorems 2.5 and 2.4,

$$\min_k\left\{\frac{kl_k + \sum_{j>k} l_j - \sum_{j=1}^{d}(l_j - l_k - \tau)_+}{\tau + l_k}\right\} \leqslant \dim_{\mathrm{H}} W_\tau \leqslant \min_k\left\{\frac{kl_k + \sum_{j>k} l_j}{\tau + l_k}\right\},$$

where the minimum in the upper bound is over those k for which $\tau + l_k > 0$, and the lower bound is only valid if all eigenvalues are outside the unit circle. We expect that the lower bound holds in more cases than those given by Theorem 2.5, but this is not always the case, which we will show in several examples.

Here, we will consider tori of several different dimensions and compare the corresponding sets W_τ. To keep them apart, we will use the notation $W_\tau(T)$, where $T : \mathbb{T}^d \to \mathbb{T}^d$.

Example 7.1. The first example is a transformation T on $\mathbb{T}^4$ and another related transformation S on $\mathbb{T}^2$. For any $m \in \mathbb{N}$, let

$$A_m = \begin{pmatrix} m+1 & m \\ 1 & 1 \end{pmatrix}.$$

Then, $\det A_m = 1$ and $\operatorname{tr} A_m = m + 2$. The eigenvalues of A_m are

$$\lambda_{m,-} = \frac{m}{2} + 1 - \sqrt{\frac{m^2}{4} + m}, \qquad \lambda_{m,+} = \frac{m}{2} + 1 + \sqrt{\frac{m^2}{4} + m}.$$

Take $m > 1$ and put

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & A_m \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_m \end{pmatrix}.$$

Thus, A is an integer matrix with no eigenvalues on the unit circle and $\det A = 1$. The transformation $T : \mathbb{T}^4 \to \mathbb{T}^4$ defined by

$$T(x) = Ax \mod 1$$

has four Lyapunov exponents

$$l_1 = \log \lambda_{m,-}, \quad l_2 = \log \frac{3 - \sqrt{5}}{2}, \quad l_3 = \log \frac{3 + \sqrt{5}}{2}, \quad l_4 = \log \lambda_{m,+}.$$

Therefore, $l_1 < l_2 < 0 < l_3 < l_4$, $l_1 + l_4 = 0$ and $l_1 + l_2 + l_3 + l_4 = 0$. We will consider $W_\tau(T)$ and prove that we do not have equality in Theorem 2.4 for $W_\tau(T)$.

Consider the transformation $S : \mathbb{T}^2 \to \mathbb{T}^2$ defined by $S(x) = A_m x \mod 1$. The dimension formula in Theorem 2.1 tells us that

$$\dim_{\mathrm{H}} W_\tau(S) = \begin{cases} \dfrac{2l_4}{\tau + l_4}, & \tau \in [0, l_4), \\ \dfrac{l_1 + l_4}{\tau + l_1} = 0, & \tau \in [l_4, \infty). \end{cases}$$

We shall now compare with $W_\tau(T)$. In this case, the upper bound from Theorem 2.4 says that

$$\dim_{\mathrm{H}} W_\tau(T) \leqslant \begin{cases} \dfrac{4l_4}{\tau + l_4}, & \tau \in [0, l_4/3), \\ \dfrac{3l_3 + l_4}{\tau + l_3} = 0, & \tau \in \left[\dfrac{l_4}{3}, \dfrac{l_3 + l_4}{2}\right), \\ \dfrac{2l_2 + l_3 + l_4}{\tau + l_2} = \dfrac{l_2 + l_4}{\tau + l_2}, & \tau \in \left[\dfrac{l_3 + l_4}{2}, l_2 + l_3 + l_4\right), \\ \dfrac{l_1 + l_2 + l_3 + l_4}{\tau + l_1} = 0, & \tau \in [l_2 + l_3 + l_4, \infty). \end{cases}$$

In fact, for $\tau \in \left(\frac{l_3 + l_4}{2}, l_4\right)$ and $z_n = 0$, the set $E_n = T^{-n} B(0, e^{-n\tau})$ is a thin and long ellipsoid that is wrapped around the torus. This ellipsoid's three shortest semi-axes shrink with an exponential speed in n and the largest grows. Therefore, we have

$$W_\tau(T) = (0,0) \times W_\tau(S).$$

Thus,

$$\dim_{\mathrm{H}} W_\tau(T) = \dim_{\mathrm{H}} W_\tau(S) = \frac{2l_4}{\tau + l_4},$$

for this τ. The upper bound for this τ, however, is

$$\dim_{\mathrm{H}} W_\tau(T) \leqslant \frac{l_2 + l_4}{\tau + l_2}.$$

It is clear that

$$\frac{l_2 + l_4}{\tau + l_2} \neq \frac{2l_4}{\tau + l_4},$$

unless $\tau = l_4$, and hence, the upper bound is not the actual value of the Hausdorff dimension in general.

When τ is close to l_4, then it is clear that the upper and lower bounds from Theorems 2.5 and 2.4 coincide, and hence, the lower bound is not valid in this case.

Note that for the range of τ considered above, the measure μ_n does not converge weakly to the Lebesgue measure on $\mathbb{T}^4$, but it does converge weakly to the Lebesgue measure on $(0,0) \times \mathbb{T}^2$.

Example 7.2. The second example is a transformation T on $\mathbb{T}^3$. For any integer $m > 2$, let

$$A = \begin{pmatrix} m & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

The eigenvalues are

$$m, \quad \frac{3+\sqrt{5}}{2}, \quad \frac{3-\sqrt{5}}{2}.$$

Hence,

$$l_1 = \log \frac{3-\sqrt{5}}{2}, \quad l_2 = \log \frac{3+\sqrt{5}}{2}, \quad l_3 = \log m.$$

Thus, $l_1 < 0 < l_2 < l_3$ and $l_1 + l_2 + l_3 = l_3 = \log m$.

Take points $(a_n, b_n, c_n) \in \mathbb{T}^3$ and consider $\tau > l_2 + \log 6$. Let z_n be such that $z_n = T^n(a_n, b_n, c_n)$. Since $\tau > l_2$, we have that $e^{-n(\tau+l_2)} < e^{-(\tau+l_1)n} < e^{-(\tau-l_2)n}$, then $T^{-n}(B(z_n, e^{-\tau n}))$ is contained in the set

$$\mathbb{T} \times (b_n - e^{-(\tau-l_2)n}, b_n + e^{-(\tau-l_2)n}) \times (c_n - e^{-(\tau-l_2)n}, c_n + e^{-(\tau-l_2)n}).$$

It is therefore possible to choose (b_n, c_n) such that $W_\tau = \emptyset$. For example, let $c_n = 0$ and b_n satisfy $b_1 = e^{-\tau - l_2}$, $b_n = 2b_{n-1} + e^{-(\tau - l_2)n}$, $n \geqslant 2$. The sequence $\{b_n\}_n$ exists, since $\sum_{n\geqslant 1} e^{-(\tau-l_2)n} = \frac{e^{-(\tau-l_2)}}{1-e^{-(\tau-l_2)}} < 1/5$. In particular, for such a choice of (b_n, c_n), we have $\dim_{\mathrm{H}} W_\tau = 0$ for $\tau > l_2 + \log 6$. In this case, the result of Hill and Velani [14, Theorem 1] is not correct. Note also that this shows that the lower bound in Theorem 2.5 does not hold in general, since it gives a positive dimension in this case.

As in the previous example, we note that for $\tau > l_2$, the measure μ_n does not converge weakly to the Lebesgue measure on $\mathbb{T}^3$, but if $(b_n, c_n) \to (b, c)$, then it converges weakly to the Lebesgue measure on $\mathbb{T} \times (b, c)$.

ACKNOWLEDGEMENTS

Zhangnan Hu is supported by Science Foundation of China University of Petroleum, Beijing (Grant No. 2462023SZBH013), by China Postdoctoral Science Foundation (Grant No. 2023M743878) and by the Postdoctoral Fellowship Program of CPSF under Grant Number GZB20240848. Wanlou Wu is supported by NSFC 12001245 and Natural Science Foundation of Jiangsu Province BK20201025. Yiwei Zhang is partially supported by NSFC Nos. 12161141002, 12271432 and Guangdong Basic and Applied Basic Research Foundation (No. 2024A1515010974).

We thank Mikael Persson Sundqvist for helping us with Figure 2.

JOURNAL INFORMATION

The *Journal of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

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Acta Mathematica Scientia, 2025, **45B**(4): 1659–1673
https://cstr.cn/32227.14.20250422
https://doi.org/10.1007/s10473-025-0422-1

Mathematica Acta Scientia 数学物理学报
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HAUSDORFF DIMENSION OF RECURRENCE SETS FOR MATRIX TRANSFORMATIONS OF TORI

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Abstract Let $T\colon \mathbb{T}^d \to \mathbb{T}^d$, defined by $Tx = Ax \pmod 1$, where A is a $d \times d$ integer matrix with eigenvalues $1 < |\lambda_1| \le |\lambda_2| \le \cdots \le |\lambda_d|$. We investigate the Hausdorff dimension of the recurrence set

$$R(\psi) := \{x \in \mathbb{T}^d \colon T^n x \in B(x, \psi(n)) \text{ for infinitely many } n\}$$

for $\alpha \ge \log|\lambda_d/\lambda_1|$, where ψ is a positive decreasing function defined on $\mathbb{N}$ and its lower order at infinity is $\alpha = \liminf\limits_{n\to\infty} \frac{-\log\psi(n)}{n}$. In the case that A is diagonalizable over $\mathbb{Q}$ with integral eigenvalues, we obtain the dimension formula.

Keywords quantitative recurrence properties; Hausdorff dimension; matrix transformations

MSC2020 37C45; 37B20; 28A80

1 Introduction

Let $(X, \mathscr{B}, T, \mu, \rho)$ be a probability measure preserving system with a compatible metric ρ. We call $(X, \mathscr{B}, T, \mu, \rho)$ a metric measure preserving system (m.m.p.s.). If (X, ρ) is separable, the Poincaré recurrence theorem shows that μ-a.e. $x \in X$ is recurrent, that is

$$\liminf_{n\to\infty} \rho(T^n x, x) = 0.$$

It shows nothing about the speed at which the orbit returns close to the initial point. One of the first general quantitative recurrence results was given by Boshernitzan [4].

Theorem 1.1 ([4]) Let $(X, \mathscr{B}, T, \mu, \rho)$ be a m.m.p.s. Assume that for some $\tau > 0$, the τ-dimensional Hausdorff measure $\mathcal{H}^\tau$ of X is σ-finite. Then for μ-a.e. $x \in X$,

$$\liminf_{n\to\infty} n^{\frac{1}{\tau}} \rho(T^n x, x) < \infty.$$

Received March 29, 2024; revised August 15, 2024. Hu's research was supported by the Science Foundation of China University of Petroleum, Beijing (2462023SZBH013), the China Postdoctoral Science Foundation (2023M743878) and the Postdoctoral Fellowship Program of CPSF (GZB20240848). Li's research was supported partially by the NSFC (12271176) and the Guangdong Natural Science Foundation (2024A1515010946).

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Futhermore, if $\mathcal{H}^\tau(X) = 0$, then for μ-almost every $x \in X$,

$$\liminf_{n\to\infty} n^{\frac{1}{\tau}} \rho(T^n x, x) = 0.$$

Later, Barreira and Saussol [3] showed that the exponent τ in Theorem 1.1 could be replaced by the lower local dimension of a measure at x. This leads us to study the size of recurrence set when the rate of recurrence is replaced by a general function. Define recurrence sets as

$$R(\psi) = \{\, x \in X : T^n(x) \in B(x, \psi(n)) \text{ for infinitely many } n \geq 1 \,\},$$

where $\psi\colon \mathbb{N} \to \mathbb{R}^+$ is a positive decreasing function, and $B(x, \psi(n))$ denotes the ball centred at x with radius $\psi(n)$.

For the measure of $R(\psi)$, Chang, Wu and Wu [5] investigated the case when X is a homogeneous self-similar set satisfying strong separation condition, and obtained results on the Hausdorff measure of $R(\psi)$. Similar results were generalised to finite conformal iterated function systems satisfying the open set condition by Baker and Farmer [2]. Later, Hussain *et al.* [14] considered more general conformal dynamical systems. When T is a piecewise expanding map, under some conditions, He and Liao [10] obtained that the measure of $R(\psi)$ obeys a full-zero law. More results about measure of $R(\psi)$ can be found in [1, 15, 16]. As for the size of $R(\psi)$ in Hausdorff dimension, Tan and Wang [20] calculated the Hausdorff dimension of $R(\psi)$ when T is the β-transformation. Seuret and Wang [19] proved similar results for self-conformal sets. There are very few results on the Hausdorff dimension when T is the matrix transformation, and as far as we know, the only results were obtained for diagonal matrix transformations [10], toral automorphisms on 2-dimensional torus [13] and recently for some special cases [22, 23].

Shrinking target problem concerns the speed at which $(T^n x)_{n\geq 1}$ returns to the neighborhood of a given point x_0 instead of the initial point x, which has many common features with the problem of quantitative recurrence. For the shrinking target problem, much more results are known. Hill and Velani [12] investigated the Hausdorff dimension of shrinking target sets in the system (X, T), where X a d-dimensional torus, and T a linear map given by an integer matrix. For a real, non-singular matrix transformation with some conditions, Li *et al.* [17] proved that the Lebesgue measure of the shrinking target set obeys a zero-one law. They also determined the Hausdorff dimension of shrinking target sets when T is a diagonal matrix transformation. One can refer to [6, 9] for more results on the measure, and [11, 18, 21] for the dimension.

Motivated by the aforementioned research, in this paper, we focus on the case where X is the d-dimensional torus $\mathbb{T}^d$ endowed with the usual quotient distance ρ_0, and T is the integer matrix transformation with the modulus of eigenvalues are strictly larger than 1. More precisely, $T\colon \mathbb{T}^d \to \mathbb{T}^d$ is defined by

$$Tx = Ax \ (\mathrm{mod}\ 1),$$

where A is a $d \times d$ integer matrix. Let $\psi\colon \mathbb{N} \to \mathbb{R}^+$ be a positive function satisfying $\psi(n) \to 0$ as $n \to \infty$. Throughout, denote the lower order of ψ at infinity by

$$\alpha := \liminf_{n\to\infty} \frac{-\log \psi(n)}{n}.$$

In this paper, $\dim_{\mathrm{H}}$ stands for the Hausdorff dimension.

Theorem 1.2 Let A be a $d\times d$ integer matrix with all eigenvalues $\lambda_1,\ \lambda_2,\cdots,\ \lambda_d$. Assume that $|\lambda_d|\geq\cdots\geq|\lambda_1|>1$. Then for $\alpha\geq\log|\lambda_d/\lambda_1|$,

$$\dim_{\mathrm{H}} R(\psi)=\min_{j\in\{1,\cdots,d\}}\Big\{\frac{j\log|\lambda_j|+\sum_{i=j+1}^{d}\log|\lambda_i|}{\alpha+\log|\lambda_j|}\Big\}.$$

Remark 1.3 Let A be as in Theorem 1.2. For $\alpha\geq 0$, we always have

$$\dim_{\mathrm{H}} R(\psi)\leq\min_{j\in\{1,\cdots,d\}}\Big\{\frac{j\log|\lambda_j|+\sum_{i=j+1}^{d}\log|\lambda_i|}{\alpha+\log|\lambda_j|}\Big\},$$

here we do not need to assume that $\alpha\geq\log|\lambda_d/\lambda_1|$. Notice that for $\alpha>0$, $\dim_{\mathrm{H}} R(\psi)\leq\frac{d\log|\lambda_d|}{\alpha+\log|\lambda_d|}<d$.

As an immediate consequence of Theorem 1.2, when the modulus of all eigenvalues of A are the same, we obtain the formula of the Hausdorff dimension of $R(\psi)$ for any $\alpha\geq 0$.

Corollary 1.4 Let A be a $d\times d$ integer matrix with all eigenvalues $\lambda_1,\ \lambda_2,\cdots,\ \lambda_d$. Assume that the modulus of all eigenvalues are the same, denoted by λ, and $\lambda>1$. Then for $\alpha\geq 0$,

$$\dim_{\mathrm{H}} R(\psi)=\frac{d\log\lambda}{\alpha+\log\lambda}.$$

When an integer matrix A is diagonalizable over $\mathbb{Z}$ or A is a diagonal real matrix with eigenvalues $\lambda_d\geq\cdots\geq\lambda_1>1$, Yuan and Wang [24] calculated the Hausdorff dimension of $R(\psi)$. However they did not consider the case when $\lambda_i<-1$ for some $1\leq i\leq d$. When A is a diagonal real matrix with $|\lambda_d|\geq\cdots\geq|\lambda_1|>1$, He and Liao [10] gave the formula of $\dim_{\mathrm{H}} R(\psi)$. If A is an integer matrix, the diagonal assumption can be relaxed to a weaker condition as the following theorem shows.

Theorem 1.5 Let A be a $d\times d$ integer matrix. Assume that A is diagonalizable over $\mathbb{Q}$ with all eigenvalues $\lambda_1,\ \lambda_2,\cdots,\ \lambda_d\in\mathbb{Z}$. Assume that $|\lambda_d|\geq\cdots\geq|\lambda_1|>1$. Then for $\alpha\geq 0$,

$$\dim_{\mathrm{H}} R(\psi)=\min_{j\in\{1,\cdots,d\}}\Big\{\frac{j\log|\lambda_j|+\sum_{k\in\mathcal{K}(j)}(\alpha+\log|\lambda_j|-\log|\lambda_i|)+\sum_{i=j+1}^{d}\log|\lambda_i|}{\log|\lambda_j|+\alpha}\Big\},$$

where

$$\mathcal{K}(j):=\{i\in\{1,\cdots,d\}\colon\ \log|\lambda_i|>\log|\lambda_j|+\alpha\}.$$

Remark 1.6 The dimension formula in Theorem 1.5 is the same as those given by [10, 24]. When $\alpha\geq\log|\lambda_d/\lambda_1|$, $\bigcup_j\mathcal{K}(j)=\varnothing$, hence in this case, the formula in Theorem 1.2 coincides with that given in Theorem 1.5.

For $n\geq 1$, write

$$R_n(\psi)=\{x\in\mathbb{T}^d:(A^n-I)x\ (\mathrm{mod}\ 1)\in B(0,\psi(n))\},$$

where I is the identity matrix. Then $R(\psi)=\limsup\limits_{n\to\infty}R_n(\psi)$.

The paper is organized as follows. Theorem 1.2 is proved in the following two sections. Section 2 is devoted to give some preparations on the geometric property of $R_n(\psi)$, which is crucial to the proof of Theorem 1.2. In Section 4, we use the preparations to construct a Cantor subset to establish the lower bound of $\dim_{\mathrm{H}} R(\psi)$. By Remark 2.4, we only prove Theorem 1.2 in the case that A is diagonalizable over $\mathbb{R}$ with eigenvalues $\lambda_i>1$, $1\leq i\leq d$. In the last section, we give the proof of Theorem 1.5.

Notation 1.7 Suppose $f(n)$ and $g(n)$ are two functions defined on natural numbers. Write $f(n) \lesssim g(n)$ or $f(n) = O(g(n))$ if and only if there exist constants $N \geq 1$ and $C > 1$ such that $f(n) \leq Cg(n)$ for all $n \geq N$. Write $f(n) \asymp g(n)$ if $f(n) \lesssim g(n)$ and $g(n) \lesssim f(n)$. The ceiling function of a real number x is denoted by $\lceil x \rceil$, and the floor function is denoted by $\lfloor x \rfloor$. For a matrix A, let $\det A$ stand for the determinant of A. We write $\#E$ for the cardinality of a finite set E.

2 Distribution of periodic points

Throughout this paper, let the eigenvalues of the matrix A be $\{\lambda_1, \lambda_2, \cdots, \lambda_d\}$ with $|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_d|$.

Definition 2.1 A point $x \in \mathbb{T}^d$ is called a **periodic point** with period n if

$$(A^n - I)x \ (\mathrm{mod}\ 1) = 0. \tag{2.1}$$

It is easy to see that (2.1) is equivalent to $A^n x \ (\mathrm{mod}\ 1) = x$. Put

$$\mathcal{P}_n = \{x \in \mathbb{T}^d\colon\ x \ \text{ is a periodic point with period } n\}.$$

Lemma 2.2 ([7, Lemma 2.3]) Let M be a non-singular integer matrix. If no eigenvalue of M is a root of unity, the number of periodic points with period n is given by

$$\#\{x \in \mathbb{T}^d\colon M^n x \ (\mathrm{mod}\ 1) = x\} = |\det(M^n - I)|.$$

Denote $|\det(A^n - I)|$ by H_n. It follows from Lemma 2.2 that

$$\#\mathcal{P}_n = H_n = \prod_{i=1}^{d} |\lambda_i^n - 1|.$$

Rewrite

$$\begin{aligned} R_n(\psi) &= \{x \in \mathbb{T}^d\colon x \in (A^n - I)^{-1}B(0, \psi(n)) + (A^n - I)^{-1}\mathbb{Z}^d\} \\ &= \bigcup_{y \in \mathcal{P}_n} \{x \in \mathbb{T}^d\colon x \in (A^n - I)^{-1}B(0, \psi(n)) + y\}. \end{aligned} \tag{2.2}$$

By (2.2) and Lemma 2.2, $R_n(\psi)$ consists of H_n ellipsoids which are translations of $(A^n - I)^{-1}B(0, \psi(n))$, denoted by $(R_{n,i})_{i=1}^{H_n}$. For $n \geq 1$ and $j = 1, 2, \cdots, d$, put

$$\ell_{n,j} = 2\psi(n)|\lambda_j^n - 1|^{-1}.$$

Let $e_{n,1} \geq e_{n,2} \geq \cdots \geq e_{n,d}$ be the lengths of semi-axes of the ellipsoid $R_{n,i}$. If A is diagonalizable, for $j = 1, \cdots, d$,

$$e_{n,j} \asymp \ell_{n,j}.$$

If A is not diagonalizable, using Jordan decomposition, we have the following lemma.

Lemma 2.3 Let A be a non-singular integer matrix. Assume that the modulus of all eigenvalues are not 1. Then there are constants $C > 1$ and $\tau > 0$ such that

$$C^{-1}n^{-\tau} \leq \frac{e_{n,j}}{\ell_{n,j}} \leq Cn^{\tau}$$

holds for all n and $1 \leq j \leq d$.

Proof By Lemma 3 in [12], for $n \ge 1$, we have $(A^n - I)^{-1} = A_1 A_2$, where all eigenvalues of A_1 have absolute value 1, the matrix A_2 is diagonalizable over $\mathbb{R}$ with eigenvalues of modulus $|\lambda_i^n - 1|^{-1}$, $1 \le i \le d$, and A_1 and A_2 commute. Applying [12, Lemma 2], there is $\tau \ge 0$ depending only on A such that for any ball $B(x, r)$,

$$B(x'', (O(n^\tau))^{-1} r) \subset A_1 B(x, r) \subset B(x', O(n^\tau) r)$$

for some x', $x'' \in \mathbb{T}^d$. Combining these, we conclude that $(A^n - I)^{-1} B(x, r)$ contains an ellipsoid with lengths of semi-axes $(O(n^\tau))^{-1} |\lambda_i^n - 1|^{-1} r$, $1 \le i \le d$, and also is contained in an ellipsoid with lengths of semi-axes $O(n^\tau) |\lambda_i^n - 1|^{-1} r$, $1 \le i \le d$. □

Remark 2.4 By Lemma 2.3, $e_{n,j}/\ell_{n,j}$ grows with polynomial speed, which does not influence the formula of Hausdorff dimension. Hence for simplicity, from now on, we assume that A is diagonalizable over $\mathbb{R}$ with eigenvalues $\lambda_i > 1$, $1 \le i \le d$, and $\psi(n) = \mathrm{e}^{-\alpha n}$, $n \ge 1$. If there exists i such that $\lambda_i < -1$, then we consider A^2 instead of A, which will not change the value of α by [17, Lemma 9].

Recall that ρ_0 is the usual quotient distance. For $i \ne k \in \{1, 2, \cdots, H_n\}$, denote

$$d_n(i, k) = \inf \big\{ \rho_0(x, y) \colon x \in R_{n,i},\ y \in R_{n,k} \big\},$$

that is, $d_n(i, k)$ is the distance between $R_{n,i}$ and $R_{n,k}$. When $i = k$, we have $d_n(i, k) = 0$. Put

$$d_n = \min_{1 \le i \ne k \le H_n} d_n(i, k).$$

The following lemma gives some information on d_n, the shortest distance between the ellipsoids.

Lemma 2.5 Let A be a $d \times d$ integer matrix. Suppose that A is diagonalizable over $\mathbb{R}$ with all eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_d$ strictly larger than 1. Then for n large enough,

$$d_n \gtrsim (\lambda_d^n - 1)^{-1}.$$

In particular, we have $R_{n,i} \cap R_{n,k} = \emptyset$ for $i \ne j$, $1 \le i, k \le H_n$.

Proof Without loss of generality, we assume that $\psi(1) < \frac{1}{3}$. For $n \ge 1$ and $x \in \mathcal{P}_n$, note that

$$B\big((A^n - I)x, \psi(n)\big) \subset B\big((A^n - I)x, \frac{1}{3}\big),$$

and $\{B\big((A^n - I)x, \frac{1}{3}\big) \colon x \in \mathcal{P}_n\}$ are disjoint, since $\{(A^n - I)x \colon x \in \mathcal{P}_n\} \subset \mathbb{Z}^d$. It follows that $\{T^{-n} B\big((A^n - I)x, \frac{1}{3}\big) \colon x \in \mathcal{P}_n\}$ are also disjoint. Then we conclude that for $1 \le i \le H_n$, the ellipsoid $R_{n,i}$ is contained in an ellipsoid $\widetilde{R}_{n,i}$ with lengths of semi-axes $\frac{1}{3}(\lambda_j^n - 1)^{-1}$, $1 \le j \le d$, and $(\widetilde{R}_{n,i})_i$ are disjoint. It implies that for any $i \ne k$,

$$d_n(i, k) \ge \min_{j = 1, 2, \cdots, d} \Big\{ 2\Big(\frac{1}{3} - \psi(n)\Big)(\lambda_j^n - 1)^{-1} \Big\},$$

which implies that $d_n \ge 2(\frac{1}{3} - \psi(n))(\lambda_d^n - 1)^{-1} > 0$. □

Recall $\mathcal{P}_n$ consists of all periodic points with period n. Now we estimate the number of periodic points with period n in a given ball.

Lemma 2.6 For any $B = B(x, r)$ in $\mathbb{T}^d$ and $n \ge 1$, we have

$$\# \mathcal{P}_n \cap B := \# \{ y \in B \colon (A^n - I) y \ (\mathrm{mod}\ 1) = 0 \} \lesssim \prod_{j \colon (\lambda_j^n - 1) r > 1} \lceil (\lambda_j^n - 1) r \rceil.$$

If $r(\lambda_1^n - 1) > 1$, then

$$\#\mathcal{P}_n \cap B \asymp r^d H_n.$$

Proof For $n \geq 1$,

$$\#\mathcal{P}_n \cap B = \#\{(A^n - I)B \cap \mathbb{Z}^d\}.$$

Notice that there is a constant $c_1 > 1$ such that $(A^n - I)B$ may be covered by a rectangle R_1 with length $2c_1(\lambda_j^n - 1)r$, $j = 1, \cdots, d$, and contains a rectangle R_2 with length $\frac{1}{c_1\sqrt{d}}(\lambda_j^n - 1)r$, $j = 1, \cdots, d$.

(i) A square of length 1/2 contains at most one point in $\mathbb{Z}^d$, and R_1 can be covered by

$$\prod_{j\colon 2c_1(\lambda_j^n - 1)r > 1/2} \lceil 4c_1(\lambda_j^n - 1)r \rceil$$

squares with length 1/2, which implies that

$$\#\mathcal{P}_n \cap B \lesssim \prod_{j\colon (\lambda_j^n - 1)r > 1} \lceil (\lambda_j^n - 1)r \rceil. \tag{2.3}$$

(ii) If $r(\lambda_1^n - 1) > 1$, then $r(\lambda_j^n - 1) > 1$, $j = 1, \cdots, d$.

Since a square of length 1 contains at least one point in $\mathbb{Z}^d$, it suffices to estimate the number of squares with length 1 which are contained in R_2. Note that R_2 contains

$$\prod_{j=1}^{d} \left\lfloor \frac{1}{c_1\sqrt{d}}(\lambda_j^n - 1)r \right\rfloor$$

squares with length 1. Combining with (3.1), we have

$$\#\mathcal{P}_n \cap B \asymp r^d \prod_{j=1}^{d} (\lambda_j^n - 1) = r^d H_n.$$

□

Remark 2.7 For any $B(x, l)$ in $\mathbb{T}^d$, given $r > 0$, if $2(\lambda_d^n - 1)l \leq r$, then $(A^n - I)B(x, l)$ can be covered by only one ball of radius r. Hence in Lemma 2.6, if $(\lambda_d^n - 1)l \leq 1/4$, we have

$$\#\{y \in B(x, l)\colon (A^n - I)y \ (\mathrm{mod}\ 1) = 0\} \leq 1.$$

The following corollary follows from Lemma 2.6, which is crucial to give the lower bound on $\dim_{\mathrm{H}} R(\psi)$. Recall that $R_n(\psi) = \bigcup_{i=1}^{H_n} R_{n,i}$.

Corollary 2.8 For $n \geq 1$, and $i \in \{1, \cdots, H_n\}$,

$$\#\mathcal{P}_m \cap R_{n,i} \asymp \psi(n)^d H_m H_n^{-1}$$

holds for $m \geq n$ large enough.

Proof Recall that $\ell_{n,j} = 2\psi(n)(\lambda_j^n - 1)^{-1}$, $j = 1, \cdots, d$. For $n \geq 1$, and $i \in \{1, \cdots, H_n\}$, $R_{n,i}$ contains a rectangle $\widetilde{R}_{n,i}$ with length $\frac{1}{\sqrt{d}}\ell_{n,j}$, $j = 1, \cdots, d$, and $\widetilde{R}_{n,i}$ contains

$$\prod_{j=1}^{d} \left\lceil \frac{\lambda_d^n - 1}{\lambda_j^n - 1} \right\rceil$$

disjoint squares with length $\frac{1}{\sqrt{d}}\ell_{n,d}$.

Taking $m > n$ large enough such that $\frac{1}{\sqrt{d}}\ell_{n,d}(\lambda_1^m - 1) > 1$, by Lemma 2.6, for any ball B with radius $\frac{1}{2\sqrt{d}}\ell_{n,d}$, we have

$$\#\mathcal{P}_m \cap B \asymp \ell_{n,d}^d H_m.$$

Therefore

$$\#\mathcal{P}_m \cap R_{n,i} \gtrsim \ell_{n,d}^d H_m \prod_{j=1}^{d} \lceil \frac{\lambda_d^n - 1}{\lambda_j^n - 1} \rceil \asymp H_m \Big(\frac{\psi(n)}{\lambda_d^n - 1}\Big)^d \prod_{j=1}^{d} \lceil \frac{\lambda_d^n - 1}{\lambda_j^n - 1} \rceil$$
$$\asymp \psi(n)^d H_m \prod_{j=1}^{d} \frac{1}{\lambda_j^n - 1} = \psi(n)^d H_m H_n^{-1}.$$

Note that $R_{n,i}$ is contained a rectangle with length $\ell_{n,j}$, $j = 1, \cdots, d$. It follows that

$$\#\mathcal{P}_m \cap R_{n,i} \lesssim \psi(n)^d H_m H_n^{-1}.$$

□

In the following, we prove Theorem 1.2. The proof is divided into two parts: Section 3 and Section 4. In Section 3, we use the natural covering to estimate the upper bound on $\dim_{\mathrm{H}} R(\psi)$, and in Section 4, we will construct a Cantor subset K of $R(\psi)$ to give a lower bound to $\dim_{\mathrm{H}} R(\psi)$.

3 Upper bound on $\dim_{\mathrm{H}} R(\psi)$

In Remark 2.4, we assume that A is diagonalizable, and hence there is a constant $c_2 > 1$ such that the quotient of singular values and eigenvalues of $(A^n - I)^{-1}$ is bounded by c_2 from above, and c_2^{-1} from below for n large enough. We may take $c_2 = 1$ both in Sections 3 and 4.

For $m \geq 1$, we have $R(\psi) \subset \bigcup_{n=m}^{\infty} R_n(\psi) = \bigcup_{n=m}^{\infty} \bigcup_{i=1}^{H_n} R_{n,i}$. The lengths of semi-axes of $R_{n,i}$ are about $\ell_{n,j} = 2\psi(n)(\lambda_j^n - 1)^{-1}$, $j = 1, 2, \cdots, d$.

For $k \in \{1, 2, \cdots, d\}$, we may use balls with radius $\ell_{n,k}$ to cover $R_{n,i}$, and the number of such balls is about

$$\prod_{j<k} \frac{\ell_{n,j}}{\ell_{n,k}} = \prod_{j=1}^{k} \frac{\lambda_k^n - 1}{\lambda_j^n - 1}. \tag{3.1}$$

For any $\delta > 0$,

$$\mathcal{H}_\delta^s(R(\psi)) \lesssim \sum_{n=m}^{\infty} H_n \ell_{n,k}^s \prod_{j=1}^{k} \frac{\lambda_k^n - 1}{\lambda_j^n - 1} \lesssim \sum_{n=m}^{\infty} \lambda_k^{n(k-s)} \psi(n)^s \prod_{j=k+1}^{d} \lambda_j^n$$
$$= \exp\Big\{ n\Big(k \log \lambda_k + \sum_{j=k+1}^{d} \log \lambda_j - s(\alpha + \log \lambda_k)\Big)\Big\}.$$

Notice that for any $s > \frac{k\log\lambda_k + \sum_{j=k+1}^d \log\lambda_j}{\alpha + \log\lambda_k}$, $\mathcal{H}^s(R(\psi)) < \infty$, and hence

$$\dim_{\mathrm{H}} R(\psi) \leq \min_k \Big\{ \frac{k\log\lambda_k + \sum_{j=k+1}^d \log\lambda_j}{\alpha + \log\lambda_k} \Big\}.$$

Remark 3.1 It follows from Lemma 2.5 that for $i \neq j$,

$$\mathrm{dist}(R_{n,i}, R_{n,j}) \gtrsim (\lambda_d^n - 1)^{-1}.$$

When $\alpha > \log(\lambda_d/\lambda_1)$, we have $\ell_{n,1} \lesssim (\lambda_d^n - 1)^{-1}$, which implies that any ball with radius $\ell_{n,k}$ intersects only one ellipsoid $R_{n,i}$. But if $\lambda_d/\lambda_1 \neq 1$, for $\alpha \in \big(0, \log(\lambda_d/\lambda_1)\big]$, the number of balls of radius $\ell_{n,k}$ covering $R_n(\psi)$ may be less than the number in (3.1), cause a ball of radius $\ell_{n,k} \leq \ell_{n,1}$ can cover many $R_{n,i}$, that is, there may exist better coverings for $R_n(\psi)$, which gives another dimension formula such as Theorem 1.5.

4 Lower bound on $\dim_{\mathrm{H}} R(\psi)$.

Recall that R_n consists of elliptical discs $(R_{n,i})_{i=1}^{H_n}$ with lengths of semi-axes $\ell_{n,j}$, $j = 1, 2, \cdots, d$, whose centres are periodic points satisfying (2.1). We first suppose that $\log(\lambda_d/\lambda_1) < \alpha < \infty$.

4.1 Construct a Cantor set

Write $\mathcal{A}_n = \{1, 2, \cdots, H_n\}$. Let $(n_j)_{j\geq 0}$ be an increasing sequence of positive integers (to be determined later).

Put $n_0 = 0$, $K_0 = \mathbb{T}^d$. Let

$$K_1 = \bigcup_{i\in\mathcal{C}_1} R_{n_1,i},$$

where $\mathcal{C}_1 := \{i \in \mathcal{A}_{n_1}\colon\ R_{n_1,i} \subset K_0\}$. For $j \geq 1$, suppose K_j has been defined, then

$$K_{j+1} = \bigcup_{i\in\mathcal{C}_{l+1}} R_{n_{j+1},i},$$

where $\mathcal{C}_{j+1} := \{i \in \mathcal{A}_{n_{j+1}}\colon R_{n_{j+1},i} \subset K_j\}$. Here choose $(n_j)_{j\geq 0}$ such that

$$\lim_{j\to\infty} \frac{\sum_{i=1}^{j-1} n_i}{n_j} = 0, \tag{4.1}$$

and for $j \geq 1$,

$$\psi(n_j)(\lambda_d^{n_j} - 1)^{-1} > (\lambda_1^{n_{j+1}} - 1)^{-1}. \tag{4.2}$$

Notice that $\mathcal{C}_j \neq \emptyset$ is guaranteed by (4.2). Then each $K_j \neq \emptyset$ is a union of finite ellipsoids in $\{R_{n_j,i}\colon 1 \leq i \leq H_{n_j}\}$ with $K_1 \supset K_2 \supset K_3 \cdots$. Let $K = \bigcap_j K_j$.

4.2 Construct the mass distribution

Now we define a mass distribution supported on K. Given a set $E \subset \mathbb{T}^d$, for $j \geq 1$, denote the collection of ellipsoids in $(R_{n_j,m})_{m=1}^{n_j}$ which are contained in E by $\mathcal{C}_j(E)$, that is,

$$\mathcal{C}_j(E) := \{1 \leq k \leq H_{n_j}\colon R_{n_j,k} \subset E\}.$$

For $j = 0$, $\mu(K_0) = 1$. For $j = 1$ and $i \in \mathcal{C}_1$,

$$\mu(R_{n_1,i}) = \frac{1}{\#\mathcal{C}_1}.$$

For $j = 2$ and $i \in \mathcal{C}_2$, there is a unique $m(i) \in \mathcal{C}_1$ such that $R_{n_2,i} \subset R_{n_1,m(i)}$, then let

$$\mu(R_{n_2,i}) = \frac{1}{\#\mathcal{C}_2(R_{n_1,m(i)})}\mu(R_{n_1,m(i)}).$$

Assume that we have defined μ on the sets $\{R_{n_k,i}\colon i \in \mathcal{C}_k\}$. Now for $j = k+1$ and $i \in \mathcal{C}_{k+1}$, since $(R_{n,i})_{i=1}^{H_n}$ do not intersect each other, there is a unique $m(i) \in \mathcal{C}_k$ such that $R_{n_{k+1},i} \subset R_{n_k,m(i)}$, then let

$$\mu(R_{n_{k+1},i}) = \frac{1}{\#\mathcal{C}_{k+1}(R_{n_k,m(i)})}\mu(R_{n_k,m(i)}).$$

Note that for $j \geq 1$

$$\mu(K_j) = 1.$$

Then the definition of μ may be extended to all subsets of $\mathbb{T}^d$ so that μ becomes a measure. The support of μ is contained in K.

4.3 Estimation on $\mu(R_{n_j,m})$.

For $j \geq 1$ and $m \in \{1, \cdots, H_{n_j}\}$, there exists $(m_k)_{k=1}^{j-1}$ with $m_k \in \{1, 2, \cdots, H_{n_k}\}$, $1 \leq k \leq j-1$ such that

$$R_{n_j,m} \subset R_{n_{j-1},m_{j-1}} \subset \cdots \subset R_{n_1,m_1}.$$

Then

$$\mu(R_{n_j,m}) = \frac{1}{\#\mathcal{C}_1}\prod_{k=1}^{j-1}\frac{1}{\#\mathcal{C}_{k+1}(R_{n_k,m_k})},$$

here $\#\mathcal{C}_1 = H_{n_1}$. By Corollary 2.8, there is a constant $C_1 > 1$ such that for any $k \geq 1$,

$$C_1^{-1}\psi(n_k)^d\frac{H_{n_{k+1}}}{H_{n_k}} \leq \#\mathcal{C}_{k+1}(R_{n_k,m_k}) \leq C_1\psi(n_k)^d\frac{H_{n_{k+1}}}{H_{n_k}}.$$

It follows that

$$\mu(R_{n_j,m}) \leq C_1^{j-1}\frac{1}{\#\mathcal{C}_1}\prod_{k=1}^{j-1}\frac{H_{n_k}}{\psi(n_k)^d H_{n_{k+1}}} = C_1^{j-1}H_{n_j}^{-1}\prod_{k=1}^{j-1}\psi(n_k)^{-d}. \tag{4.3}$$

Also

$$\mu(R_{n_j,m}) \geq C_1^{-j+1}H_{n_j}^{-1}\prod_{k=1}^{j-1}\psi(n_k)^{-d}. \tag{4.4}$$

4.4 Estimate the local dimension of μ

Lemma 4.1 Let $B := B(x, r_1)$ and $\widetilde{B} := B(y, r_2)$. Then for $n \geq 1$, the set $B \cap (A^n - I)^{-1}\widetilde{B}$ can be covered by

$$\prod_{i:r_1\leq(\lambda_i^n-1)^{-1}r_2}\lceil\frac{r_1}{r_3}\rceil\prod_{i:r_1>(\lambda_i^n-1)^{-1}r_2>r_3}\lceil\frac{(\lambda_i^n-1)^{-1}r_2}{r_3}\rceil$$

balls of radius $r_3 \leq r_1$.

Proof Note that $(A^n - I)^{-1}\widetilde{B}$ is contained in a rectangle R with lengths $2(\lambda_i^n - 1)^{-1}r_2$, $i = 1, \cdots, d$, and

$$\begin{aligned}\{1, \cdots, d\} &= \{i : r_1 \leq 2(\lambda_i^n-1)^{-1}r_2\} \cup \{i : r_1 > 2(\lambda_i^n-1)^{-1}r_2 > r_3\} \cup \{i : r_3 > 2(\lambda_i^n-1)^{-1}r_2\}\\ &=: I_1 \cup I_2 \cup I_3.\end{aligned}$$

Take

$$L_i = \begin{cases} 2r_1 & \text{if } i \in I_1, \\ 2r_2(\lambda_i^n - 1)^{-1} & \text{if } i \in I_2 \cup I_3.\end{cases}$$

Then $(A^n-I)^{-1}\widetilde{B}\cap B$ can be covered by a rectangle $\widetilde{R}$ with length L_i, $i=1,\cdots,d$. It implies that $\widetilde{R}$ may be covered by

$$\prod_{i\in I_1\cup I_2}\lceil\frac{L_i}{2r_3}\rceil\prod_{i\in I_3}1\asymp\prod_{i:r_1\le(\lambda_i^n-1)^{-1}r_2}\lceil\frac{r_1}{r_3}\rceil\prod_{i:r_1>(\lambda_i^n-1)^{-1}r_2>r_3}\lceil\frac{(\lambda_i^n-1)^{-1}r_2}{r_3}\rceil$$

squares with length r_3. $\square$

For $x\in K$, $x\in K_j$ for $j\ge 1$, and by the construction of K_j, there is a unique sequence of ellipsoids containing x, denoted by $(E_j)_j$. For $r>0$, there is some j such that

$$(\lambda_d^{n_j}-1)^{-1}\psi(n_j)\le r<(\lambda_d^{n_{j-1}}-1)^{-1}\psi(n_{j-1}),\tag{4.5}$$

since $\psi(n_j)(\lambda_d^{n_j}-1)^{-1}$ is decreasing as $j\to\infty$. And there are two cases:

(A) $\exists k\ge 1$ such that

$$(\lambda_{k+1}^{n_j}-1)^{-1}\psi(n_j)\le r<(\lambda_k^{n_j}-1)^{-1}\psi(n_j),$$

(B) $(\lambda_1^{n_j}-1)^{-1}\psi(n_j)\le r$.

First we consider Case **(A)** : $(\lambda_{k+1}^{n_j}-1)^{-1}\psi(n_j)\le r<(\lambda_k^{n_j}-1)^{-1}\psi(n_j)$.

Let $B=B(x,r)$. For j in (4.5), by (4.2) and (4.5), we have $\operatorname{diam}(R_{n_{j+1},m})=2(\lambda_1^{n_{j+1}}-1)^{-1}\psi(n_{j+1})<(\lambda_d^{n_j}-1)^{-1}\psi(n_j)<r$, hence up to a set of zero measure,

$$B\subset\bigcup_{\substack{B\cap R_{n_{j+1},m}\ne\emptyset\\ m\in\mathcal{C}_{j+1}}}R_{n_{j+1},m}\subset\bigcup_{\substack{R_{n_{j+1},m}\subset 2B\\ m\in\mathcal{C}_{j+1}}}R_{n_{j+1},m}.$$

Then

$$\mu(B)\le\sum_{\substack{R_{n_{j+1},m}\subset 2B\\ m\in\mathcal{C}_{j+1}}}\mu(R_{n_{j+1},m}).$$

Since $r<(\lambda_1^{n_j}-1)^{-1}\psi(n_j)$, by Remark 2.7 and $\alpha>\log(\lambda_d/\lambda_1)$, E_j is the unique ellipsoid of degree n_j which intersects $2B$. Then

$$\#\mathcal{C}_{j+1}(2B)=\#\mathcal{C}_{j+1}(2B\cap E_j),$$

which implies that

$$\mu(B)\le(\#\mathcal{C}_{j+1}(2B\cap E_j))\mu(R_{n_{j+1},m})\le(\#\mathcal{C}_{j+1}(2B\cap E_j))\frac{\mu(E_j)}{\#\mathcal{C}_{j+1}(E_j)}.$$

Now we estimate $\#\mathcal{C}_{j+1}(2B\cap E_j)$. By Lemma 4.1, $2B\cap E_j$ can be covered by

$$C\prod_{i:2r\le(\lambda_i^{n_j}-1)^{-1}\psi(n_j)}\lceil\frac{2r}{\ell}\rceil\prod_{i:2r>(\lambda_i^{n_j}-1)^{-1}\psi(n_j)>\ell}\lceil\frac{(\lambda_i^{n_j}-1)^{-1}\psi(n_j)}{\ell}\rceil$$

balls with radius ℓ, and by Lemma 2.6, if $\ell>(\lambda_1^{n_{j+1}}-1)^{-1}$, then any ball with radius ℓ contains about $\ell^dH_{n_{j+1}}$ periodic points in $\mathcal{P}_{n_{j+1}}$. Take $\ell=\psi(n_j)(\lambda_d^{n_j}-1)^{-1}$, then

$$\begin{aligned}\#\mathcal{C}_{j+1}(2B\cap E_j)&\lesssim\ell^dH_{n_{j+1}}\prod_{i:2r<(\lambda_i^{n_j}-1)^{-1}\psi(n_j)}\lceil\frac{r}{\ell}\rceil\prod_{i:2r\ge(\lambda_i^{n_j}-1)^{-1}\psi(n_j)>\ell}\lceil\frac{(\lambda_i^{n_j}-1)^{-1}\psi(n_j)}{\ell}\rceil\\&=\psi(n_j)^d(\lambda_d^{n_j}-1)^{-d}H_{n_{j+1}}\prod_{i=1}^k\frac{r(\lambda_d^{n_j}-1)}{\psi(n_j)}\prod_{i=k+1}^d\frac{(\lambda_d^{n_j}-1)}{(\lambda_i^{n_j}-1)}.\end{aligned}$$

By Corollary 2.8,

$$\#\mathcal{C}_{j+1}(E_j) \asymp \psi(n_j)^d H_{n_{j+1}} H_{n_j}^{-1}.$$

It follows that

$$\frac{\#\mathcal{C}_{j+1}(2B\cap E_j)}{\#\mathcal{C}_{j+1}(E_j)} \lesssim \prod_{i=1}^{k} \frac{r(\lambda_i^{n_j}-1)}{\psi(n_j)}.$$

Combining the assumption that $\psi(n)=\mathrm{e}^{-\alpha n}$, it derives from these estimates, (4.3) and (4.4) that

$$\begin{aligned}\frac{\log\mu(B)}{\log r} &\geq \frac{1}{\log r}\Big(\log\mu(E_j)+\log\frac{\#\mathcal{C}_{j+1}(2B\cap E_j)}{\#\mathcal{C}_{j+1}(E_j)}\Big)\\ &\geq \frac{1}{\log r}\Big(-\log H_{n_j}+d\alpha\sum_{i=1}^{j-1}n_i+k\log r+\sum_{i=1}^{k}\log(\lambda_i^{n_j}-1)-\sum_{i=1}^{k}\log\psi(n_j)+O(j)\Big)\\ &= k+\frac{1}{\log r}\Big(d\alpha\sum_{i=1}^{j-1}n_i-\sum_{i=k+1}^{d}\log(\lambda_i^{n_j}-1)+\sum_{i=1}^{k}\alpha n_j+O(j)\Big)\\ &= k+\frac{1}{\log r}\Big(d\alpha\sum_{i=1}^{j-1}n_i-n_j\sum_{i=k+1}^{d}\log\lambda_i+k\alpha n_j+O(j+1)\Big).\end{aligned}$$

Notice that if $k\alpha-\sum_{i=k+1}^{d}\log\lambda_i>0$, we have

$$\frac{\log\mu(B)}{\log r}\geq k+\frac{n_j(k\alpha-\sum_{i=k+1}^{d}\log\lambda_i)}{\log((\lambda_k^{n_j}-1)^{-1}\psi(n_j))}+\frac{1}{\log r}\big(d\alpha\sum_{i=1}^{j-1}n_i+O(j+1)\big).$$

Note that For $\epsilon>0$, there is some constant $j_1>1$ such that $j\geq j_1$

$$\frac{n_j(k\alpha-\sum_{i=k+1}^{d}\log\lambda_i)}{\log((\lambda_k^{n_j}-1)^{-1}\psi(n_j))}\geq -\frac{k\alpha-\sum_{i=k+1}^{d}\log\lambda_i}{\alpha+\log\lambda_k}-\frac{\epsilon}{2}.$$

and by equation (4.1),

$$\frac{1}{\log r}\big(d\alpha\sum_{i=1}^{j-1}n_i+O(j+1)\big)\gtrsim -\frac{1}{n_j(\alpha+\log\lambda_k)}\big(d\alpha\sum_{i=1}^{j-1}n_i+O(j+1)\big)$$

increasingly tends to 0, as $j\to\infty$, hence there is some constant $j_2>1$ such that for all $j\geq j_2$,

$$\frac{1}{\log r}\Big(d\alpha\sum_{i=1}^{j-1}n_i+O(j+1)\Big)\geq -\frac{\epsilon}{2}.$$

Combining these inequalities, we have

$$\frac{\log\mu(B)}{\log r}\geq \frac{k\log\lambda_k+\sum_{i=k+1}^{d}\log\lambda_i}{\alpha+\log\lambda_k}-\epsilon \tag{4.6}$$

for all $j>\max\{j_1,j_2\}$.

If $k\alpha-\sum_{i=k+1}^{d}\log\lambda_i\leq 0$, then for j large enough, we get

$$\begin{aligned}\frac{\log\mu(B)}{\log r}&\geq k+\frac{1}{\log((\lambda_{k+1}^{n_j}-1)^{-1}\psi(n_j))}\Big(k\alpha n_j-n_j\sum_{i=k+1}^{d}\log\lambda_i\Big)-\frac{\epsilon}{2}\\ &\geq k-\frac{k\alpha-\sum_{i=k+1}^{d}\log\lambda_i}{\alpha+\log\lambda_{k+1}}-\epsilon=\frac{(k+1)\log\lambda_{k+1}-\sum_{i=k+2}^{d}\log\lambda_i}{\alpha+\log\lambda_{k+1}}-\epsilon.\end{aligned} \tag{4.7}$$

It follows from inequalities (4.6) and (4.7) that

$$\frac{\log\mu(B)}{\log r}\ge\min\Big\{\frac{(k+1)\log\lambda_{k+1}-\sum_{i=k+2}^{d}\log\lambda_i}{\alpha+\log\lambda_{k+1}},\frac{k\log\lambda_k-\sum_{i=k+1}^{d}\log\lambda_i}{\alpha+\log\lambda_k}\Big\}-\epsilon.$$

Now we consider Case (**B**): $(\lambda_1^{n_j}-1)^{-1}\psi(n_j)\le r<(\lambda_d^{n_{j-1}}-1)^{-1}\psi(n_{j-1})$.

Since $2B$ does not intersect ellipsoids of degree n_{j-1} except E_{j-1}, we have

$$\mu(B)\le\sum_{\substack{m\in\mathcal{C}_j\\ R_{n_j,m}\cap B\neq\varnothing}}\mu(R_{n_j,m})\le\sum_{\substack{m\in\mathcal{C}_j\\ R_{n_j,m}\subset 2B}}\mu(R_{n_j,m})\le(\#\mathcal{C}_j(2B\cap E_{j-1}))\mu(R_{n_j,m}).$$

There are three cases to consider.

Case (i) : $r\ge(\lambda_1^{n_j}-1)^{-1}$.

Applying Lemma 2.6, $\#\mathcal{C}_j(2B\cap E_{j-1})\asymp r^dH_{n_j}$. By (4.3) and (4.4),

$$\log\mu(R_{n_j,m})=-\log H_{n_j}+d\alpha\sum_{i=1}^{j-1}n_i+O(j).$$

Therefore

$$\begin{aligned}\frac{\log\mu(B)}{\log r}&\ge d+\frac{1}{\log r}\Big(d\alpha\sum_{i=1}^{j-1}n_i+O(j)\Big)\ge d+\frac{d\alpha\sum_{i=1}^{j-1}n_i+O(j)}{\log\lambda_d^{-n_{j-1}}\psi(n_{j-1})}\\&\ge d-\frac{d\alpha}{\alpha+\log\lambda_d}-\epsilon=\frac{d\log\lambda_d}{\alpha+\log\lambda_d}-\epsilon.\end{aligned}$$

Case (ii) : $(\lambda_1^{n_j}-1)^{-1}\psi(n_j)\le r<(\lambda_d^{n_j}-1)^{-1}$.

By Lemma 2.6, $\#\mathcal{C}_j(2B\cap E_{j-1})\le1$, hence $\mu(B)\le\mu(E_j)$ which derives that for j large enough

$$\begin{aligned}\frac{\log\mu(B)}{\log r}&\ge\frac{1}{\log r}(-\log H_{n_j}+d\alpha\sum_{i=1}^{j-1}n_i+O(j))\\&\ge\frac{\sum_{i=1}^{d}\log(\lambda_i^{n_j}-1)}{\log((\lambda_1^{n_j}-1)^{-1}\psi(n_j))}-\epsilon\ge\frac{\sum_{i=1}^{d}\log\lambda_i}{\log\lambda_1+\alpha}-\epsilon.\end{aligned}$$

Case (iii) : There exists $1\le k\le d-1$ such that

$$(\lambda_{k+1}^{n_j}-1)^{-1}\le r<(\lambda_k^{n_j}-1)^{-1}.$$

By Lemma 2.6, $\#\mathcal{C}_j(2B\cap E_{j-1})\lesssim\prod_{i=k+1}^{d}(\lambda_i^{n_j}-1)r$, which derives that

$$\mu(B)\le\mu(R_{n_j,m})\prod_{i=k+1}^{d}(\lambda_i^{n_j}-1)r.$$

Then

$$\begin{aligned}\frac{\log\mu(B)}{\log r}&\ge\frac{1}{\log r}\Big(-\log H_{n_j}+d\alpha\sum_{i=1}^{j-1}n_i+O(j)+\sum_{i=k+1}^{d}\log(\lambda_1^{n_j}-1)+(d-k)\log r\Big)\\&=d-k+\frac{1}{\log r}\Big(-\sum_{i=1}^{k}\log(\lambda_i^{n_j}-1)+d\alpha\sum_{i=1}^{j-1}n_i+O(j)\Big)\\&\ge\frac{(d-k)(\alpha+\log\lambda_1)+\sum_{i=1}^{k}\log\lambda_i}{\alpha+\log\lambda_1}-\epsilon\end{aligned}$$

$$= \frac{\sum_{i=1}^{d} \log \lambda_i + \sum_{i=k+1}^{d} (\alpha - \log(\lambda_i/\lambda_1))}{\alpha + \log \lambda_1} - \epsilon \geq \frac{\sum_{i=1}^{d} \log \lambda_i}{\alpha + \log \lambda_1} - \epsilon,$$

where the last inequality holds since $\alpha > \log(\lambda_d/\lambda_1)$.

Combining Case (**A**) and Case (**B**), for j large enough, we have

$$\frac{\log \mu(B)}{\log r} \geq \min_k \Big\{ \frac{k \log \lambda_k + \sum_{i=k+1}^{d} \lambda_i}{\alpha + \log \lambda_k} - \epsilon \Big\}.$$

Letting $\epsilon \to 0$ and applying the mass distribution principle [8, Proposition 2.3], we have

$$\dim_{\mathrm{H}} R(\psi) \geq \min_k \Big\{ \frac{k \log \lambda_k + \sum_{i=k+1}^{d} \lambda_i}{\alpha + \log \lambda_k} \Big\}. \tag{4.8}$$

Now we deal with the case that $\alpha = \log(\lambda_d/\lambda_1)$. Given $\eta > \log(\lambda_d/\lambda_1)$, define $\psi_\eta: \mathbb{R}^+ \to \mathbb{R}^+$ as

$$\psi_\eta \colon x \mapsto \mathrm{e}^{-x\eta}.$$

There exists an infinite set $\mathcal{N}_\eta \subset \mathbb{N}$ such that

$$\psi(n) > \psi_\eta(n) = \mathrm{e}^{-n\eta}, \quad n \in \mathcal{N}_\eta.$$

Let

$$W_\eta := \{x \in \mathbb{T}^d \colon T^n x \in B(x, \psi_\eta(n)) \text{ i.m. } n \in \mathcal{N}_\eta\}.$$

Note that

$$\liminf_{\substack{n \to \infty \\ n \in \mathcal{N}_\eta}} \frac{-\log \psi_\eta(n)}{n} = \eta,$$

and

$$W_\eta \subset R(\psi).$$

From these and inequality (4.8), we have

$$\dim_{\mathrm{H}} R(\psi) \geq \dim_{\mathrm{H}} W_\eta \geq \min_k \Big\{ \frac{k \log \lambda_k + \sum_{i=k+1}^{d} \lambda_i}{\eta + \log \lambda_k} \Big\}$$

for any $\eta > \log(\lambda_d/\lambda_1)$. Letting $\eta \to \alpha = \log(\lambda_d/\lambda_1)$, we get

$$\dim_{\mathrm{H}} R(\psi) \geq \min_k \Big\{ \frac{k \log \lambda_k + \sum_{i=k+1}^{d} \lambda_i}{\alpha + \log \lambda_k} \Big\}.$$

Next we consider the case $\alpha = \infty$. Given $M > 0$, define $\psi_M: x \mapsto \mathrm{e}^{-xM}$. Then for n large enough, we have

$$\psi(n) < \psi_M(n) = \mathrm{e}^{-nM}.$$

Let

$$W_M := \{x \in \mathbb{T}^d \colon T^n x \in B(x, \psi_M(n)) \text{ i.m. } n \geq 1\}.$$

Since $W_M \supset R(\psi)$, by Section 3, it follows that

$$0 \leq \dim_{\mathrm{H}} R(\psi) \leq \dim_{\mathrm{H}} W_M \leq \min_k \Big\{ \frac{k \log \lambda_k + \sum_{i=k+1}^{d} \lambda_i}{M + \log \lambda_k} \Big\}.$$

Letting $M \to \infty$, we get

$$\dim_{\mathrm{H}} R(\psi) = 0.$$

Then combining these with Section 3, we finish the proof of Theorem 1.2.

5 The proof of Theorem 1.5

Since A is diagonalizable over $\mathbb{Q}$, there is an invertible matrix P such that

$$A = P^{-1}DP,$$

where $D = \mathrm{diag}\{\lambda_1, \cdots, \lambda_{\mathrm{d}}\}$ is a diagonal matrix. The matrix P consists of d^2 rational numbers. Denote by β the least common multiple of the denominators of elements of P. Write $\widetilde{P} = \beta P$, and $\widetilde{P}$ is an integer matrix. Then

$$A = \widetilde{P}^{-1}D\widetilde{P}.$$

Define $f\colon \mathbb{T}^d \to \mathbb{T}^d$ as

$$f\colon x \mapsto \widetilde{P}x \pmod 1.$$

Then $f \circ T = D \pmod 1 \circ f$. Since $\widetilde{P}$ is nonsingular, its singular values $e_1, \cdots, e_d$ are strictly larger than 0. Denote

$$e_{min} = \min\{e_1, \cdots, e_d\}, \quad e_{max} = \max\{e_1, \cdots, e_d\}.$$

Then

$$e_{min}^d|x-y| \le |f(x) - f(y)| \le e_{max}^d|x-y|. \tag{5.1}$$

It shows that f is a bi-Lipschitz mapping.

For $n \ge 1$,

$$\begin{aligned} R_n(\psi) &= \{x \in \mathbb{T}^d\colon (A^n - I)x \pmod 1 \in B(0, \psi(n))\} \\ &= \{x \in \mathbb{T}^d\colon \widetilde{P}^{-1}(D^n - I)\widetilde{P}x \pmod 1 \in B(0, \psi(n))\} \\ &= f^{-1}\{y \in \mathbb{T}^d\colon (D^n - I)y \pmod 1 \in f(B(0, \psi(n)))\} =: f^{-1}E_n. \end{aligned}$$

By inequalities (5.1),

$$B(0, e_{min}^d\psi(n)) \subset f(B(0, \psi(n))) \subset B(0, e_{max}^d\psi(n)).$$

Hence $\dim_{\mathrm{H}}\big(\limsup\limits_{n\to\infty} E_n\big)$ equals to the Hausdorff dimension of

$$\limsup_{n\to\infty}\Big\{y \in \mathbb{T}^d\colon (D^n - I)y \pmod 1 \in B(0, a\psi(n))\Big\} =: \limsup_{n\to\infty} E'_n$$

for any given constant $a > 0$. It implies that

$$\dim_{\mathrm{H}}\big(\limsup_{n\to\infty} R_n(\psi)\big) = \dim_{\mathrm{H}}\big(\limsup_{n\to\infty} f(R_n(\psi))\big) = \dim_{\mathrm{H}}\big(\limsup_{n\to\infty} E'_n\big).$$

Note that

$$\liminf_{n\to\infty}\frac{-\log(a\psi(n))}{n} = \alpha.$$

Combining the fact that D is an integer matrix and Theorem 1.7 in [10], we have

$$\dim_{\mathrm{H}} R(\psi) = \min_{1\le j\le d}\Big\{\frac{j\log|\lambda_j| + \sum_{k\in\mathcal{K}(j)}(\alpha + \log|\lambda_j| - \log|\lambda_i|) + \sum_{i=j+1}^d \log|\lambda_i|}{\log|\lambda_j| + \alpha}\Big\},$$

where

$$\mathcal{K}(j) := \{1 \le i \le d\colon\ \log|\lambda_i| > \log|\lambda_j| + \alpha\}.$$

Conflict of Interest The authors declare no conflict of interest.

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IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **37** (2024) 055010 (22pp) https://doi.org/10.1088/1361-6544/ad3597

Hausdorff dimension of recurrence sets

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Received 10 March 2023; revised 5 January 2024
Accepted for publication 19 March 2024
Published 2 April 2024

Recommended by Dr Lorenzo J Diaz

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Abstract
We consider linear mappings on the 2-dimensional torus, defined by $T(x) = Ax \pmod 1$, where A is an invertible 2×2 integer matrix, with no eigenvalues on the unit circle. In the case $\det A = \pm 1$, we give a formula for the Hausdorff dimension of the set

$$\left\{x \in \mathbb{T}^2 : d\left(T^n\left(x\right), x\right) < e^{-\alpha n} \text{ for infinitely many } n\right\}.$$

Keywords: Hausdorff dimension, recurrence, linear maps

Mathematics Subject Classification numbers: 37C45, 37D20, 28A80

1. Introduction

Let $(X, \mathscr{B}, T, \mu, d)$ be a metric measure preserving system (m.m.p.s.). If (X, d) is a separable metric space, then the well-known Poincaré recurrence theorem shows that μ-a.e. $x \in X$ is recurrent, that is

$$\liminf_{n\to\infty} d\left(T^n x, x\right) = 0.$$

It tells us that for μ-almost every $x \in X$, the orbit returns to a sequence of shrinking targets of the initial point infinitely many times. However, the theorem tells us nothing about the speed at which the orbit can return to the initial point or the shrinking targets of the the initial point. Boshernitzan [3] investigated the rate of recurrence for general systems.

Theorem 1.1 ([3]). *Let $(X, \mathscr{B}, T, \mu, d)$ be a m.m.p.s. Assume that for some $\tau > 0$, the τ-dimensional Hausdorff measure $\mathcal{H}^\tau$ of X is σ-finite. Then for μ-a.e. $x \in X$,*

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$$\liminf_{n\to\infty} n^{\frac{1}{\tau}} d(T^n x, x) < \infty.$$

Futhermore, if $\mathcal{H}^\tau(X) = 0$*, then for* μ*-almost every* $x \in X$*,*

$$\liminf_{n\to\infty} n^{\frac{1}{\tau}} d(T^n x, x) = 0.$$

Later, Barreira and Saussol [2] related the rate of recurrence to the lower pointwise dimension.

Theorem 1.2 ([2]). *If* $T\colon X \to X$ *is a Borel measurable map on a measurable subset* $X \subset \mathbb{R}^m$*, and* μ *is a T-invariant Borel probability measure on X, then for* μ*-almost every* $x \in X$*, we have*

$$\liminf_{n\to\infty} n^{1/\tau} d(T^n x, x) = 0$$

holds for μ*-almost every* $x \in X$ *such that* $\tau > \underline{d}_\mu(x)$*, where*

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

Hence a natural question is how large is the set of recurrent points when the rate of recurrence is replaced by a general function. More precisely, let $(X, \mathscr{B}, T, \mu, d)$ be a m.m.p.s. and $r_n(x)$ be some positive function on $\mathbb{N} \times X$. Define the recurrence set as

$$E = E(r_n) = \{x \in X : T^n x \in B(x, r_n(x)) \text{ for infinitely many } n \geqslant 1\}.$$

Tan and Wang [16] calculated the Hausdorff dimension of $E(r_n)$ when T is the β-transformation with $\beta > 1$. Later, Seuret and Wang [17] proved a similar result for conformal iterated function systems. Chang *et al* [4] considered the recurrence set on a self-similar set with the strong separation condition. Baker and Farmer [1] generalised their results to finite conformal iterated function systems. Hussein *et al* [11] showed that the measure of $E(r_n)$ obeys a zero–full law for some conformal and expanding systems. Kirsebom *et al* [12] investigated the measure of $E(r_n)$ for a class of mixing interval maps and some linear maps on tori.

The recurrence set is a limsup set which often has a large intersection property, originally introduced by Falconer [6]. Given $s \in (0, m]$, he defined $\mathcal{G}^s(\mathbb{R}^m)$ to be the class of all G_δ sets F in $\mathbb{R}^d$ such that the Hausdorff dimension of any set in $\mathcal{G}^s(\mathbb{R}^m)$ is at least s, and closed under similarity transformations and countable intersections. Defining the corresponding class of sets on the d-dimensional torus $\mathbb{T}^d$ is straightforward.

Persson and Reeve [14] used Riesz potentials to determine if a limsup set belongs to the class $\mathcal{G}^s(\mathbb{T}^m)$. The following lemma is important for the proof of our results. The Lebesgue measure on $\mathbb{T}^d$ is denoted by $\mathcal{L}$.

Lemma 1.3 (lemma 2.1 in [15]. See also [14]). *Let* E_n *be open sets in* $\mathbb{T}^m$ *and let* μ_n *be measures with* $\mu_n(\mathbb{T}^m \setminus E_n) = 0$*. If there is a constant* $C > 1$ *such that*

$$C^{-1} \leqslant \liminf_{n\to\infty} \frac{\mu_n(B)}{\mathcal{L}(B)} \leqslant \limsup_{n\to\infty} \frac{\mu_n(B)}{\mathcal{L}(B)} \leqslant C$$

for any ball B, and

$$\iint |x-y|^{-s} d\mu_n(x)\, d\mu_n(y) < C$$

for all n, then $\limsup_{n\to\infty} E_n \in \mathcal{G}^s(\mathbb{T}^m)$*, and in particular we have* $\dim_{\mathrm{H}}(\limsup_{n\to\infty} E_n) \geqslant s$*.*

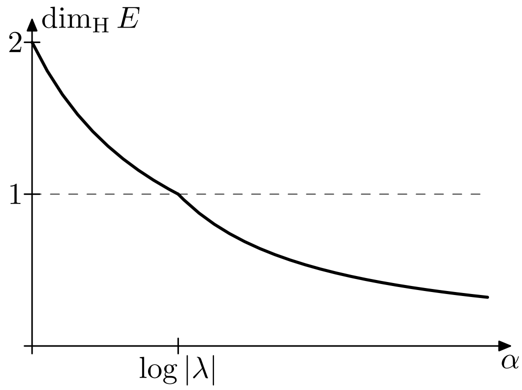

Figure 1. The graph of $\dim_{\mathrm{H}} E$ as a function of α, with $A = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$..

Motivated by above results, we focus on the Hausdorff dimension of $E(r_n)$ when T is a linear mapping on $\mathbb{T}^2$ and the rate r_n does not depend on the initial point, that is, the set

$$E = \left\{ x \in \mathbb{T}^2 : T^n(x) \in B(x, r_n) \text{ for infinitely many } n \geqslant 1 \right\}.$$

Theorem 1.4. *Let A be a 2×2 integer matrix with $|\det A| = 1$ and an eigenvalue $|\lambda| > 1$. Let $T(x) = Ax \pmod 1$, and for $n \geqslant 1$, $r_n = e^{-\alpha n}$, $\alpha > 0$. Then*

$$\dim_{\mathrm{H}} E = s_0,$$

where

$$s_0 = \min\left\{ \frac{2\log|\lambda|}{\alpha + \log|\lambda|}, \frac{\log|\lambda|}{\alpha} \right\}.$$

Moreover, for $\alpha > 0$, we have $E \in \mathcal{G}^{s_0}(\mathbb{T}^2)$.

By Theorem 1.4, the dimension formula of E is a continuous and decreasing function as a function of α. Figure 1 shows the graph of $\dim_{\mathrm{H}} E$ as a function of α when $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. He and Liao [10, theorem 1.7] gave a formula for the dimension of E when A is a diagonal matrix, not necessarily integer, and with all diagonal elements of modulus larger than 1. Our result seems to be the first result of this type when there is both contraction and expansion present.

Our result is only for the two dimensional case, $\mathbb{T}^2$. It is however natural to expect that a similar result holds in any dimension. To get an upper bound on the Hausdorff dimension of E, our method works well in any dimension. However, to get a lower bound on the Hausdorff dimension on E, we need to get good control on the periodic points of T. We have only been able to achieve such control when the dimension of the space is at most 2. For similar reasons, we have only considered the case when A is an integer matrix.

Remark 1.5. Theorem 1.4 does not only hold for $r_n = e^{-\alpha n}$. With the following adjustments, the result is valid for more general non-increasing sequences $\{r_n\}$. Let

$$\alpha = \liminf_{n \to \infty} \frac{-\log r_n}{n}.$$

With the definition of α and [13, lemma 9], theorem 1.4 holds for a general non-increasing sequence of positive real numbers $\{r_n\}_{n\geqslant 1}$.

Let A and $\{r_n\}_{n\geqslant 1}$ be as in theorem 1.4. Rewrite the recurrence set as

$$E = \left\{ x \in \mathbb{T}^2 : A^n x \pmod 1 \in B(x, r_n) \text{ i.o.} \right\},$$

and let

$$\begin{aligned} \mathcal{E}_n &:= \left\{ x \in \mathbb{T}^2 : A^n x \pmod 1 \in B(x, r_n) \right\} \\ &= \left\{ x \in \mathbb{T}^2 : (A^n - I) x \pmod 1 \in B(0, r_n) \right\}, \end{aligned}$$

where I is the identity matrix. Then $E = \limsup_{n\to\infty} \mathcal{E}_n$.

We may rewrite $\mathcal{E}_n$ as

$$\mathcal{E}_n = \left\{ x \in \mathbb{T}^2 : x \in (A^n - I)^{-1} B(0, r_n) + (A^n - I)^{-1} \mathbb{Z}^2 \right\}. \tag{1.1}$$

Therefore, $\mathcal{E}_n$ consists of a union of elliptical discs $(A^n - I)^{-1} B(0, r_n)$ translated by the vectors in $(A^n - I)^{-1}\mathbb{Z}^2$. We denote the elliptical discs in $\mathcal{E}_n$ by $\{\mathcal{E}_n^i\}_i$.

A point $x \in \mathbb{T}^2$ is called a *periodic point* with period n, or an *n-periodic point*, if $T^n(x) = x$, or equivalenty if

$$(A^n - I) x \pmod 1 = 0. \tag{1.2}$$

Therefore, $x \in \mathbb{T}^2$ is an n-periodic point if and only if $x \in (A^n - I)^{-1}\mathbb{Z}^2$, and we see that the centres of the elliptical discs in $\mathcal{E}_n$ are exactly the n-periodic points.

The paper is organised as follows. In next section, we will discuss the periodic points of $T(x) = Ax \pmod 1$,

$$\left\{ x \in \mathbb{T}^2 : (A^n - I) x \pmod 1 = 0 \right\},$$

which is crucial to our proof, since in order to understand $\mathcal{E}_n$, we need to understand the distribution of these periodic points. The proof of our main result is divided into two parts. In section 3, we give the upper bound on the Hausdorff dimension of E. In the last section we prove that E has a large intersection property, which gives the lower bound on the Hausdorff dimension of E.

Without loss of generality, we only prove theorem 1.4 for $\det A = 1$ and $\lambda > 1$, since for $\lambda < -1$ or $\det A = -1$, we can consider A^2 instead of A, whose eigenvalues are $\lambda^2 > 1$ and λ^{-2}, and $\det A^2 = 1$. Hence we omit the proof for other cases.

Notation 1.6. *Write $f_n \lesssim g_n$, $n \in \mathbb{N}$, if there is an absolute constant $0 < c < \infty$ such that for all $n \in \mathbb{N}$, $f_n \leqslant c g_n$. If $f_n \lesssim g_n$ and $g_n \lesssim f_n$ for $n \in \mathbb{N}$, then we write $f_n \asymp g_n$.*

2. Periodic points

Let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ be an integer matrix with $\det A = 1$ and eigenvalue $\lambda > 1$. Write $\gamma = \frac{\lambda - a}{b}$ and $\beta = \frac{\lambda^{-1} - a}{b}$. Then

$$\begin{bmatrix} 1 \\ \gamma \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ \beta \end{bmatrix}$$

are eigenvectors with eigenvalues λ and λ^{-1}. We can diagonalise A as $A = TDT^{-1}$, where

$$T = \begin{bmatrix} 1 & 1 \\ \gamma & \beta \end{bmatrix}, \; T^{-1} = \frac{1}{\beta - \gamma} \begin{bmatrix} \beta & -1 \\ -\gamma & 1 \end{bmatrix} \text{and } D = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}.$$

Therefore, for $n \geqslant 1$,

$$\begin{aligned} A^n &= TD^nT^{-1} \\ &= \frac{1}{\beta - \gamma} \begin{bmatrix} \beta\lambda^n - \gamma\lambda^{-n} & -\lambda^n + \lambda^{-n} \\ \gamma\beta\left(\lambda^n - \gamma^{-n}\right) & -\gamma\lambda^n + \beta\lambda^{-n} \end{bmatrix}. \end{aligned}$$

It follows that

$$A^n - I = \begin{bmatrix} \frac{\beta\lambda^n - \gamma\lambda^{-n}}{\beta - \gamma} - 1 & \frac{-\lambda^n + \lambda^{-n}}{\beta - \gamma} \\ \frac{\gamma\beta\left(\lambda^n - \lambda^{-n}\right)}{\beta - \gamma} & \frac{-\gamma\lambda^n + \beta\lambda^{-n}}{\beta - \gamma} - 1 \end{bmatrix}.$$

Put

$$H_n := \det\left(A^n - I\right).$$

From the above expressions of $A^n - I$, we get

$$\left(A^n - I\right)^{-1} = \frac{1}{H_n} \begin{bmatrix} \frac{-\gamma\lambda^n + \beta\lambda^{-n}}{\beta - \gamma} - 1 & \frac{\lambda^n - \lambda^{-n}}{\beta - \gamma} \\ \frac{\gamma\beta\left(-\lambda^n + \lambda^{-n}\right)}{\beta - \gamma} & \frac{\beta\lambda^n - \gamma\lambda^{-n}}{\beta - \gamma} - 1 \end{bmatrix}. \tag{2.1}$$

We will use this formula in the proof of lemma 2.4.

In this section, we investigate the periodic points in order to understand the distribution of the elliptical discs $\{\mathcal{E}_n^i\}_i$.

2.1. The number of periodic points

Lemma 2.1. *For every $n \geqslant 1$, $H_n = 2 - \lambda^n - \lambda^{-n}$, and H_n is an integer.*

Proof. Since $\det A^n = 1$, we have $H_n = 2 - \operatorname{tr} A^n = 2 - \lambda^n - \lambda^{-n}$. □

Lemma 2.2 (lemma 2.3 in [5]). *If M is a $d \times d$ integer matrix with no eigenvalue being a root of unity, and $T_M(x) = Mx \pmod 1$, then the number of periodic points with period n under T_M is $|\det(M^n - I)|$. In particular*

$$\#\left\{x \in \mathbb{T}^2 : (A^n - I)x \pmod 1 = 0\right\} = |H_n|.$$

2.2. Periodic points when n is odd

To obtain the lower bound on $\dim_{\mathrm{H}} E$, it suffices to study the distribution of periodic points when n is odd, since $\limsup_{k\to\infty} \mathcal{E}_{2k+1} \subset \limsup_{k\to\infty} \mathcal{E}_k$. In the following, we only consider the case when n is odd. Assume that $n = 2k+1$. Put

$$S_k = 1 + \operatorname{tr} A + \ldots + \operatorname{tr} A^k = 1 + \operatorname{tr} D + \ldots + \operatorname{tr} D^k.$$

Lemma 2.3. *For every $n = 2k+1$, $H_n = -(tr\, A - 2)S_k^2$.*

Proof. For $k \geqslant 0$, it follows that

$$S_k = \sum_{j=-k}^{k} \lambda^j = \frac{\lambda^{-k} - \lambda^{k+1}}{1-\lambda}. \tag{2.2}$$

Since $\det(A-I) = \det(D-I) = 2 - \operatorname{tr} D = 2 - \operatorname{tr} A$, for every $k \geqslant 0$,

$$\begin{aligned} H_n = \det\left(A^{2k+1} - I\right) &= \det(A-I)\det\left(I + A + \ldots + A^{2k}\right) \\ &= (2 - \operatorname{tr} A)\det\left(I + D + \ldots + D^{2k}\right). \end{aligned}$$

It remains to prove that $S_k^2 = \det(I + D + \ldots + D^{2k})$. Using the first equality of (2.2), we find that

$$\begin{aligned} \det\left(I + D + \ldots + D^{2k}\right) &= \left(\sum_{j=0}^{2k} \lambda^j\right)\left(\sum_{j=0}^{2k} \lambda^{-j}\right) \\ &= \left(\lambda^k \sum_{j=-k}^{k} \lambda^j\right)\left(\lambda^{-k} \sum_{j=-k}^{k} \lambda^j\right) = \left(\sum_{j=-k}^{k} \lambda^j\right)^2 = S_k^2, \end{aligned}$$

and this finishes the proof. □

Let $S_k' = (\operatorname{tr} A - 2)S_k$. Since we have assumed $\lambda > 1$, we have $\operatorname{tr} A - 2 > 0$ so that $S_k' > 0$. Let

$$\mathscr{P}_o = \left\{ \left(\frac{m}{S_k'}, \frac{j}{S_k'} \right) : 0 \leqslant m, j \leqslant S_k' - 1 \right\}.$$

Lemma 2.4. *For $n = 2k+1$ with $k \geqslant 0$, the n-periodic points are all contained in $\mathscr{P}_o$.*

Proof. We prove that $B_{2k+1} := (\operatorname{tr} A - 2)S_k(A^{2k+1} - I)^{-1}$ is an integer matrix. (Recall that $(\operatorname{tr} A - 2) > 0$.) If B_{2k+1} is an integer matrix, then any periodic point $x = (A^{2k+1} - I)^{-1}[m_1\ m_2]^t$ for $m_1, m_2 \in \mathbb{Z}$, can be rewritten as

$$x = \frac{1}{S_k'} B_{2k+1} [m_1\ m_2]^t.$$

Since $B_{2k+1}[m_1\ m_2]^t$ is an integer vector, this implies that $x \in \mathscr{P}_o$.

Now we prove that the matrix B_{2k+1} is integral. Since $H_n = -(\operatorname{tr} A - 2)S_k^2$, the formula (2.1) allows us to compute

$$B_{2k+1} = -\frac{1}{S_k} \begin{bmatrix} \frac{-\gamma\lambda^{2k+1} + \beta\lambda^{-2k-1}}{\beta - \gamma} - 1 & \frac{\lambda^{2k+1} - \lambda^{-2k-1}}{\beta - \gamma} \\ \frac{\gamma\beta\left(-\lambda^{2k+1} + \lambda^{-2k-1}\right)}{\beta - \gamma} & \frac{\beta\lambda^{2k+1} - \gamma\lambda^{-2k+1}}{\beta - \gamma} - 1 \end{bmatrix},$$

and since $\det(D^n - I) = H_n$, we also have

$$(\operatorname{tr} A - 2)\, S_k \left(D^{2k+1} - I\right)^{-1} = \frac{-1}{S_k} \begin{bmatrix} \lambda^{-2k-1} - 1 & 0 \\ 0 & \lambda^{2k+1} - 1 \end{bmatrix}. \tag{2.3}$$

Put

$$R_k := (-1)^k + (-1)^{k-1} \text{ tr } A + \ldots - \text{ tr } A^{k-1} + \text{ tr} A^k$$

which is an integer, and observe that

$$\begin{aligned} R_k &= \lambda^k - \lambda^{k-1} + \ldots + (-1)^k + (-1)^{k-1} \lambda^{-1} + \ldots - \lambda^{-k+1} + \lambda^{-k} \\ &= \frac{\lambda^{k+1} + \lambda^{-k}}{1+\lambda}. \end{aligned} \tag{2.4}$$

By the definition of B_{2k+1}, (2.2) and (2.3),

$$\begin{aligned} \operatorname{tr} B_{2k+1} &= \operatorname{tr}\left(\left(\operatorname{tr} A - 2\right) S_k \left(D^{2k+1} - I\right)^{-1}\right) \\ &= -\frac{\lambda^{2k+1} - 1}{S_k} - \frac{\lambda^{-2k-1} - 1}{S_k} \\ &= -\left(\lambda - 1\right) \lambda^k + \lambda^{-k-1} \frac{\lambda^{k+1} - \lambda^{-k}}{S_k} \\ &= \left(1 - \lambda\right)\left(\lambda^k - \lambda^{-k-1}\right). \end{aligned}$$

Moreover,

$$\begin{aligned} \det B_{2k+1} &= \det\left(\left(\operatorname{tr} A - 2\right) S_k \left(D^{2k+1} - I\right)^{-1}\right) \\ &= \frac{\left(\lambda^{2k+1} - 1\right)\left(\lambda^{-2k-1} - 1\right)}{S_k^2} \\ &= \left(\lambda - 1\right)^2 \frac{2 - \lambda^{2k+1} - \lambda^{-2k-1}}{\left(\lambda^{k+1} - \lambda^{-k}\right)^2} \lambda^k = -\frac{\left(\lambda - 1\right)^2}{\lambda}. \end{aligned}$$

Hence, $\operatorname{tr} B_{2k+1} = \det(B_{2k+1}) S_k$. Since $\det B_{2k+1}$ does not depend on k, we have $\det B_{2k+1} = \det B_1 = \det(A - I) = 2 - \operatorname{tr} A$, which is an integer. It follows that $\operatorname{tr} B_{2k+1} = (2 - \operatorname{tr} A) S_k$ is an integer as well.

Finally, we let $e_1 = [1 \quad 0]^t$ and $e_2 = [0 \quad 1]^t$, compute $e_2^t B_{2k+1} e_2$, $e_2^t B_{2k+1} e_1$ and $e_1^t B_{2k+1} e_2$ and show that they are integers. Recalling $\gamma = \frac{\lambda - a}{b}$, $\beta = \frac{\lambda^{-1} - a}{b}$, after some simplifications, we obtain that

$$\begin{aligned} e_2^t B_{2k+1} e_2 &= -\frac{1}{S_k}\left(\frac{\beta \lambda^{2k+1} - \gamma \lambda^{-2k-1}}{\beta - \gamma} - 1\right) \\ &= \frac{1}{1+\lambda}\left(\frac{\lambda^{2k+1} - \lambda^{-2k+1} + \lambda^2 - 1}{\lambda^{1+k} - \lambda^{-k}} + a \frac{\lambda^{-2k} - \lambda^{2k+2}}{\lambda^{1+k} - \lambda^{-k}}\right) \\ &= \frac{1}{1+\lambda}\left(\frac{\left(\lambda^{1+k} - \lambda^{-k}\right)\left(\lambda^k + \lambda^{-k+1}\right)}{\lambda^{1+k} - \lambda^{-k}} - a \frac{\lambda^{2k+2} - \lambda^{-2k}}{\lambda^{1+k} - \lambda^{-k}}\right) \\ &= \frac{1}{1+\lambda}\left(\lambda^k + \lambda^{-k+1} - a\left(\lambda^{k+1} + \lambda^{-k}\right)\right) \\ &= R_{k-1} - a R_k, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} e_1^t B_{2k+1} e_2 &= -\frac{1}{S_k} \frac{\lambda^{2k+1} - \lambda^{-2k-1}}{\beta - \gamma} \\ &= \frac{\lambda^{-1}(1-\lambda)}{\beta-\gamma}\left(\lambda^{k+1} + \lambda^{-k}\right) \\ &= \frac{\lambda^{-1}(1-\lambda)}{\beta-\gamma}(1+\lambda) R_k \\ &= bR_k. \end{aligned} \tag{2.6}$$

Similarly,

$$\begin{aligned} e_2^t B_{2k+1} e_1 &= -\frac{1}{S_k} \frac{\gamma\beta\left(\lambda^{-2k-1} - \lambda^{2k+1}\right)}{\beta - \gamma} \\ &= -\frac{\lambda^{-1}(1-\lambda)\gamma\beta}{(\beta-\gamma)(\lambda^{-k} - \lambda^{k+1})}\left(\lambda^{-2k} - \lambda^{2k+2}\right) \\ &= -\frac{\lambda^{-1}(1-\lambda)\gamma\beta}{\beta-\gamma}\left(\lambda^{k+1} + \lambda^{-k}\right) \\ &= -\frac{(1-\lambda)\gamma\beta}{\lambda(\beta-\gamma)}(1+\lambda) R_k. \end{aligned} \tag{2.7}$$

Notice that $e_2^t B_1 e_1 = c$, that is

$$c = \frac{\gamma\beta(\lambda-1)}{\lambda(\beta-\gamma)}(1+\lambda).$$

Because of this, (2.7) can be rewritten as

$$e_2^t B_{2k+1} e_1 = cR_k. \tag{2.8}$$

By (2.5), (2.6) and (2.8), the matrix elements $e_2^t B_{2k+1} e_1$, $e_1^t B_{2k+1} e_2$, $e_2^t B_{2k+1} e_2$ are all integers. Since $\operatorname{tr} B_{2k+1}$ is an integer, the lower right element of B_{2k+1} must be an integer as well. □

Lemma 2.5. *For $n = 2k+1$ with $k \geqslant 0$, all the points $\left\{ \left(\frac{m}{S_k}, \frac{j}{S_k}\right) : m, j = 0, \ldots, S_k - 1 \right\}$ are n-periodic points.*

Proof. Let us prove that $B'_{2k+1} := \frac{1}{S_k}(A^{2k+1} - I)$ is an integer matrix, with $\det B'_{2k+1} = 2 - \operatorname{tr} A$. This will imply that if $x = \frac{1}{S_k}[x_1\ x_2]^t$ where x_1 and x_2 are integers, then $(A^{2k+1} - I)x$ is an integer vector, and hence $T^{2k+1}(x) = x$.

Notice that

$$e_1^t B'_{2k+1} e_1 = -e_2^t B_{2k+1} e_2, \quad e_2^t B'_{2k+1} e_2 = -e_1^t B_{2k+1} e_1,$$

$$e_1^t B'_{2k+1} e_2 = e_1^t B_{2k+1} e_2, \quad e_2^t B'_{2k+1} e_1 = e_2^t B_{2k+1} e_1,$$

where $B_{2k+1} = (trA - 2) S_k (A^{2k+1} - I)^{-1}$. From the proof of lemma 2.4, it follows that the matrix elements of B'_{2k+1} are all integers. □

Remark 2.6. The number of periodic points found in lemma 2.5 is clearly S_k^2. This is enough for our needs, but we note that by lemma 2.2, the number of periodic points are $|H_{2k+1}|$. Recall from lemma 2.3 that

$$S_k^2 = -\frac{1}{\operatorname{tr} A - 2} H_{2k+1}.$$

Hence the periodic points found in lemma 2.5 are all the periodic points if and only if $\operatorname{tr} A - 2 = \pm 1$.

Remark 2.7. When n is even, that is, $n = 2k$, $k \geqslant 1$, the periodic points with period n are contained in

$$\mathscr{P}_e = \left\{ \left(\frac{m}{g_k}, \frac{j}{g_k} \right) : 0 \leqslant m, j \leqslant g_k - 1 \right\},$$

where $g_k = \sqrt{(a+d)^2 - 4}(\lambda^k - \lambda^{-k})$.

Notation 2.8. *We are going to prove that each ellipse $\mathcal{E}_n^i$ contains a parallelogram and is contained in a parallelogram. Both parallelograms are comparable in size. As we shall see, the semiaxes of the ellipses are comparable to $\lambda_{n,1}$ and $\lambda_{n,2}$ as follows. For $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ we write*

$$\begin{aligned}
\lambda_{n,1} &= r_n \frac{\lambda^n - 1}{|H_n|} = \frac{r_n}{1 - \lambda^{-n}}, \\
\lambda_{n,2} &= r_n \frac{1 - \lambda^{-n}}{|H_n|} = \frac{r_n}{\lambda^n - 1},
\end{aligned}$$

where $\gamma = \frac{\lambda - a}{b}$, $\beta = \frac{\lambda^{-1} - a}{b}$, and we let $e_{n,1} > e_{n,2}$ be the semiaxes of the ellipse $\mathcal{E}_n^i$.

We are going to show that $e_{n,i}$ are comparable to $\lambda_{n,i}$. Since each $\mathcal{E}_n^i$ is a translation of the ellipse $(A^n - I)^{-1} B(0, r_n)$, we can inscribe a parallelogram in $\mathcal{E}_n^i$ as follows. Consider the points

$$\pm \frac{r_n}{\sqrt{1+\gamma^2}} \begin{bmatrix} 1 \\ \gamma \end{bmatrix}, \qquad \pm \frac{r_n}{\sqrt{1+\beta^2}} \begin{bmatrix} 1 \\ \beta \end{bmatrix}.$$

which are boundary points of $B(0, r_n)$. Since the above vectors are eigenvectors of A it follows that the ellipse $(A^n - I)^{-1} B(0, r_n)$ has an inscribed parallelogram with vertices given by

$$\begin{aligned}
&\frac{1}{\sqrt{1+\gamma^2}} \begin{bmatrix} 1 \\ \gamma \end{bmatrix} \lambda_{n,2}, &\qquad -&\frac{1}{\sqrt{1+\gamma^2}} \begin{bmatrix} 1 \\ \gamma \end{bmatrix} \lambda_{n,2}, \\
&\frac{1}{\sqrt{1+\beta^2}} \begin{bmatrix} 1 \\ \beta \end{bmatrix} \lambda_{n,1}, &\qquad -&\frac{1}{\sqrt{1+\beta^2}} \begin{bmatrix} 1 \\ \beta \end{bmatrix} \lambda_{n,1},
\end{aligned}$$

In the same way, by replacing r_n by Cr_n for a sufficiently large C, we get a larger parallelogram in which the ellipse is contained. Therefore the quotients $e_{n,i}/\lambda_{n,i}$ are bounded from above and from below by constants independent of n and i.

We obtain the following lemma.

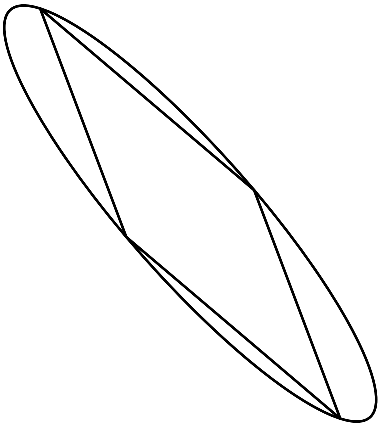

Figure 2. Illustration of an ellipse $\mathcal{E}_n^i$ and its inscribed parallelogram, for the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ when $n = 1$.

Lemma 2.9. *For $i \geqslant 1$, the elliptical disc $\mathcal{E}_n^i$, contains an inscribed parallelogram E_n^i with vertices*

$$x_{n,i} \pm \frac{1}{\sqrt{1+\gamma^2}} \begin{bmatrix} 1 \\ \gamma \end{bmatrix} \lambda_{n,2}, \qquad x_{n,i} \pm \frac{1}{\sqrt{1+\beta^2}} \begin{bmatrix} 1 \\ \beta \end{bmatrix} \lambda_{n,1},$$

whose centre $x_{n,i}$ is an n-periodic point. Also the lengths of the diagonal lines of E_n^i are $2\lambda_{n,1}$ and $2\lambda_{n,2}$.

There is a constant $c > 1$ such that

$$c^{-1} \leqslant \frac{e_{n,1}}{\lambda_{n,1}} \leqslant c, \qquad c^{-1} \leqslant \frac{e_{n,2}}{\lambda_{n,2}} \leqslant c$$

holds for all n.

The parallelograms in lemma 2.9 will be used in section 4 to get a lower bound on the Hausdorff dimension. They are illustrated in figure 2.

3. The upper bound on $\dim_{\mathrm{H}} E$

In this section, we will give the upper bound on $\dim_{\mathrm{H}} E$.

Lemma 3.1. *Let A be a 2×2 integer matrix with $\det A = 1$ and an eigenvalue $\lambda > 1$. Let $T(x) = Ax \pmod 1$, and for $n \geqslant 1$, $r_n = e^{-\alpha n}$, $\alpha \geqslant 0$. Then*

$$\dim_{\mathrm{H}} E \leqslant s_0,$$

where

$$s_0 = \min\left\{ \frac{2\log\lambda}{\alpha + \log\lambda}, \frac{\log\lambda}{\alpha} \right\}.$$

Proof. Recall that $\mathcal{E}_n$ consists of $|H_n|$ elliptical discs with same shape. For every $k \geqslant 1$, $\{\mathcal{E}_n\}_{n \geqslant k}$ is a cover of E and therefore, all the elliptical discs of $\mathcal{E}_n$ $(n \geqslant k)$ form a cover of E. We will get two upper bounds for $\dim_H E$ by using directly these elliptical discs to cover E, or using suitable squares to first cover each elliptical disc and then to cover E.

Case 1: For $\delta > 0$, there is some $k_0 \geqslant 1$ such that $r_n < \delta$ for all $n \geqslant k_0$. Since $\mathcal{E}_n^i \subset B(x_{n,i}, c\lambda_{n,1})$ for an absolute constant $c > 0$,

$$\mathcal{H}_\delta^s(E) \leqslant \sum_{n \geqslant k_0} \sum_{i=1}^{|H_n|} |\mathcal{E}_n^i|^s \lesssim \sum_{n \geqslant k_0} |H_n| \lambda_{n,1}^s. \tag{3.1}$$

It follows from the facts $\lambda_{n,1} \asymp r_n$ and $H_n \asymp \lambda^n$ that

$$\mathcal{H}_\delta^s(E) \lesssim \sum_{n \geqslant k_0} r_n^s \lambda^n.$$

Therefore, for $s > \frac{\log \lambda}{\alpha}$, we conclude that $\mathcal{H}^s(E) < \infty$, which implies that $\dim_H E \leqslant \frac{\log \lambda}{\alpha}$.

Case 2: For $i \geqslant 1$, $\mathcal{E}_n^i$ is contained in a rectangle, the length of whose sides are comparable to $\lambda_{n,1}$ and $\lambda_{n,2}$. From this $\mathcal{E}_n^i$ can be covered by

$$\Gamma_{n,i} := C_1 \frac{\lambda_{n,1}}{\lambda_{n,2}} \asymp \lambda^n.$$

squares with length of each side $\lambda_{n,2}$, where $C_1 > 0$ is a constant only depending on A. For any $\delta > 0$, there is some $n_0 \geqslant 1$ such that $\sqrt{2}\lambda_{n,2} < \delta$ for all $n \geqslant n_0$. Then

$$\mathcal{H}_\delta^s(E) \leqslant \sum_{n \geqslant k_0} \sum_{i=1}^{|H_n|} \Gamma_{n,i} \lambda_{n,2}^s \lesssim \sum_{n \geqslant k_0} \lambda^{2n} \left(r_n \lambda^{-n}\right)^s,$$

where the second inequality follows from $\lambda_{n,2} \asymp r_n \lambda^{-n}$. Therefore for any $s > \frac{2\log\lambda}{\alpha + \log\lambda}$, $\mathcal{H}^s(E) < \infty$, which implies that $\dim_H E \leqslant \frac{2\log\lambda}{\alpha+\log\lambda}$.

Combining Cases 1 and 2, we have

$$\dim_H E \leqslant \min\left\{ \frac{2\log\lambda}{\alpha + \log\lambda}, \frac{\log\lambda}{\alpha} \right\}.$$

□

4. Lower bound on Hausdorff dimension

Recall that the s-dimensional Riesz potential of a measure μ is defined by

$$R_s\mu(x) = \int |x-y|^{-s} \, \mathrm{d}\mu(y).$$

The s-dimensional Riesz energy of μ is

$$I_s(\mu) = \int R_s\mu \, \mathrm{d}\mu = \iint |x-y|^{-s} \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y).$$

Falconer [7] introduced singular values and singular value function of linear mappings T on $\mathbb{R}^d$. The singular values β_i $(1 \leqslant i \leqslant d)$ of T are the lengths of the semi-axes of the ellips

TB, where B is the unit ball in $\mathbb{R}^d$. Assume that $\beta_1 \geqslant \beta_2 \geqslant \ldots \geqslant \beta_d$. For $0 \leqslant s \leqslant d$, the singular value function $\phi^s(T)$ is defined by

$$\phi^s(T) = \beta_1\beta_2\ldots\beta_m\beta_{m+1}^{s-m},$$

where m is the largest integer not larger than s.

Let A be a 2×2 integer matrix with $\det A = 1$ and an eigenvalue $\lambda > 1$. Lemma 2.5 tells us that $\left\{\left(\frac{m}{S_k}, \frac{j}{S_k}\right) : m, j = 0, \ldots, S_k - 1\right\}$ are some of the periodic points with period $n = 2k+1$. After re-enumeration, we denote these periodic points by $\{x_{n,i}\}_{i=1}^{N_n}$, where $N_n = S_k^2$.

For $n = 2k+1$, define

$$E_n = \bigcup_{i=1}^{N_n} E_n^i, \tag{4.1}$$

and recall from lemma 2.9 that $E_n^i \subset \mathcal{E}_n^i$ is the parallelogram with centre $x_{n,i}$, and lengths of diagonal lines $2\lambda_{n,1}$ and $2\lambda_{n,2}$. Note that $E_n \subset \bigcup_{j=1}^{|H_n|}\mathcal{E}_n^j$. Hence $\limsup_n E_n \subset E$.

Now we estimate the shortest distance between E_n^i and E_n^j with $i \neq j$.

Lemma 4.1. *For $i \neq j$, we assume that $x_{n,i} = (\frac{i_1}{S_k}, \frac{i_2}{S_k})$, $x_{n,j} = (\frac{j_1}{S_k}, \frac{j_2}{S_k})$, then*

$$d\left(E_n^i, E_n^j\right) \begin{cases} \gtrsim \frac{1}{S_k|i_1 - j_1|} \gtrsim \lambda^{-n} & \textit{if } i_1 \neq j_1, \\ \asymp S_k^{-1} & \textit{if } i_1 = j_1. \end{cases} \tag{4.2}$$

Proof. Note that for $i \neq j$,

$$d\left(E_n^i, E_n^j\right) \geqslant d(x_{n,i}, l_{n,j}) - 2r_n\lambda_{n,2},$$

where $d(x_{n,i}, l_{n,j})$ is the distance between $x_{n,i}$ and the line $l_{n,j}$ given by $f(x) = \beta(x - \frac{j_1}{S_k}) + \frac{j_2}{S_k}$. When $i_1 = j_1$, we get that $d(E_n^i, E_n^j) \asymp S_k^{-1}$. When $i_1 \neq j_1$, since $\frac{\lambda^{-1}-a}{b}$ is an algebraic number of degree 2, by Liouville's theorem on diophantine approximation,

$$\begin{aligned} d(x_{n,i}, l_{n,j}) &\gtrsim \left|\beta\left(\frac{i_1 - j_1}{S_k}\right) + \frac{j_2 - i_2}{S_k}\right| \\ &\geqslant \frac{c_1}{S_k|i_1 - j_1|}, \end{aligned}$$

where $c_1 > 0$ is a constant only depending on A. Hence the distance between E_n^i and E_n^j, $i \neq j$, satisfies

$$d\left(E_n^i, E_n^j\right) \begin{cases} \gtrsim \frac{1}{S_k|i_1 - j_1|} \gtrsim \lambda^{-n} > 0 & \text{if } i_1 \neq j_1, \\ \asymp S_k^{-1} & \text{if } i_1 = j_1. \end{cases}$$

□

We conclude that for large enough n, the parallelograms $\{E_n^i\}_{i=1}^{N_n}$ do not intersect each other. Therefore

$$\mathcal{L}\left(E_n^i\right) \asymp \frac{r_n^2}{|H_n|}, \quad \mathcal{L}(E_n) = \sum_{i=1}^{N_n}\mathcal{L}\left(E_n^i\right) \asymp r_n^2.$$

4.1. Estimate when α is large

In this subsection, we give a lower bound on the Hausdorff dimension of E when $\alpha \geqslant \frac{1}{2}\log\lambda$.

Since $\limsup_{k\to\infty} E_{2k+1} \subset E = \limsup_{k\to\infty} E_k$, the following theorem implies that $E \in \mathcal{G}^{s_0}(\mathbb{T}^2)$.

Theorem 4.2. *For $\alpha \geqslant \frac{1}{2}\log\lambda$, let E_n be defined as in (4.1). Then*

$$\limsup_{k\to\infty} E_{2k+1} \in \mathcal{G}^{s_0}\left(\mathbb{T}^2\right),$$

where $s_0 = \min\left\{\frac{2\log\lambda}{\alpha+\log\lambda}, \frac{\log\lambda}{\alpha}\right\}$.

Proof of theorem 4.2. Let

$$\mu_n = \frac{1}{\mathcal{L}(E_n)}\mathcal{L}|_{E_n}.$$

The distribution of the periodic points $\{x_{n,i}\}_i$ is very regular, since they are grid points, and they are in fact distributed according to Lebesgue measure as $n\to\infty$. Hence, it follows that $\#\mathcal{P}_n \cap B \asymp r^2 S_k^2$ for large n. Therefore it is clear that there is a constant $C > 1$ such that for any ball $B \subset \mathbb{T}^2$,

$$C^{-1} \leqslant \liminf_{k\to\infty} \frac{\mu_{2k+1}(B)}{\mathcal{L}(B)} \leqslant \limsup_{k\to\infty} \frac{\mu_{2k+1}(B)}{\mathcal{L}(B)} \leqslant C.$$

(In fact, we may take $C = 1$, since the points $\{x_{n,i}\}_i$ are distributed according to the Lebesgue measure as $n\to\infty$.)

Now we show that $I_s(\mu_n)$ is finite for some $s > 0$, here $n = 2k+1$. Write

$$I_1 = \sum_{i=1}^{N_n} \int_{E_n^i}\int_{E_n^i} \frac{1}{|x-y|^s}\,\mathrm{d}\mu_n(x)\,\mathrm{d}\mu_n(y),$$

and

$$I_2 = \sum_{i=1}^{N_n}\sum_{\substack{j=1\\ j\neq i}}^{N_n} \int_{E_n^i}\int_{E_n^j} \frac{1}{|x-y|^s}\,\mathrm{d}\mu_n(x)\,\mathrm{d}\mu_n(y).$$

(a) Put $T_n\colon x \mapsto r_n(A^n - I)^{-1}x$ and $\phi^s(T_n)$ be the singular value function of T_n. Then

$$\begin{aligned} I_1 &= \sum_{i=1}^{N_n} \frac{1}{\mathcal{L}(E_n)^2} \int_{E_n^i}\int_{E_n^i} \frac{1}{|x-y|^s}\,\mathrm{d}x\,\mathrm{d}y \\ &\lesssim \frac{1}{r_n^4}\sum_{i=1}^{N_n} \frac{\mathcal{L}\left(\mathcal{E}_n^i\right)^2}{\phi^s(T_n)} = \frac{1}{r_n^4}\sum_{i=1}^{N_n}\frac{\pi^2 r_n^4}{S_k^4\phi^s(T_n)} \\ &= S_k^{-2}\phi^{-s}(T_n), \end{aligned}$$

where the first inequality follows from lemma 2.2 in [7].

(b) Now we estimate I_2.

It suffices to prove that the Riesz potential

$$R(x) = \sum_{j \neq i_0} \int_{E_n^j} \frac{1}{|x-y|^s} \, \mathrm{d}\mu_n(y)$$

is uniformly bounded for $x \in E_n^{i_0}$,

Since $\alpha > \frac{1}{2}\log\lambda$, for any $i \geqslant 1$, the number of cubes $\{(\frac{l}{S_k}, \frac{l+1}{S_k}) \times (\frac{m}{S_k}, \frac{m+1}{S_k})\}$ which E_n^i intersects is uniformly bounded. Hence $|i_1 - j_1| \lesssim 1$, where $\frac{i_1}{S_k}$, $\frac{j_1}{S_k}$ are the first coordinate of $x_{n,i}$ and $x_{n,j}$ respectively. By (4.2), the shortest distance d_n between E_n^i and E_n^j, $i \neq j$ satisfies

$$d_n \gtrsim \lambda^{-\frac{n}{2}}.$$

Since the centres of parallelograms E_n^j are placed along a lattice, for $j \neq i_0$, $x \in E_n^{i_0}$ and $y \in E_n^j$, there are $0 \leqslant m \leqslant d_n^{-1}$ and $1 \leqslant l \leqslant d_n^{-1}$ such that $|x-y| \geqslant (m^2+l^2)^{1/2} d_n$. Therefore,

$$\begin{aligned}
R(x) &\leqslant \sum_{m=0}^{d_n^{-1}} \sum_{l=1}^{d_n^{-1}} \left(\left(m^2+l^2\right)^{1/2} d_n \right)^{-s} \mu_n\left(E_n^j\right) \\
&\lesssim \frac{1}{r_n^2} S_k^s \int_0^{d_n^{-1}} \int_0^{d_n^{-1}} \left(u^2+v^2\right)^{-\frac{s}{2}} \mathrm{d}u \, \mathrm{d}v \\
&\lesssim r_n^{-2} \lambda^{\frac{1}{2}ns} \lambda^{\frac{1}{2}n(2-s)} = e^{n(-2\alpha+\log\lambda)} < C_1,
\end{aligned}$$

where C_1 is a constant which is independent of n and x. It hence follows that

$$I_2 = \sum_{i=1}^{N_n} \int_{E_n^i} R(x) \, \mathrm{d}\mu_n(x) \asymp C_1 \lambda^n r_n^{-2} < C_2.$$

where C_2 is a constant which is independent of n.

Combining (a) and (b), we have

$$I_s(\mu_n) = I_1 + I_2 \leqslant S_k^{-2} \phi^{-s}(T_n) + C_2.$$

Put $s_0 = \min\{\frac{\log\lambda}{\alpha}, \frac{2\log\lambda}{\alpha+\log\lambda}\} \leqslant 2$ and let m be such that $m < s_0 \leqslant m+1$. Take s such that $m < s < s_0$. Recall that by lemma 2.9, the singular values of T_n are approximately $\lambda_{n,2} \leqslant \lambda_{n,1}$.

If $m = 0$, then $s_0 = \frac{\log\lambda}{\alpha}$, and

$$\phi^s(T_n) \asymp (\lambda_{n,1})^s \asymp r_n^s.$$

Here $\lambda_{n,1} \asymp r_n$. Then for large n,

$$S_k^{-2} \phi^{-s}(T_n) \asymp r_n^{-s} \lambda^{-n} = \exp\{n(\alpha s - \log\lambda)\}.$$

From this and since $s < \frac{1}{\alpha}\log\lambda$, it follows that $\exp\{n(\alpha s - \log\lambda)\} < M$. This implies that $S_k^{-2}\phi^{-s}(T_n) < M$, where $M > 0$ is an absolute constant.

If $m = 1$, then $s_0 = \frac{2\log\lambda}{\alpha+\log\lambda}$ and

$$\phi^s(T_n) \asymp \lambda_{n,1} (\lambda_{n,2})^{s-1} \asymp r_n \left(r_n \lambda^{-n}\right)^{s-1} = r_n^s \lambda^{n(1-s)},$$

here $\lambda_{n,2} \asymp r_n \lambda^{-n}$. For n large enough, we have

$$S_k^{-2}\phi^{-s}(T_n) \asymp \lambda^{-n} r_n^{-s} \lambda^{n(s-1)} = e^{\alpha s n} \lambda^{n(s-2)}$$
$$= \exp\{n(\alpha s - (2-s)\log\lambda)\}.$$

When $s < \frac{2\log\lambda}{\alpha+\log\lambda}$, the expression $\exp\{n(\alpha s - (2-s)\log\lambda)\}$ is bounded and it follows that $S_k^{-2}\phi^{-s}(T_n) < M$ for some number M.

Therefore, for any

$$s < s_0 := \min\left\{\frac{2\log\lambda}{\alpha+\log\lambda}, \frac{1}{\alpha}\log\lambda\right\} \leqslant 2,$$

the energy $I_s(\mu_{2k+1})$ is uniformly bounded for all large enough k. We conclude by lemma 1.3 that $\limsup_k E_{2k+1} \in \mathcal{G}^{s_0}(\mathbb{T}^2)$. It follows that when $\alpha \geqslant \frac{1}{2}\log\lambda$,

$$E \in \mathcal{G}^{s_0}\left(\mathbb{T}^2\right).$$

Moreover,

$$\dim_{\mathrm{H}} E \geqslant \dim_{\mathrm{H}}\left(\limsup_{k\to\infty} E_{2k+1}\right) \geqslant \min\left\{\frac{2\log\lambda}{\alpha+\log\lambda}, \frac{1}{\alpha}\log\lambda\right\}.$$

□

4.2. Estimate when α is small

In this subsection, we give the lower bound on $\dim_{\mathrm{H}} \limsup_n E_n$ when $0 < \alpha < \frac{1}{2}\log\lambda$. We denote by $\mathcal{L}^1$ the Lebesgue measure restricted on $\mathbb{T}$.

Let $L_{x_0} := \{(x_0, y) \in \mathbb{T}^2 : y \in \mathbb{T}\}$, $x_0 \in \mathbb{T}$ and

$$\mathcal{F}_n(x_0) = \left\{\mathcal{E}_{n,i} \cap L_{x_0} : 1 \leqslant i \leqslant N_n,\ \lambda_{n,2}/3 \leqslant |E_n^i \cap L_{x_0}| \leqslant 2\lambda_{n,2}\right\}.$$

Assume that $\mathcal{F}_n(x_0) = \{I_{n,i} : 1 \leqslant i \leqslant M_n\}$, and put $M_n = \#\mathcal{F}_n(x_0)$, and

$$F_n(x_0) = \bigcup_{i=1}^{M_n} I_{n,i}.$$

Theorem 4.3. *For $0 < \alpha < \frac{1}{2}\log\lambda$, given $x_0 \in \mathbb{T}$, we have*

$$\limsup_{n\to\infty} F_n(x_0) \in \mathcal{G}^{s_1}(L_{x_0}),$$

where $s_1 = \frac{\log\lambda - \alpha}{\log\lambda + \alpha}$. In particular, for any $x_0 \in \mathbb{T}$,

$$\dim_{\mathrm{H}}\left(\limsup_{n\to\infty} F_n(x_0)\right) \geqslant s_1.$$

Observe that for any $x \in \mathbb{T}$, the set $F_n(x)$ may be regarded as a subset of the intersection of E_n with the line L_x on $\mathbb{T}^2$. Thus, applying proposition 7.9 in [8], we deduce that

$$\dim_{\mathrm{H}} \left(\limsup_{n\to\infty} E_n \right) \geqslant s_1 + 1 = s_0.$$

Then we get a lower bound on $\dim_{\mathrm{H}} E$, which coincides with its upper bound. However, we will show more about $\limsup_k E_{2k+1}$, namely the following.

Theorem 4.4. *For $0 < \alpha < \frac{1}{2}\log\lambda$, we have*

$$\limsup_{n\to\infty} E_n \in \mathcal{G}^{s_0}\left(\mathbb{T}^2\right),$$

where $s_0 = \frac{2\log\lambda}{\log\lambda + \alpha}$.

Since it suffices to study $\limsup_k F_{2k+1}(x_0)$ and $\limsup_k E_{2k+1}$ instead of $\limsup_n F_n(x_0)$ and $\limsup_n E_n$, from now on, we always assume that $n = 2k+1$, $k \geqslant 0$.

Lemma 4.5. *Given any segment with $L_l \subset L_{x_0}$, of length ℓ, for large enough $n = 2k+1$, we have*

$$\#\left\{ i : 1 \leqslant i \leqslant N_n,\ \frac{\lambda_{n,2}}{4} \leqslant |E_n^i \cap L_l| \leqslant 2\lambda_{n,2} \right\} \asymp \ell r_n \lambda^n.$$

In particular, $\# M_n \asymp r_n \lambda^n$.

Proof. Assume that $L_l = \{ (x_0, y) : y \in (y_0, y_0 + \ell) \}$. Note that $i \geqslant 1$ with $\frac{\lambda_{n,2}}{4} \leqslant |E_n^i \cap L_l| \leqslant 2\lambda_{n,2}$ if and only if $x_{n,i} \in P_l$, where $x_{n,i}$ is the centre of $\mathcal{E}_n^i$, and P_l is the parallelogram with vertices (a', b'), (d', c'), $(a', b' + \ell)$, $(d', c' + \ell)$, and

$$\begin{aligned} a' &= x_0 - K_0 \lambda_{n,1}, & d' &= x_0 + K_0 \lambda_{n,1}, \\ b' &= y_0 - K_0' \lambda_{n,1}, & c' &= y_0 + K_0' \lambda_{n,1} \end{aligned}$$

where $K_0,\ K_0' > 0$ are absolute constants. We observe that

$$\#\{ x_i : 1 \leqslant i \leqslant N_n,\ x_i \in P_l \} = \#\left\{ (p,q) \in \mathbb{Z}^2 : (p,q) \in S_k P_l \right\}.$$

The coordinates of the vertices of $S_k P_l$ are $(a' S_k, c' S_k)$, $(d' S_k, b' S_k)$, $(a' S_k, c' S_k + \ell S_k)$, and $(d' S_k, b' S_k + \ell S_k)$. By Pick's theorem [9]

$$\begin{aligned} \#\left\{ (p,q) \in \mathbb{Z}^2 : (p,q) \in S_k P_l \right\} &\leqslant \left(\lceil d' S_k \rceil - \lfloor a' S_k \rfloor \right) \lceil l S_k \rceil \\ &\leqslant (l S_k + 1) \left(2 K_0 r_n S_k \frac{\lambda^n - 1}{|H_n|} + 2 \right) \lesssim \ell r_n \lambda^n, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} \#\left\{ (p,q) \in \mathbb{Z}^2 : (p,q) \in S_k P_l \right\} &\geqslant \left(\lfloor d' S_k \rfloor - \lceil a' S_k \rceil \right) \lfloor l S_k \rfloor \\ &\geqslant (l S_k - 1) \left(2 K_0 r_n S_k \frac{\lambda^n - 1}{|H_n|} - 2 \right) \gtrsim \ell r_n \lambda^n. \end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4), we get that

$$\#\left\{ i : 1 \leqslant i \leqslant N_n,\ \frac{\lambda_{n,2}}{4} \leqslant |E_n^i \cap L_l| \leqslant 2\lambda_{n,2} \right\} \asymp \ell r_n \lambda^n.$$

Take $\ell = 1$. Then $M_n \asymp r_n \lambda^n$. □

Now we give the proof of theorem 4.3.

Proof of theorem 4.3. Given $x_0 \in \mathbb{T}$, in this proof, we write $F_n(x_0)$ as F_n for convenience. For $n = 2k+1$, let

$$\mu_n = \frac{1}{\mathcal{L}^1(F_n)} \mathcal{L}^1|_{F_n}.$$

It follows from lemma 4.5 that $\mathcal{L}^1(F_n) \asymp r_n^2$.

(i) For any ball $B \subset L_{x_0}$, we have

$$\begin{aligned}\mu_n(B) &= r_n^{-2} \sum_{\substack{i=1 \\ I_{n,i} \cap B \neq \varnothing}}^{M_n} \mathcal{L}^1(B \cap I_{n,i}) \\ &\lesssim r_n^{-2} \mathcal{L}^1\left(\left(1 + \frac{2\lambda_{n,2}}{r_B}\right) B\right) r_n \lambda^n r_n \lambda^{-n} \leqslant \mathcal{L}^1\left(\left(1 + \frac{2\lambda_{n,2}}{r_B}\right) B\right).\end{aligned}$$

Since $\max_i |I_{n,i}| \to 0$, as $n \to \infty$, then

$$\limsup_{n\to\infty} \frac{\mu_n(B)}{\mathcal{L}^1(B)} \lesssim 1.$$

And for large enough n we estimate that

$$\begin{aligned}\mu_n(B) &= \sum_{\substack{i=1 \\ I_{n,i} \cap B \neq \varnothing}}^{M_n} \frac{\mathcal{L}^1(B \cap I_{n,i})}{\mathcal{L}^1(F_n)} \geqslant \sum_{\substack{i=1 \\ I_{n,i} \subset B}}^{M_n} \frac{\mathcal{L}^1(I_{n,i})}{\mathcal{L}^1(F_n)} \\ &\asymp r_n^{-2} \# \{ i : I_{n,i} \subset B \} r_n \lambda^{-n}.\end{aligned}$$

Since

$$\# \{ i : I_{n,i} \subset B \} \supset \{ i : \mathcal{E}_n^i \cap (1 - 2\lambda_{n,2}) B \neq \varnothing \},$$

Lemma 4.5 gives that

$$\# \{ i : I_{n,i} \subset B \} \gtrsim \mathcal{L}^1((1 - 2\lambda_{n,2}) B) r_n \lambda^n.$$

Hence,

$$\mu_n(B) \gtrsim \mathcal{L}^1((1 - 2\lambda_{n,2}) B),$$

and it follows that for large enough n,

$$\liminf_{n\to\infty} \frac{\mu_n(B)}{\mathcal{L}^1(B)} \gtrsim 1.$$

Therefore

$$C_3^{-1} \leqslant \liminf_{n\to\infty} \frac{\mu_n(B)}{\mathcal{L}^1(B)} \leqslant \limsup_{n\to\infty} \frac{\mu_n(B)}{\mathcal{L}^1(B)} \leqslant C_3,$$

where $C_3 > 0$ is a constant independent of n and B.

(2) Now we show that the energy of μ_n is bounded for some s. We observe that

$$\begin{aligned} I_s(\mu_n) &= \iint |x-y|^{-s}\,\mathrm{d}\mu_n(x)\,\mathrm{d}\mu_n(y) \\ &= \sum_{i=1}^{M_n}\sum_{j=1}^{M_n}\int_{I_{n,i}}\int_{I_{n,j}} |x-y|^{-s}\,\mathrm{d}\mu_n(x)\,\mathrm{d}\mu_n(y) \\ &= \sum_{i=1}^{M_n}\int_{I_{n,i}}\int_{I_{n,i}} |x-y|^{-s}\,\mathrm{d}\mu_n(x)\,\mathrm{d}\mu_n(y) \\ &\quad + \sum_{i=1}^{M_n}\sum_{\substack{j=1\\ j\neq i}}^{M_n}\int_{I_{n,i}}\int_{I_{n,j}} |x-y|^{-s}\,\mathrm{d}\mu_n(x)\,\mathrm{d}\mu_n(y). \end{aligned}$$

(1) When $i=j$, for $x\in I_{n,i}$ for some $i\geqslant 1$, we have

$$I_1(x) := \int_{I_{n,i}} |x-y|^{-s}\,\mathrm{d}\mu_n(y) \leqslant \frac{1}{r_n^2}|I_{n,i}|^{1-s} \lesssim e^{n\{(1+s)\alpha-(1-s)\log\lambda\}}.$$

If $s < \frac{\log\lambda-\alpha}{\log\lambda+\alpha}$, then $I_1(x)$ is bounded as a function of n, which implies that

$$I_1 := \sum_{i=1}^{M_n}\int_{I_{n,i}}\int_{I_{n,i}} |x-y|^{-s}\,\mathrm{d}\mu_n(x)\,\mathrm{d}\mu_n(y) \lesssim r_n\lambda^n r_n^{-1}\lambda^{-n} < C_4, \tag{4.5}$$

for some constant C_4.

(2) When $x\in I_{n,i}$ for some $i\geqslant 1$, we let

$$I_2(x) := \sum_{\substack{j=1\\ j\neq i}}^{M_n}\int_{I_{n,j}} |x-y|^{-s}\,\mathrm{d}\mu_n(y).$$

For $p\geqslant 0$, let

$$A_p = \left\{ j : I_{n,j} \cap \left\{ (x_0,y) : \frac{p+1}{S_k} > |y-x| \geqslant \frac{p}{S_k} \right\} \neq \varnothing \right\}.$$

Then

$$\begin{aligned} I_2(x) &\leqslant S_1(x) + S_2(x) \\ &:= \sum_{p=1}^{S_k-1}\sum_{j\in A_p}\int_{I_{n,j}} |x-y|^{-s}\,\mathrm{d}\mu_n(y) + \sum_{\substack{j\in A_0\\ j\neq 1}}\int_{I_{n,j}} |x-y|^{-s}\,\mathrm{d}\mu_n(y). \end{aligned}$$

For $p \geqslant 1$, $j \in A_p$, and $y \in I_{n,j}$, we see that $|x-y| \geqslant \frac{p}{S_k}$. Then it follows from lemma 4.5 that for any $s \leqslant 1$,

$$\begin{aligned} S_1(x) &= \sum_{p=1}^{S_k-1} \sum_{j\in A_p} \mu_n(I_{n,j}) \left(\frac{p}{S_k}\right)^{-s} \\ &\asymp \sum_{p=1}^{S_k-1} \# A_p p^{-s} r_n^{-1} \lambda^{-n} S_k^s \\ &\asymp \lambda^{\frac{n}{2}(s-1)} \sum_{p=1}^{S_k-1} p^{-s} \leqslant \lambda^{\frac{n}{2}(s-1)} \int_0^{S_k} u^{-s}\,\mathrm{d}u \\ &\leqslant \lambda^{\frac{n}{2}(s-1)} \lambda^{\frac{n}{2}(1-s)} < C_5. \end{aligned} \tag{4.6}$$

Now we estimate $S_2(x)$. Recall (4.2) which states that

$$d\left(E_n^m, E_n^j\right) \gtrsim \lambda^{-n}.$$

We denote the shortest distance between intervals $\{I_{n,i}\}$ by d_n. Then $d(I_{n,j_1}, I_{n,j_2}) \geqslant d_n \gtrsim \lambda^{-n}$ for any $j_1 \neq j_2 \in A_0$. Therefore

$$\begin{aligned} S_2(x) &\leqslant \sum_{q=1}^{\# A_0} (q d_n)^{-s} \mu_n(I_{n,j}) \lesssim r_n^{-2} r_n \lambda^{-n} \lambda^{ns} \sum_{q=1}^{\# A_0} q^{-s} \\ &\leqslant r_n^{-1} \lambda^{n(s-1)} \int_0^{\# A_0} u^{-s}\,\mathrm{d}u \leqslant r_n^{-1} \lambda^{n(s-1)} (r_n S_k)^{1-s} \\ &\lesssim r_n^{-s} \lambda^{\frac{n}{2}(s-1)} = e^{n\left\{\alpha s + \frac{1}{2}\log\lambda (s-1)\right\}}. \end{aligned} \tag{4.7}$$

We see that $S_2(x) \lesssim 1$ is bounded as a function of n when $s < \frac{\log\lambda}{\log\lambda + 2\alpha}$. Inequalities (4.6) and (4.7) yields that for any $s < \frac{\log\lambda}{\log\lambda+2\alpha}$,

$$I_2 := \sum_{i=1}^{M_n} \sum_{\substack{j=1 \\ j\neq i}}^{M_n} \int_{I_{n,i}} \int_{I_{n,j}} |x-y|^{-s}\,\mathrm{d}\mu_n(x)\,\mathrm{d}\mu_n(y) < C_6. \tag{4.8}$$

Notice that

$$\min\left\{\frac{\log\lambda-\alpha}{\log\lambda+\alpha}, \frac{\log\lambda}{\log\lambda+2\alpha}\right\} = \frac{\log\lambda-\alpha}{\log\lambda+\alpha}$$

holds for all $\alpha > 0$. Combining (4.5) and (4.8), for any $s < \frac{\log\lambda-\alpha}{\log\lambda+\alpha}$, we have $I_s(\mu_n) < C_4 + C_6$ for n large enough.

Write $s_1 = \frac{\log\lambda-\alpha}{\log\lambda+\alpha}$. Applying lemma 1.3, we get that $\limsup_{n\to\infty} F_n \in \mathcal{G}^{s_1}(L_{x_0})$. □

Suppose that μ is a probability measure on $[0,1]^2$ which can be disintegrated as

$$\mu(A) = \int \mu_x(A \cap \{x\} \times [0,1])\,\mathrm{d}\nu(x) = \int \mu_x(A)\,\mathrm{d}\nu(x),$$

for any $A \in \mathcal{B}$, where μ_x is a measure with support in $\{x\} \times [0,1]$. We then say $\{\mu_x\}_{x\in[0,1]}$ is a disintegration of μ over ν. We assume moreover that ν is a probability measure and that μ_x is a probability measure for ν a.e. x.

Lemma 4.6. *With μ as above, we have for $x = (x_1, x_2)$ that*

$$R_{s+t}\mu(x) \leqslant \int R_s\mu_{y_1}(x_2)\,|x_1 - y_1|^{-t}\,\mathrm{d}\nu(y_1).$$

In particular, if the s-dimensional Riesz potentials of μ_x are uniformly bounded and if ν has a bounded t-dimensional potential, then

$$I_{s+t}(\mu) \leqslant \sup_x \|R_s\mu_x\|_\infty \|R_t\nu\|_\infty.$$

Proof. Writing $x = (x_1, x_2)$ and $y = (y_1, y_2)$, we have

$$\begin{aligned} R_{s+t}\mu(x) &= \iint |(x_1,x_2) - (y_1,y_2)|^{-s-t}\,\mathrm{d}\mu_{y_1}(y_2)\,\mathrm{d}\nu(y_1) \\ &\leqslant \int \left(\int |x_2 - y_2|^{-s}\,\mathrm{d}\mu_{y_1}(y_2)\right)|x_1 - y_1|^{-t}\,\mathrm{d}\nu(y_1) \\ &= \int R_s\mu_{y_1}(x_2)\,|x_1 - y_1|^{-t}\,\mathrm{d}\nu(y_1). \end{aligned}$$

Hence

$$\|R_{s+t}\mu\|_\infty \leqslant \sup_x \|R_s\mu_x\|_\infty \|R_t\nu\|_\infty.$$

Since μ is a probability measure, we obtain $I_{s+t}(\mu) \leqslant \|R_{s+t}\mu\|_\infty$. □

Now we finish the proof of our main result theorem 1.4.

Proof of theorem 1.4. For $\alpha \geqslant \frac{1}{2}\log\lambda$, it follows from theorem 4.2 that

$$E \in \mathcal{G}^{s_0}\left(\mathbb{T}^2\right),$$

where $s_0 = \min\left\{\frac{2\log\lambda}{\alpha+\log\lambda}, \frac{1}{\alpha}\log\lambda\right\}$. Hence

$$\dim_{\mathrm{H}} E \geqslant s_0.$$

For $0 < \alpha < \frac{1}{2}\log\lambda$, let $\tilde{\mu}_n$ be defined as

$$\tilde{\mu}_n(A) = \int \mu_{x,n}(A)\,\mathrm{d}x,$$

for $A \in \mathcal{B}$, where $\mu_{x,n} = \frac{1}{\mathcal{L}^1(F_n(x))}\mathcal{L}^1|_{F_n(x)}$. For $n \geqslant 1$,

$$\tilde{\mu}_n\left(\mathbb{T}^2 \setminus E_n\right) = 0,$$

that is, the support of $\tilde{\mu}_n$ is in E_n. Then $\{\mu_{x,n}\}_{x\in\mathbb{T}}$ is a disintegration of $\tilde{\mu}_n$ over Lebesgue measure $\mathcal{L}^1$.

By theorem 4.3, the $(s_1 - \frac{\varepsilon}{2})$-dimensional Riesz potentials of μ_{2k+1} are uniformly bounded for k large enough, and note that $\mathcal{L}^1$ has a bounded $(1 - \frac{\varepsilon}{2})$-dimensional potential for any $\varepsilon > 0$. Applying theorem 4.3 and lemma 4.6, for k large enough, we have

$$I_{s_1+1-\varepsilon}\left(\tilde{\mu}_{2k+1}\right) < \infty,$$

where $s_0 - \varepsilon = s_1 + 1 - \varepsilon$ when $\alpha < \frac{1}{2}\log\lambda$. It is easy to show that for any ball $B \subset \mathbb{T}^2$,

$$1 \lesssim \liminf_{k\to\infty} \frac{\tilde{\mu}_{2k+1}(B)}{\mathcal{L}(B)} \leqslant \limsup_{k\to\infty} \frac{\tilde{\mu}_{2k+1}(B)}{\mathcal{L}(B)} \lesssim 1.$$

Applying lemma 1.3, we get that $\limsup_{k\to\infty} E_{2k+1} \in \mathcal{G}^{s_0-\varepsilon}(\mathbb{T}^2)$. Since $\varepsilon > 0$ is arbitrary, it follows that $\limsup_{k\to\infty} E_{2k+1} \in \mathcal{G}^{s_0}$ and in particular, $\dim_{\mathrm{H}} E \geqslant s_0$.

Therefore, for $\alpha > 0$, we conclude that

$$E \in \mathcal{G}^{s_0}\left(\mathbb{T}^2\right),$$

and $\dim_{\mathrm{H}} E \geqslant s_0$. This, together with lemma 3.1 gives us that $\dim_{\mathrm{H}} E = s_0$.

□

Data availability statement

No new data were created or analysed in this study.

Acknowledgments

We thank the two referees for their useful comments which helped improving the manuscript.

This research of Zhangnan Hu was supported by China Scholarship Council and Science Foundation of China University of Petroleum, Beijing (No. 2462023SZBH013). This research of Zhangnan Hu was supported by China Scholarship Council and Science Foundation of China University of Petroleum, Beijing (No. 2462023SZBH013), and China Postdoctoral Science Foundation (No. 2023M743878).

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Potential Analysis (2024) 60:341–363
https://doi.org/10.1007/s11118-022-10052-7

Large Intersection Property for Limsup Sets in Metric Space

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Received: 6 April 2022 / Accepted: 20 October 2022 / Published online: 15 November 2022

Abstract

We show that limsup sets generated by a sequence of open sets in compact Ahlfors s-regular $(0 < s < \infty)$ space $(X, \mathscr{B}, \mu, \rho)$ belong to the classes of sets with large intersections with index λ, denoted by $\mathcal{G}^{\lambda}(X)$, under some conditions. In particular, this provides a lower bound on Hausdorff dimension of such sets. These results are applied to obtain that limsup random fractals with indices γ_2 and δ belong to $\mathcal{G}^{s-\delta-\gamma_2}(X)$ almost surely, and random covering sets with exponentially mixing property belong to $\mathcal{G}^{s_0}(X)$ almost surely, where s_0 equals to the corresponding Hausdorff dimension of covering sets almost surely. We also investigate the large intersection property of limsup sets generated by rectangles in metric space.

Keywords Limsup sets · Large intersection property · Metric space · Limsup random fractals · Random covering sets

Mathematics Subject Classification (2010) Primary 37A50; Secondary 28A78 · 28A80

1 Introduction

Sets with large intersection were introduced by Falconer in [9]. Given $s \in (0, d]$, he defined $\mathcal{G}^s(\mathbb{R}^d)$ to be the class of all G_δ sets F in $\mathbb{R}^d$ such that

$$\dim_{\mathrm{H}} \bigcap_{n=1}^{\infty} f_n(F) \geq s$$

holds for all sequences of similarity transformations $\{f_n\}_{n\geq 1}$, where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension. It is known that $\mathcal{G}^s(\mathbb{R}^d)$ is the maximal class of G_δ sets satisfying (i)

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Hausdorff dimension larger than s, and (ii) closed under similarity transformations and countable intersections. The packing dimension of sets in $\mathcal{G}^s(\mathbb{R}^d)$ is d from the fact that they are dense G_δ sets. In [9], Falconer also gave several equivalent definitions and established various properties of $\mathcal{G}^s(\mathbb{R}^d)$. In 2004, Bugeaud [3] generalized the class for more general gauge functions. Later, Negreira and Sequeira [21] extended the class of sets in Euclidean space with large intersection property to metric space endowed with doubling measure.

There are many applications of large intersection property. Lots of mathematicians applied this property for estimating Hausdorff dimension from below in the study of Diophantine approximations, and refer to [3, 4, 9, 21] for more details. In 2007, Durand [5] investigated the size and large intersection properties of limsup sets generated by homogeneous ubiquitous systems in $\mathbb{R}^d$. In 2010, Durand [8] showed that random covering sets $\limsup_{n\to\infty} B(\xi_n, r_n)$ are sets with large intersection, where $\{\xi_n\}_{n\geq 1}$ are independent and uniformly distributed random variables on the circle. In 2019, Persson [22] proved that dynamical covering sets $\limsup_{n\to\infty} B(T^n x, n^{-\alpha})$ has large intersection property for dynamical systems (T, μ) which have summable decay of correlations. Ding [4] showed that under a full Hausdorff measure assumption, the limsup sets generated by rectangles with some conditions in compact metric space are sets with large intersection. In 2021, Persson [23] considered various sequences of open sets with general shapes, proved the corresponding limsup sets have large intersection properties, and obtained the lower bound on the Hausdorff dimension. Aubry and Jaffard [1] noticed that large intersection property also occurred in probability theory, such as the multifractal analysis of random wavelet series. See also [6, 7, 10] for more study in fractals, dynamical systems and the multifractal analysis of other stochastic processes.

Limsup sets, the upper limits of sequences of sets, play an important role in many areas, such as random covering problem [16], shrinking target problem [13], the study of Brownian motion [18] and so on. Motivated by the study on sets with large intersections, we are interested in the large intersection properties of limsup sets generated by open sets in metric spaces $(X, \mathscr{B}, \mu, \rho)$ and those applications.

Definition 1.1 A Borel measure μ on metric space (X, ρ) is Ahlfors s-regular ($0 < s < \infty$) if there exists a constant $1 \leq C < \infty$ such that

$$C^{-1} r^s \leq \mu\big(B(x, r)\big) \leq C r^s \tag{1.1}$$

holds for all $x \in X$ and $0 < r \leq \operatorname{diam} X$, where $\operatorname{diam} X$ is the diameter of X. Here $B(x, r) = \{y \in X : \rho(x, y) < r\}$ is an open ball with centre x and radius r. A space $(X, \mathscr{B}, \mu, \rho)$ is said to be Ahlfors s-regular if μ satisfies formula (1.1).

In this paper, we consider a probability space $(X, \mathscr{B}, \mu, \rho)$ where μ is Ahlfors s-regular ($0 < s < \infty$). For $B = B(x, r)$, we adopt the convention that $cB = B(x, cr)$, $B^t = B(x, r^{t/s})$ and $B = \varnothing$ if $r = 0$. If we say simply balls, they are open balls.

Definition 1.2 For $0 \leq t \leq s$, the t-potential at $y \in X$ of the measure μ is defined as

$$\phi_t(\mu, y) = \int_X \rho(x, y)^{-t} d\mu(x).$$

The t-energy of the measure μ is defined as

$$I_t(\mu) = \int_X \int_X \rho(x, y)^{-t} d\mu(x) d\mu(y).$$

Denote

$$\phi_t(\mu, U, y) = \int_U \rho(x, y)^{-t} d\mu(x),$$
$$I_t(\mu, U) = \int_U \int_U \rho(x, y)^{-t} d\mu(x) d\mu(y)$$

for $U \in \mathscr{B}$.

The definition of $\mathcal{G}^\lambda(X)$ in Theorem 1.1 is stated in Definition 2.3.

Theorem 1.1 *Let $(X, \mathscr{B}, \mu, \rho)$ be a compact Ahlfors s-regular space $(0 < s < \infty)$, and $\{B_n\}_{n\geq 1}$ be a sequence of balls in X with* $\operatorname{diam} B_n$ *decreasing to 0 as $n \to \infty$. For $n \geq 1$, E_n is an open subset of B_n and let*

$$\lambda = \sup\left\{t \geq 0 \colon \sup_{n\geq 1} \frac{I_t(\mu, E_n)\mu(B_n)}{\mu(E_n)^2} < \infty\right\}.$$

Then $\mu\Big(\limsup_{n\to\infty} B_n\Big) = 1$ implies that $\limsup_{n\to\infty} E_n \in \mathcal{G}^\lambda(X)$.

Corollary 1.1 *Under the setting in Theorem 1.1, if $\mu\Big(\limsup_{n\to\infty} B_n\Big) = 1$, then we have* $\dim_{\mathrm{H}}\Big(\limsup_{n\to\infty} E_n\Big) \geq \lambda$ *and* $\dim_{\mathrm{P}}\Big(\limsup_{n\to\infty} E_n\Big) = s$, *where* $\dim_{\mathrm{P}}$ *denotes the packing dimension.*

In Theorem 1.1 and Corollary 1.1, we consider E_n are open sets with arbitrary shapes. In particular, when we consider a special case that $\{E_n\}_{n\geq 1}$ is a sequence of balls, an important tool in determining the Hausdorff dimension of $\limsup_{n\to\infty} E_n$ is the Mass Transference Principle, which was established by Beresnevich and Velani [2].

Theorem 1.2 (Mass Transference Principle [2]) *Let $(X, \mathscr{B}, \mu, \rho)$ be a locally compact Ahlfors s-regular space $(0 < s < \infty)$, Let $\{B_n\}_{n\geq 1}$ be a sequence of balls in X with* $\operatorname{diam} B_n \to 0$ *as $n \to \infty$. Let $t > 0$ and suppose that*

$$\mathcal{H}^s(\limsup_{n\to\infty} B_n^t) = \mathcal{H}^s(X).$$

Then,

$$\mathcal{H}^t(\limsup_{n\to\infty} B_n) = \mathcal{H}^t(X).$$

Here $\mathcal{H}^t(F)$ denotes the Hausdorff t-measure of a set $F \subset X$.

For an Ahlfors s-regular space $(X, \mathscr{B}, \mu, \rho)$, Heinonen [12, Section 8.7] proved that $C'^{-1}\mathcal{H}^s \leq \mu \leq C'\mathcal{H}^s$, where $C' \geq 1$ is a constant. Then by Theorem 1.2, given $t \in (0, s]$, we can deduce that $\dim_{\mathrm{H}}\Big(\limsup_{n\to\infty} B_n\Big) \geq t$, if $\mu\Big(\limsup_{n\to\infty} B_n^t\Big) = 1$. Such estimation can be derived by Corollary 1.1 when X is compact. In addition, as the following corollary shows, $\limsup_{n\to\infty} B_n$ also has large intersection property.

Corollary 1.2 *Let $(X, \mathscr{B}, \mu, \rho)$ be a compact Ahlfors s-regular space $(0 < s < \infty)$, and $\{B_n\}_{n\geq 1}$ be a sequence of balls with* $\operatorname{diam} B_n$ *decreasing to 0 as $n \to \infty$. For $t \in (0, s]$, if $\mu\Big(\limsup_{n\to\infty} B_n^t\Big) = 1$, then $\limsup_{n\to\infty} B_n \in \mathcal{G}^t(X)$.*

Applying Theorem 1.1 and Corollary 1.2, we prove that limsup random fractals, random covering sets and limsup sets generated by rectangles have large intersection property (Theorems 4.1, 4.3 and 4.4).

The rest of this paper is organized as follows. In Section 2 we give a brief review of the class of sets with large intersection properties. In Section 3, we give the proofs of Theorem 1.1 and Corollary 1.2 which are our main results. In the last section, there are some examples. We apply our results to the study of the large intersection properties of limsup random fractals under some conditions and random covering sets $\limsup_{n\to\infty} B(\xi_n, r_n)$, where the centers $\{\xi_n\}_{n\geq 1}$ are uniformly distributed and exponentially mixing random variables, are sets with large intersection properties. We also show limsup sets generated by rectangles have large intersection property.

2 Preliminaries

In this section, we refer (X, ρ) to general metric space, and μ is a Borel measure on X. We denote $\overline{B}(x, r)$ the closure of $B(x, r)$.

2.1 Generalized Dyadic Cubes in Metric Spaces

Definition 2.1 A metric space (X, ρ) has the finite doubling property if every closed ball $\overline{B}(x, 2r) \subset X$ may be covered by finitely many closed balls of radius r. Furthermore, such a space is doubling if there exists $N \in \mathbb{N}$ independent of x and r such that $\overline{B}(x, 2r)$ can be covered by at most N balls of radius r.

Letting (X, ρ) be a metric space with the finite doubling property, Käenmäki, Rajala and Suomala [17] showed that there exists a nesting family of "cubes" which are similar as the dyadic cubes of Euclidean spaces.

Theorem 2.1 ([17]) *Let (X, ρ) be a metric space with the finite doubling property and let $0 < b < \frac{1}{3}$ be a constant. Then there exists a collection $\{Q_{n,i} : n \in \mathbb{Z}, i \in \mathbb{N}_n \subset \mathbb{N}\}$ of Borel sets that have the following properties:*

1. *$X = \bigcup_{i\in\mathbb{N}_n} Q_{n,i}$ for every $n \in \mathbb{Z}$.*
2. *$Q_{n,i} \cap Q_{m,j} = \varnothing$ or $Q_{n,i} \subset Q_{m,j}$, where $n, m \in \mathbb{Z}$, $n \geq m$, $i \in \mathbb{N}_n$ and $j \in \mathbb{N}_m$.*
3. *For every $n \in \mathbb{Z}$ and $i \in \mathbb{N}_n$, there exists a point $x_{n,i} \in X$ such that*
$$B(x_{n,i}, c_1 b^n) \subset Q_{n,i} \subset \overline{B}(x_{n,i}, c_1' b^n), \tag{2.1}$$
where $c_1 = \frac{1}{2} - \frac{b}{1-b}$, $c_1' = \frac{1}{1-b}$.
4. *There exists a point $x_0 \in X$ so that for every $n \in \mathbb{Z}$, there is an index $i \in \mathbb{N}_n$ with $B(x_0, c_1 b^n) \subset Q_{n,i}$.*
5. *$\{x_{n,i} : i \in \mathbb{N}_n\} \subset \{x_{n+1,i} : i \in \mathbb{N}_{n+1}\}$ for all $n \in \mathbb{Z}$.*

Proposition 2.1 *Let (X, ρ) be a metric space with an Ahlfors s-regular measure μ.*

1. *Suppose that μ is a probability measure, then for $n \in \mathbb{Z}$, we have*
$$C^{-1} {c_1'}^{-s} b^{-ns} \leq \#\mathbb{N}_n \leq C c_1^{-s} b^{-ns}.$$
2. *The metric space (X, ρ) has the doubling property.*

Proof (1) From the construction of $\{Q_{n,i} : n \in \mathbb{Z}, i \in \mathbb{N}_n \subset \mathbb{N}\}$ in [17], we see that for every $n \in \mathbb{Z}$, $\{Q_{n,i}\}_{i\in\mathbb{N}_n}$ are pairwise disjoint, that is, $Q_{n,i} \cap Q_{n,j} = \varnothing$ for $i \neq j \in \mathbb{N}_n$. Since μ is Ahlfors s-regular, combining (1), (3) in Theorem 2.1 and the inequalities (1.1), we have

$$\mu(X) = \sum_{i\in\mathbb{N}_n} \mu(Q_{n,i}) \geq \sum_{i\in\mathbb{N}_n} \mu(B(x_{n,i}, c_1 b^n)) \geq C^{-1}\#\mathbb{N}_n(c_1 b^n)^s,$$

and

$$\mu(X) = \sum_{i\in\mathbb{N}_n} \mu(Q_{n,i}) \leq \sum_{i\in\mathbb{N}_n} \mu(\overline{B}(x_{n,i}, c_1' b^n)) \leq C\#\mathbb{N}_n(c_1' b^n)^s,$$

which implies that

$$C^{-1}c_1'^{-s}b^{-ns} \leq \#\mathbb{N}_n \leq Cc_1^{-s}b^{-ns}.$$

(2) Given $x \in X$ and $r > 0$, let $n_0 = \min\{n \geq 1 : c_1' b^n < r\}$. Let

$$I = \{i \in \mathbb{N}_{n_0} : Q_{n_0,i} \cap \overline{B}(x, 2r) \neq \varnothing\}.$$

Then $\{\overline{B}(x_{n_0,i}, r)\}_{i\in I}$ is a cover of $\overline{B}(x, 2r)$. Note that $\bigcup_{i\in I} Q_{n_0,i} \subset B(x, 4r)$ and $r \leq c_1' b^{n_0-1}$, then by (2.1), we have

$$\begin{aligned} C4^s r^s &\geq \mu(B(x, 4r)) \geq \mu\Big(\bigcup_{i\in I} Q_{n_0,i}\Big) \\ &= \sum_{i\in I} \mu(Q_{n_0,i}) \geq \sum_{i\in I} \mu(B(x_{n_0,i}, c_1 b^{n_0})) \\ &\geq C^{-1}c_1^s b^{n_0 s}\#I \geq C^{-1}(bc_1/c_1')^s r^s \#I, \end{aligned}$$

which follows that $\#I \leq C^2\left(\frac{4c_1'}{bc_1}\right)^s$. □

Therefore for a compact Ahlfors s-regular space $(X, \mathscr{B}, \mu, \rho)$, there exists the family $\{Q_{n,i} : n \geq 0, i \in \mathbb{N}_n\}$ satisfying the properties in Theorem 2.1, which are called "generalized dyadic cubes". For convenience, we write $\mathcal{Q}_0 = \{X\}$ and $\mathcal{Q}_n = \{Q_{n,i} : i \in \mathbb{N}_n\}$ for $n \geq 1$, and $\mathcal{Q} = \bigcup_{n\geq 0} \mathcal{Q}_n$.

2.2 Large Intersection Properties in Metric Spaces

Definition 2.2 Let (X, ρ) be a metric space. A Borel measure μ on X is said to be doubling if it is finite and positive in every ball and there exists a constant $1 \leq c_2 < \infty$ such that for all $x \in X$ and $r > 0$,

$$0 < \mu\Big(\overline{B}(x, 2r)\Big) \leq c_2\mu\Big(\overline{B}(x, r)\Big) < \infty.$$

Let (X, ρ) be a metric space endowed with doubling measure μ. Denote $\tau = \dim_{\mathrm{H}} X$, and τ is finite, since $\tau \leq \dim_{\mathrm{A}} X < \infty$, and see [11, Sections 4, 13] for more details about Assouad dimension $\dim_{\mathrm{A}}$. In [21], Negreira and Sequeira gave the definition of the classes of G_δ sets with large intersection property. Recall that a G_δ set is a countable intersection of open sets.

Definition 2.3 Let $0 < t \leq \tau$. Given $F \subset X$, define the net content

$$\mathscr{M}_\infty^t(F) = \inf\Big\{\sum_{i\geq 1} \mu(Q_i)^{t/\tau} : F \subset \bigcup_{i\geq 1} Q_i \text{ where } Q_i \in \mathcal{Q}\Big\}.$$

We denote $\mathcal{G}^t(X)$ the class of all G_δ sets $F \subset X$ such that for every $t' < t$,

$$\mathcal{M}_\infty^{t'}(F \cap Q) = \mathcal{M}_\infty^{t'}(Q)$$

holds for all $Q \in \mathcal{Q}$.

Remark 2.1 (1) Negreira and Sequeira [21] showed that the net contents given by different dyadic decomposition are equivalent, and pointed out that the net content is defined by $\mu(Q)^{t/\tau}$ instead of $(\operatorname{diam} Q)^t$, since the function $Q \mapsto (\operatorname{diam} Q)^t$ is not sub-additive if $t > 1$ which is important in the study of $\mathcal{M}_\infty^t$ and large intersection property.

(2) Notice that the class $\mathcal{G}^t(X)$ given in Definition 2.3 depends on μ. Since Ahlfors s-regular measures on X are equivalent, therefore the classes $\mathcal{G}^t(X)$ defined by different Ahlfors s-regular measures are same. In this paper we only consider such measures, hence we use the notation $\mathcal{G}^t(X)$ instead of $\mathcal{G}_\mu^t(X)$.

There are several equivalent definitions of $\mathcal{G}^t(X)$, see [21]. In this paper, we will use the following equivalent definition.

Theorem 2.2 ([21]) *Let (X, ρ) be a metric space enowed with doubling measure μ, and $\tau = \dim_{\mathrm{H}} X$. Take a G_δ subset $F \subset X$ and $0 < t \leq \tau$. Then the following statements are equivalent:*

1. *For all generalized dyadic cubes $Q \in \mathcal{Q}$, we have*

$$\mathcal{M}_\infty^{t'}(F \cap Q) = \mathcal{M}_\infty^{t'}(Q) \quad (\forall t' < t).$$

2. *There exists a constant $0 < c \leq 1$ such that for all generalized dyadic cubes $Q \in \mathcal{Q}$, we have*

$$\mathcal{M}_\infty^{t'}(F \cap Q) \geq c\mathcal{M}_\infty^{t'}(Q) \quad (\forall t' < t). \tag{2.2}$$

Remark 2.2 Note that, given $0 < t \leq \tau$, $\mathcal{M}_\infty^t(Q) = \mu(Q)^{t/\tau}$ holds for all generalized dyadic cubes $Q \in \mathcal{Q}$ by the subadditivity in Remark 2.1(1). Then (2.2) can be rewritten as

$$\mathcal{M}_\infty^{t'}(F \cap Q) \geq c\mu(Q)^{t'/\tau} \quad (\forall t' < t). \tag{2.3}$$

The following are some properties of the class $\mathcal{G}^t(X)$.

Proposition 2.2 ([21]) *Let (X, ρ) be a metric space endowed with doubling measure μ, and $\tau = \dim_{\mathrm{H}} X$ and take $0 < t \leq \tau$.*

1. *If $0 < t_1 \leq t$, then $\mathcal{G}^t(X) \subset \mathcal{G}^{t_1}(X)$.*
2. *If $F \subset E \subset X$, where E, F are G_δ sets, and $F \in \mathcal{G}^t(X)$, then $E \in \mathcal{G}^t(X)$.*
3. *If (X, ρ) is complete, then $\mathcal{G}^t(X)$ is closed under countable intersections.*
4. *When μ is Ahlfors τ-regular, then $F \in \mathcal{G}^t(X)$ implies that $\dim_{\mathrm{H}} F \geq t$.*

Remark 2.3 Let (X, ρ) be a metric space endowed with doubling measure μ, and $\tau = \dim_{\mathrm{H}} X$. Take $0 < t \leq \tau$, then from the definition of $\mathcal{G}^t(X)$, we have

$$\mathcal{G}^t(X) = \bigcap_{t'<t} \mathcal{G}^{t'}(X).$$

3 Proofs of Theorem 1.1 and Corollary 1.2

In 2019, Persson [22] used potentials and energies of measures to prove that some sets have large intersection property, which is a new and useful method in the study of large intersection property. Later, Persson [23] applied this idea to show that the limsup sets, generated by sequences of open sets, belong to $\mathcal{G}^{\gamma}(\mathbb{T}^d)$ for some $\gamma > 0$. In this paper, we borrowed some ideas from [23] to prove Theorem 1.1. Before that, we give some related lemmas.

In this section, we always assume that $(X, \mathscr{B}, \mu, \rho)$ is a probability space where the metric space (X, ρ) is compact and μ is an Ahlfors s-regular Borel measure ($0 < s < \infty$). Let $\mathcal{Q} = \bigcup_{k\geq 0} \mathcal{Q}_k$ be the collection of generalized dyadic cubes in (X, ρ) given in Section 2.1.

Lemma 3.1 *For* $0 \leq t < s$ *and* $U \in \mathscr{B}$ *with* $\operatorname{diam} U > 0$*, we have*

$$(\operatorname{diam} U)^{-t}\mu(U)^2 \leq I_t(\mu, U) \leq C_1(\operatorname{diam} U)^{s-t}\mu(U),$$

where $C_1 > 0$ *is a constant which only depends on* s *and* t.

Proof Write $l = \operatorname{diam} U$, $l_j = 2^{-j}l$. For $t < s$ and $y \in U$, we have

$$\begin{aligned}
\phi_t(\mu, U, y) &= \int_U \rho(x, y)^{-t} d\mu(x) \leq \int_{\overline{B}(y,l)} \rho(x, y)^{-t} d\mu(x) \\
&= \sum_{j\geq 0} \int_{\overline{B}(y,l_j)\setminus\overline{B}(y,l_{j+1})} \rho(x, y)^{-t} d\mu(x) \\
&\leq \sum_{j\geq 0} l_{j+1}^{-t}\mu\Big(\overline{B}(y, l_j) \setminus \overline{B}(y, l_{j+1})\Big) \\
&= \sum_{j\geq 0} l_{j+1}^{-t}\Big(\mu(\overline{B}(y, l_j)) - \mu(\overline{B}(y, l_{j+1}))\Big) \\
&\leq \sum_{j\geq 0} l_{j+1}^{-t}\Big(Cl_j^s - C^{-1}l_{j+1}^s\Big) \\
&= l^{s-t}\sum_{j\geq 0} 2^{(1+j)(t-s)}(C2^s - C^{-1}) \\
&= C_1(\operatorname{diam} U)^{s-t},
\end{aligned}$$

where $C_1 > 0$ is a constant. Then

$$I_t(\mu, U) = \int_U \phi_t(\mu, U, y)d\mu(y) \leq C_1(\operatorname{diam} U)^{s-t}\mu(U).$$

Also

$$\begin{aligned}
I_t(\mu, U) &= \int_U\int_U \rho(x, y)^{-t} d\mu(x)d\mu(y) \\
&\geq \int_U\int_U (\operatorname{diam} U)^{-t} d\mu(x)d\mu(y) = (\operatorname{diam} U)^{-t}\mu(U)^2.
\end{aligned}$$

□

Remark 3.1 For $0 \leq t < s$, since (X, ρ) is compact, we have $I_t(\mu) \leq C_1 \max\{1, (\operatorname{diam} X)^s\} < \infty$.

The following two lemmas play important roles in the proof of Theorem 1.1, which are inspired by [23, Lemmas 2.1 and 5.1], but more general than those in [23] since here we consider compact metric spaces and generalized dyadic cubes.

Lemma 3.2 *Let $\{\mu_n\}_{n\geq 1}$ be a sequence of Borel measures on X. Suppose there exists a constant $M > 0$ such that for every ball $B \subset X$,*

$$M^{-1} \leq \liminf_{n\to\infty} \frac{\mu_n(B)}{\mu(B)} \leq \limsup_{n\to\infty} \frac{\mu_n(B)}{\mu(B)} \leq M, \tag{3.1}$$

and $\sup_{n\geq 1} I_\gamma(\mu_n) < M$ for some $\gamma \in (0, s]$. Then for $t < \gamma$, there exists a constant $M_1 > 0$ such that

$$M_1^{-1} \leq \liminf_{n\to\infty} \frac{I_t(\mu_n, Q)}{I_t(\mu, Q)} \leq \limsup_{n\to\infty} \frac{I_t(\mu_n, Q)}{I_t(\mu, Q)} \leq M_1$$

holds for all $Q \in \mathcal{Q}$.

Proof Fix $Q \in \mathcal{Q}$. For every $\alpha > 0$, given $y \in Q$, let $Q_\alpha(y) = \{x \in Q\colon \rho(x, y) < \alpha^{-1/\gamma}\}$. Then

$$\phi_t(\mu_n, Q, y) = \int_{Q_\alpha(y)} \rho(x, y)^{-t} d\mu_n(x) + \int_{Q\setminus Q_\alpha(y)} \rho(x, y)^{-t} d\mu_n(x).$$

Since $I_\gamma(\mu_n) < M$, we have

$$\begin{aligned} M &> \iint \rho(x, y)^{-\gamma} d\mu_n(x) d\mu_n(y) \\ &\geq \int_Q \Big(\int_{Q_\alpha(y)} \rho(x, y)^{-\gamma} d\mu_n(x) \Big) d\mu_n(y) \\ &> \alpha \int_Q \mu_n(Q_\alpha(y)) d\mu_n(y), \end{aligned}$$

and it follows that

$$\begin{aligned} &\int_Q d\mu_n(y) \int_{Q_\alpha(y)} \rho(x, y)^{-t} d\mu_n(x) = \int_Q d\mu_n(y) \int_{Q_\alpha(y)} d\mu_n(x) \int_0^{\rho(x,y)^{-t}} du \\ &= \int_Q d\mu_n(y) \int_{Q_\alpha(y)} d\mu_n(x) \int_0^{\alpha^{t/\gamma}} du + \int_Q d\mu_n(y) \int_{Q_\alpha(y)} d\mu_n(x) \int_{\alpha^{t/\gamma}}^{\rho(x,y)^{-t}} du \\ &= \alpha^{t/\gamma} \int_Q \mu_n(Q_\alpha(y)) d\mu_n(y) + \int_{\alpha^{t/\gamma}}^{\infty} du \int_Q d\mu_n(y) \int_{Q_{u^{\gamma/t}}(y)} d\mu_n(x) \\ &\leq M\alpha^{t/\gamma - 1} + M \int_{\alpha^{t/\gamma}}^{\infty} u^{-\gamma/t} du \\ &= M\alpha^{t/\gamma-1} + \frac{Mt}{(\gamma - t)} \alpha^{t/\gamma - 1} = \frac{M\gamma}{(\gamma - t)} \alpha^{t/\gamma-1}, \end{aligned}$$

where we apply Fubini's theorem to derive the third equality. For $\epsilon > 0$, for α large enough, $\int_Q d\mu_n(y) \int_{Q_\alpha(y)} \rho(x, y)^{-t} d\mu_n(x) < \epsilon$. Observe that

$$\int_Q d\mu_n(y) \int_{Q \setminus Q_\alpha(y)} \rho(x, y)^{-t} d\mu_n(x) \le \int_Q \int_Q \min\{\rho(x, y)^{-t}, \alpha^{t/\gamma}\} d\mu_n(x) d\mu_n(y).$$

Denote $f(x, y) = \min\{\rho(x, y)^{-t}, \alpha^{t/\gamma}\}$. Note that $f(x, y)$ is continuous on $Q \times Q$. Let $\{Q_{k,i} \in \mathcal{Q}_k : 1 \le i \le N(k)\}$ be a partition of Q by Borel sets satisfying $\max_{1 \le i \le N(k)} \operatorname{diam}(Q_{k,i}) \to 0$, $k \to \infty$, and take $(x_i, y_j) \in Q_{k,i} \times Q_{k,j}$, then for all $(x, y) \in Q \times Q$ we have

$$\sum_{i,j}^{N(k)} f(x_i, y_j) \chi_{\{Q_{k,i} \times Q_{k,j}\}}(x, y) \to f(x, y), \ as\ k \to \infty.$$

Since $f(x, y) \le \alpha^{t/\gamma}$, using Dominated Convergence Theorem, we obtain

$$\begin{aligned}
\int_Q \int_Q f(x, y) d\mu_n(x) d\mu_n(y) &= \lim_{k\to\infty} \int_Q \int_Q \sum_{i,j}^{N(k)} f(x_i, y_j) \chi_{\{Q_{k,i} \times Q_{k,j}\}}(x, y) d\mu_n(x) d\mu_n(y) \\
&= \lim_{k\to\infty} \sum_{i,j}^{N(k)} f(x_i, y_j) \mu_n(Q_{k,i}) \mu_n(Q_{k,j}) \\
&\le \lim_{k\to\infty} \sum_{i,j}^{N(k)} f(x_i, y_j) \mu_n(\overline{B}_{k,i}) \mu_n(\overline{B}_{k,j}),
\end{aligned}$$

where χ denotes the indicator function, and $\overline{B}_{k,i} = \overline{B}(x_{k,i}, c_1' b^k)$. Recall that $B(x_{k,i}, c_1 b^k) \subset Q_{k,i} \subset \overline{B}_{k,i}$, there exists a constant $C_2 > 0$ such that $\mu(\overline{B}_{k,i}) \le C_2 \mu(Q_{k,i})$. We derive from (3.1) that

$$\limsup_{n\to\infty} \frac{\mu_n(\overline{B})}{\mu(\overline{B})} \le C^2 M.$$

Then for $\theta > 0$, for n large enough, we have

$$\begin{aligned}
\lim_{k\to\infty} \sum_{i,j}^{N(k)} f(x_i, y_j) \mu_n(\overline{B}_{k,i}) \mu_n(\overline{B}_{k,j}) &\le (C^2 M + \theta)^2 \lim_{k\to\infty} \sum_{i,j}^{N(k)} f(x_i, y_j) \mu(\overline{B}_{k,i}) \mu(\overline{B}_{k,j}) \\
&\le (C^2 M + \theta)^2 C_2^2 \lim_{k\to\infty} \sum_{i,j}^{N(k)} f(x_i, y_j) \mu(Q_{k,i}) \mu(Q_{k,j}) \\
&= (C^2 M + \theta)^2 C_2^2 \int_Q \int_Q f(x, y) d\mu(x) d\mu(y) \\
&\le (C^2 M + \theta)^2 C_2^2 \int_Q \int_Q \rho(x, y)^{-t} d\mu(x) d\mu(y).
\end{aligned}$$

Since $\epsilon > 0$ and $\theta > 0$ are arbitrary, we prove that

$$\limsup_{n\to\infty} \frac{I_t(\mu_n, Q)}{I_t(\mu, Q)} \le C^4 M^2 C_2^2.$$

Notice that

$$I_t(\mu_n, Q) \ge (\operatorname{diam} Q)^{-t} \mu_n(Q)^2$$

and

$$\mu_n(Q) > C^{-1}2^{-s-1}M^{-1}(c_1/c_1')^s(\text{diam } Q)^s,$$

then combining these with Lemma 3.1, there exists a constant $C_3 > 0$ which only depends on $s,\ t$ such that

$$\frac{I_t(\mu_n, Q)}{I_t(\mu, Q)} \geq C_3.$$

Let $M_1 = \max\{C_3^{-1}, C^4M^2C_2^2\}$, then we proved this lemma. □

Remark 3.2 In Lemma 3.2, the condition $\sup_{n\geq 1} I_\gamma(\mu_n) < M$ can be weaken to $\sup_{n\geq N_0} I_\gamma(\mu_n) < M$ for some $N_0 \in \mathbb{N}$.

Lemma 3.3 *Under the same conditions as Lemma 3.2, let $\{A_n\}_{n\geq 1}$ be a sequence of open sets in X satisfying $\mu_n(X \setminus A_n) = 0$, $n \geq 1$. Then $\limsup\limits_{n\to\infty} A_n \in \mathcal{G}^\gamma(X)$.*

Proof Let $Q \in \mathcal{Q}$ and $t < \gamma$. By (3.1), $\mu_n(Q) > 0$ for n large enough. Define

$$\nu_n(F) = \int_{F\cap Q} (\phi_t(\mu_n, Q, y))^{-1} d\mu_n(y)$$

for every $F \in \mathscr{B}$. Since $\mu_n(X \setminus A_n) = 0$, $\nu_n(Q) = \nu_n(Q \cap A_n)$. Given $\epsilon > 0$, combining Jensen's inequality [20] and Lemma 3.2, for n large enough, we have

$$\begin{aligned}
\nu_n(F) &= \mu_n(F\cap Q)\int_{F\cap Q}\Big(\int_Q \rho(x,y)^{-t}d\mu_n(x)\Big)^{-1}\frac{1}{\mu_n(F\cap Q)}d\mu_n(y)\\
&\geq \mu_n(F\cap Q)\Big(\int_{F\cap Q}\Big(\int_Q \rho(x,y)^{-t}d\mu_n(x)\Big)\frac{1}{\mu_n(F\cap Q)}d\mu_n(y)\Big)^{-1}\\
&\geq (\mu_n(F\cap Q))^2\Big(\int_Q\int_Q \rho(x,y)^{-t}d\mu_n(y)d\mu_n(x)\Big)^{-1}\\
&> (\mu_n(F\cap Q))^2(M_1+\epsilon)^{-1}\Big(I_t(\mu,Q)\Big)^{-1}. \qquad (3.2)
\end{aligned}$$

Hence ν_n is a nonzero measure and absolutely continuous with respect to μ_n. By Lemma 3.1 and (3.2), for n large enough, we obtain

$$\nu_n(Q) \geq (M_1+\epsilon)^{-1}\frac{(\mu_n(Q))^2}{I_t(\mu,Q)} \geq C_1^{-1}(M_1+\epsilon)^{-1}\frac{(\mu_n(Q))^2}{(\text{diam } Q)^{s-t}\mu(Q)} \geq c_3\mu(Q)^{t/s}, \quad (3.3)$$

where the constant $c_3 > 0$ depends on $s,\ t$ and is independent from Q and n. By Jensen's inequality,

$$\phi_t^{-1}(\mu_n, Q, y) = \frac{1}{\mu_n(Q)}\Big(\int_Q \frac{\rho(x,y)^{-t}}{\mu_n(Q)}d\mu_n(x)\Big)^{-1} \leq \frac{1}{(\mu_n(Q))^2}\int_Q \rho(x,y)^t d\mu_n(x).$$

Then for every $J \in \mathcal{Q}$ and $J \subset Q$, we get that

$$\begin{aligned}
\nu_n(J) &= \int_J (\phi_t(\mu_n, Q, y))^{-1}d\mu_n(y) \leq \int_J (\phi_t(\mu_n, Q, y))^{-1}d\mu_n(y)\\
&\leq \frac{1}{(\mu_n(J))^2}\int_J\int_J \rho(x,y)^t d\mu_n(x)d\mu_n(y)\\
&\leq (\text{diam } J)^t \leq c_4\mu(J)^{t/s} \qquad (3.4)
\end{aligned}$$

for n large enough. The constant $c_4 > 0$ depends on $s,\ t$, and is independent of J and n.

Let $\{Q_k\}_{k\geq 1}$ be a cover of $Q \cap A_n$, where $Q_k \in \mathcal{Q}$. Without loss of generality, we assume $Q_k \cap Q_j = \varnothing$ for $k \neq j$ and $Q_k \subset Q$. For n large enough, we derive from (3.3) and (3.4) that

$$\sum_k \mu(Q_k)^{t/s} \geq c_4^{-1} \sum_k \nu_n(Q_k) \geq c_4^{-1} \nu_n(Q \cap A_n) = c_4^{-1} \nu_n(Q) \geq c_3 c_4^{-1} \mu(Q)^{t/s}.$$

It follows that

$$\liminf_{n\to\infty} \mathscr{M}^t_\infty(Q \cap A_n) \geq c_3 c_4^{-1} \mu(Q)^{t/s}$$

for n large enough. Therefore, for every $m \geq 1$ and generalized dyadic cube Q,

$$\mathscr{M}^t_\infty\Big(\bigcup_{n\geq m} A_n \cap Q\Big) \geq \sup_{n\geq m} \mathscr{M}^t_\infty(A_n \cap Q) \geq c_3 c_4^{-1} \mu(Q)^{t/s}. \quad (\forall t < \gamma)$$

So $\bigcup_{n\geq m} A_n \in \mathcal{G}^\gamma(X)$ by (2.3). Thus from (3) in Proposition 2.2, we prove that

$$\limsup_{n\to\infty} A_n = \bigcap_{m\geq 1} \bigcup_{n\geq m} A_n \in \mathcal{G}^\gamma(X).$$

□

Recall the 5r-covering theorem in metric space.

Lemma 3.4 ([19]) *Let (X, ρ) be a separable metric space and $\mathcal{A}$ be a family either of closed balls or open balls such that*

$$\sup\{\operatorname{diam} B : B \in \mathcal{A}\} < \infty.$$

Then there is a finite or countable sequence $\{B_i\}_{i\in I}$ of pairwise disjoint balls such that

$$\bigcup_{B\in\mathcal{A}} B \subset \bigcup_{i\in I} 5B_i.$$

Now we prove Theorem 1.1.

Proof of Theorem 1.1 Since $\mu\Big(\limsup_{n\to\infty} B_n\Big) = 1$, for $n \geq 1$, there is some N_n such that

$$\mu\Big(\bigcup_{m=n}^{N_n} B_m\Big) > 1 - \frac{1}{2n}.$$

Denote $A_n = \bigcup_{m=n}^{N_n} B_m$. By Lemma 3.4, there is $\mathcal{I}_n \subset \{n, n+1, \ldots N_n\}$ such that

$$B_i \cap B_j = \varnothing, \text{ for } i \neq j \in \mathcal{I}_n \quad \text{and} \quad A_n \subset \bigcup_{m\in\mathcal{I}_n} 5B_m.$$

(i) Construct measures

Let $A'_n = \bigcup_{m\in\mathcal{I}_n} B_m$. Since

$$\mu(A'_n) = \sum_{m\in\mathcal{I}_n} \mu(B_m) \geq C^{-2} 5^{-s} \sum_{m\in\mathcal{I}_n} \mu(5B_m) > C^{-2} 5^{-s} (1 - \frac{1}{2n}) > 0, \tag{3.5}$$

and $\mu(A'_n) < 1$, we define

$$\eta_n(F) = \frac{1}{\mu(A'_n)} \mu(F \cap A'_n)$$

for every $F \in \mathscr{B}$. Then η_n is a probability measure supported on A_n' satisfying $\eta_n(A_n') = \eta_n(X) = 1$.

Let $F_n = \bigcup_{m\in\mathcal{I}_n} E_m$. Define

$$\mu_n(F) = \sum_{m\in\mathcal{I}_n} \eta_n(B_m)\frac{\mu(E_m\cap F)}{\mu(E_m)}$$

for $F \in \mathscr{B}$. Then μ_n is a probability measure supported on F_n satisfying $\mu_n(F_n) = \mu_n(X) = 1$.

Let $B \subset X$ be a ball. Firstly we will show the following inequality

$$\limsup_{n\to\infty} \frac{\eta_n(B)}{\mu(B)} \le C^4 5^s. \tag{3.6}$$

Denote $B = B(x_B, r_B)$. Denote the radius of B_n by r_n. Since r_n decreases to 0 as $n \to \infty$, for $\theta > 0$, there is some $N > 0$ such that for $n \ge N$, $m \in \mathcal{I}_n$, we have $r_m < r_B\theta/2$. Note that

$$\bigcup_{\substack{m\in\mathcal{I}_n\\ B\cap B_m\neq\varnothing}} B_m \subset (1+\theta)B,$$

hence

$$\begin{aligned}\eta_n(B) &= \frac{1}{\mu(A_n')}\mu(B\cap A_n') \le \frac{1}{\mu(A_n')}\mu\Big(\bigcup_{\substack{m\in\mathcal{I}_n\\ B\cap B_m\neq\varnothing}} B_m\Big)\\ &\le \frac{\mu((1+\theta)B)}{\mu(A_n')} < \frac{C^2(1+\theta)^s\mu(B)}{C^{-2}5^{-s}(1-1/2n)},\end{aligned}$$

which implies

$$\limsup_{n\to\infty} \frac{\eta_n(B)}{\mu(B)} \le C^4 5^s.$$

Secondly we will prove that

$$C^{-4}5^{-s} \le \liminf_{n\to\infty} \frac{\mu_n(B)}{\mu(B)} \le \limsup_{n\to\infty} \frac{\mu_n(B)}{\mu(B)} \le C^6 5^s$$

hold for every ball B. Given $\theta > 0$, since $\bigcup_{\substack{m\in\mathcal{I}_n\\ B_m\cap B\neq\varnothing}} B_m \subset (1+\theta)B$ holds for n large enough, we obtain

$$\begin{aligned}\mu_n(B) &= \sum_{m\in\mathcal{I}_n} \eta_n(B_m)\frac{\mu(E_m\cap B)}{\mu(E_m)} = \sum_{\substack{m\in\mathcal{I}_n\\ B_m\cap B\neq\varnothing}} \eta_n(B_m)\frac{\mu(E_m\cap B)}{\mu(E_m)}\\ &\le \sum_{\substack{m\in\mathcal{I}_n\\ B_m\cap B\neq\varnothing}} \eta_n(B_m) \le \eta_n((1+\theta)B) \le (C^4 5^s+\theta)(1+\theta)^s C^2\mu(B).\end{aligned}$$

The last inequality follows from (3.6). So $\limsup\limits_{n\to\infty} \frac{\mu_n(B)}{\mu(B)} \le C^6 5^s$.

Let $\mathcal{I}_n(B) = \{m \in \mathcal{I}_n : B_m \subset B\}$, then we have

$$\begin{aligned}
\liminf_{n\to\infty} \frac{\mu_n(B)}{\mu(B)} &= \liminf_{n\to\infty} \frac{1}{\mu(B)} \sum_{m\in\mathcal{I}_n} \eta_n(B_m) \frac{\mu(E_m \cap B)}{\mu(E_m)} \\
&\geq \liminf_{n\to\infty} \frac{1}{\mu(B)} \sum_{m\in\mathcal{I}_n(B)} \eta_n(B_m) \\
&\geq \liminf_{n\to\infty} \frac{1}{\mu(B)} \sum_{m\in\mathcal{I}_n(B)} \frac{\mu(B_m)}{\mu(A'_n)} \\
&\geq \liminf_{n\to\infty} \frac{1}{\mu(B)} \sum_{m\in\mathcal{I}_n(B)} \mu(B_m). \qquad (3.7)
\end{aligned}$$

We claim that

$$\liminf_{n\to\infty} \sum_{m\in\mathcal{I}_n(B)} \mu(B_m) \geq C^{-4}5^{-s}\mu(B). \qquad (3.8)$$

Then combining (3.7) and (3.8), we have

$$\liminf_{n\to\infty} \frac{\mu_n(B)}{\mu(B)} \geq \liminf_{n\to\infty} \frac{1}{\mu(B)} \sum_{m\in\mathcal{I}_n(B)} \mu(B_m) \geq C^{-4}5^{-s}.$$

Now we obtain the claim. For $0 < \epsilon < 1/2$, take n large enough such that $r_m < \epsilon r_B/10,\ m \in \mathcal{I}_n$. Then if $m \in \mathcal{I}_n \setminus \mathcal{I}_n(B)$, that is, $B^c \cap B_m \neq \varnothing$, we have $5B_m \cap (1-\epsilon)B = \varnothing$. Therefore

$$\begin{aligned}
\mu\Big((1-\epsilon)B \cap A_n\Big) &\leq \mu\Big(\bigcup_{m\in\mathcal{I}_n(B)} 5B_m\Big) + \mu\Big(\bigcup_{m\in\mathcal{I}_n\setminus\mathcal{I}_n(B)} 5B_m \cap (1-\epsilon)B\Big) \\
&= \mu\Big(\bigcup_{m\in\mathcal{I}_n(B)} 5B_m\Big) \leq \sum_{m\in\mathcal{I}_n(B)} \mu(5B_m) \\
&\leq C^2 5^s \mu\Big(\bigcup_{m\in\mathcal{I}_n(B)} B_m\Big).
\end{aligned}$$

We also note that

$$\mu\Big((1-\epsilon)B \cap A_n\Big) \geq \mu\Big((1-\epsilon)B\Big) - \frac{1}{2n} \geq C^{-2}(1-\epsilon)^s\mu(B) - \frac{1}{2n}.$$

Combining the inequalities above, we prove the claim.

(ii) Show that $\sup_{n\geq N} I_t(\mu_n) < \infty$ for some $N \geq 1$

Note that

$$\begin{aligned}
I_t(\mu_n) &= \int_X \int_X \rho(x,y)^{-t} d\mu_n(x) d\mu_n(y) = \int_{F_n} \int_{F_n} \rho(x,y)^{-t} d\mu_n(x) d\mu_n(y) \\
&= \sum_{m\in\mathcal{I}_n} \sum_{i\in\mathcal{I}_n} \int_{E_m} \int_{E_i} \rho(x,y)^{-t} d\mu_n(x) d\mu_n(y).
\end{aligned}$$

When $i = m$, since $\mu_n \mid_{E_m} = \frac{\eta_n(B_m)}{\mu(E_m)}\mu \mid_{E_m}$ and $\eta_n = \frac{1}{\mu(A'_n)}\mu \mid_{A'_n}$, by (3.5) we have

$$\int_{E_m}\int_{E_m} \rho(x,y)^{-t} d\mu_n(x) d\mu_n(y) = \Big(\frac{\eta_n(B_m)}{\mu(E_m)}\Big)^2 \int_{E_m}\int_{E_m} \rho(x,y)^{-t} d\mu(x) d\mu(y)$$
$$= \Big(\frac{\eta_n(B_m)}{\mu(E_m)}\Big)^2 I_t(\mu, E_m) = \Big(\frac{\mu(B_m)}{\mu(A'_n)\mu(E_m)}\Big)^2 I_t(\mu, E_m)$$
$$\le C^4 5^{2s}\Big(\frac{2n}{2n-1}\Big)^2\Big(\frac{\mu(B_m)}{\mu(E_m)}\Big)^2 I_t(\mu, E_m)$$

for $n \ge 1$. Due to $t < \lambda$, there is some absolute constant $M_3 > 0$ depending t such that

$$\sup_{n\ge 1} \frac{I_t(\mu, E_n)\mu(B_n)}{\mu(E_n)^2} < M_3.$$

Hence

$$\int_{E_m}\int_{E_m} \rho(x,y)^{-t} d\mu_n(x) d\mu_n(y) \le C^4 5^{2s}\Big(\frac{2n}{2n-1}\Big)^2\Big(\frac{\mu(B_m)}{\mu(E_m)}\Big)^2 I_t(\mu, E_m)$$
$$\le C^4 5^{2s}\Big(\frac{2n}{2n-1}\Big)^2 M_3 \mu(B_m).$$

For $i \ne m$, we suppose that there exists $0 < c < 1$ such that $E_m \subset cB_m$ for every $m \ge 1$, which is assumed for technical reasons, and can be referred to [23]. Otherwise, we consider $c^{-1}B_m$ instead of B_m from the beginning of this proof. For $x \in E_m,\ y \in E_i$, we get

$$\rho(x,y) \ge \rho(x_i, x_m) - c(r_m + r_i) \ge (1-c)\rho(x_i, x_m).$$

Then

$$\int_{E_m}\int_{E_i} \rho(x,y)^{-t} d\mu_n(x) d\mu_n(y) \le \int_{E_m}\int_{E_i} ((1-c)\rho(x_i,x_m))^{-t} d\mu_n(x) d\mu_n(y)$$
$$= ((1-c)\rho(x_i,x_m))^{-t}\mu_n(E_m)\mu_n(E_i) = ((1-c)\rho(x_i,x_m))^{-t}\eta_n(B_m)\eta_n(B_i)$$
$$= ((1-c)\rho(x_i,x_m))^{-t}\frac{\mu(B_m)\mu(B_i)}{(\mu(A'_n))^2} \le (1-c)^{-t}\frac{C^4 5^{2s}}{(1-\frac{1}{2n})^2}\rho(x_i,x_m)^{-t}\mu(B_m)\mu(B_i)$$
$$\le (1-c)^{-t}\frac{C^4 5^{2s}}{(1-\frac{1}{2n})^2} 2^t \int_{B_m}\int_{B_i} \rho(x,y)^{-t} d\mu(x) d\mu(y).$$

The last inequality follows from $\rho(x,y) \le \rho(x_i,x_m) + r_i + r_m \le 2\rho(x_i,x_m)$ for $x \in B_m,\ y \in B_i$.

Therefore by Remark 3.1 and the fact that $\{B_m\}_{m\in\mathcal{I}_n}$ are disjoint balls, for n large enough, we obtain

$$I_t(\mu_n) \le C^4 5^{2s}\Big(\frac{2n}{2n-1}\Big)^2 \sum_{m\in\mathcal{I}_n}\Big(\frac{\mu(B_m)}{\mu(E_m)}\Big)^2 I_t(\mu, E_m)$$
$$+ \frac{2^t}{(1-c)^t} C^4 5^{2s}\Big(\frac{2n}{2n-1}\Big)^2 \sum_{m\in\mathcal{I}_n}\sum_{\substack{i\in\mathcal{I}_n\\ i\ne m}} \int_{B_m}\int_{B_i} \rho(x,y)^{-t} d\mu(x) d\mu(y)$$
$$\le 2C^4 5^{2s} M_3 \sum_{m\in\mathcal{I}_n}\mu(B_m) + (1-c)^{-t} 2C^4 5^{2s} 2^t I_t(\mu)$$
$$\le 2C^4 5^{2s} M_3 + (1-c)^{-t} 2C^4 5^{2s} 2^t C_1 \max\{1, (\operatorname{diam} X)^s\} < \infty.$$

By Lemma 3.3, $\limsup_{n\to\infty} F_n \in \mathcal{G}^t(X)$ for all $t < \lambda$, hence $\limsup_{n\to\infty} F_n \in \mathcal{G}^\lambda(X)$. Applying Proposition 2.2 (2), we have $\limsup_{n\to\infty} E_n \in \mathcal{G}^\lambda(X)$. $\square$

Proof of Corollary 1.2 Without loss of generality, we assume that $\operatorname{diam} B_n < 1$. Given $t \in (0, s]$, write $F_n = B_n^t$, then $\mu\Big(\limsup_{n\to\infty} F_n\Big) = 1$. For $\beta \geq 0$, by Lemma 3.1, we have

$$C^{-1}2^{-\beta}r_n^{t-\beta} \leq \frac{I_\beta(\mu, B_n)\mu(F_n)}{\mu(B_n)^2} \leq C^2 C_1 2^{s-\beta} r_n^{t-\beta}.$$

Notice that if $\beta \leq t$, we have $\sup_{n\geq 1} \frac{I_\beta(\mu,B_n)\mu(F_n)}{\mu(B_n)^2} < \infty$, and if $\beta > t$, then $\limsup_{n\to\infty} \frac{I_\beta(\mu,B_n)\mu(F_n)}{\mu(B_n)^2} = \infty$, since r_n decreases to 0 as $n \to \infty$. Hence

$$t = \sup\Big\{\beta \geq 0\colon \sup_{n\geq 1} \frac{I_\beta(\mu, B_n)\mu(F_n)}{\mu(B_n)^2} < \infty\Big\},$$

and by Theorem 1.1, we conclude that $\limsup_{n\to\infty} B_n \in \mathcal{G}^t(X)$. $\square$

4 Applications

Let (X, ρ) be a compact metric space endowed with an Ahlfors s-regular probability measure μ. In this section we investigate the large intersection property of limsup random fractals, random covering sets, and limsup sets generated by rectangles.

4.1 Limsup Random Fractals

Given $0 < b < 1/3$, there is a family of generalized dyadic cubes of X defined in Subsection 2.1. In this subsection, we consider a model of limsup random fractals which is constructed by using generalized dyadic cubes $\mathcal{Q} = \bigcup_{n\geq 0} \mathcal{Q}_n$.

For $n \geq 1$, let $\Big\{Z_n(Q),\, Q \in \mathcal{Q}_n\Big\}$ be a sequence of random variables, each taking values in $\{0, 1\}$. Let

$$A(n) = \bigcup_{\substack{Q\in\mathcal{Q}_n,\\ Z_n(Q)=1}} Q^o,$$

where Q^o is the interior of Q. The random set

$$A = \limsup_{n\to\infty} A(n)$$

is called a limsup random fractal associated to $\big\{Z_n(Q),\, Q \in \mathcal{Q}_n, n \geq 1\big\}$.

For $n \geq 1$, and $Q \in \mathcal{Q}_n$, denote the probability $P_n(Q) := \mathbb{P}(Z_n(Q) = 1)$, and

$$\gamma_1 := -\limsup_{n\to\infty} \frac{\max_{Q\in\mathcal{Q}_n} \log_{b^{-1}} P_n(Q)}{n}, \tag{4.1}$$

$$\gamma_2 := -\limsup_{n\to\infty} \frac{\min_{Q\in\mathcal{Q}_n} \log_{b^{-1}} P_n(Q)}{n}. \tag{4.2}$$

We refer to γ_1 and γ_2 as the indices of the limsup random fratcal A.

We assume the following condition, which is similar as the condition (H2) in [16].

Correlation Condition Suppose that there is a constant $\delta \geq 0$ such that for all $\epsilon > 0$,

$$\limsup_{n\to\infty} \frac{1}{n} \log_{b^{-1}} f(n,\epsilon) \leq \delta, \tag{4.3}$$

where

$$f(n,\epsilon) = \max_{Q\in\mathcal{Q}_n} \#\{Q' \in \mathcal{Q}_n : \mathrm{Cov}(Z_n(Q), Z_n(Q')) \geq \epsilon P_n(Q)P_n(Q')\}.$$

Theorem 4.1 *Let $(X, \mathscr{B}, \mu, \rho)$ be a compact Ahlfors s-regular space $(0 < s < \infty)$. Let $A = \limsup_{n\to\infty} A(n)$ be a limsup random set with indices γ_1, γ_2 and satisfy the Correlation Condition. If $\gamma_2 + \delta < s$, then $A \in \mathcal{G}^{s-\gamma_2-\delta}(X)$ a.s.*

Corollary 4.1 *Under the setting in Theorem 4.1, we have*

$$\max\{0, s - \gamma_2 - \delta\} \leq \dim_{\mathrm{H}} A \leq \max\{0, s - \gamma_1\} \quad a.s.$$

Remark 4.1 We can also use the hitting probability of limsup random fractals to get the estimation on Hausdorff dimension in Corollary 4.1. For instance, let A be a limsup random fratcal in $\mathbb{T}$ with $\gamma_1 = \gamma_2 < 1$, $\delta = 0$, and if for every $n \geq 1$, there exists a constant $p_n \in [0, 1]$ such that $P_n(Q) = p_n$ for every $Q \in \mathcal{Q}_n$, Khoshnevisan, Peres and Xiao [18] showed that $\dim_{\mathrm{H}} A = 1 - \gamma_1$ a.s. For more related research, see [14, 16].

Before proving Theorem 4.1, we give some lemmas.

Lemma 4.1 *Let C and c_1' be the constants in (1.1) and (2.1) respectively. Let $B = B(x_B, r)$ be a ball with $r > 0$ and*

$$\mathcal{Q}_n(B) = \{Q \in \mathcal{Q}_n : Q \subset B\}.$$

Then there exists some $N \geq 1$ such that for all $n \geq N$, we have

$$\#\mathcal{Q}_n(B) \geq C^{-2}(r{c'_1}^{-1}b^{-n} - 2)^s.$$

Proof For $Q \in \mathcal{Q}_n(B)$, by (2.1),there exists $x_Q \in Q$ such that

$$Q \subset \overline{B}(x_Q, c_1' b^n).$$

For $r > 0$, there exists some $N \geq 1$ such that for all $n \geq N$, we have $r > 2c_1' b^n$. Then for $Q \in \mathcal{Q}_n$, observe that if $Q \notin \mathcal{Q}_n(B)$, then $Q \cap B(x_B, r - 2c_1' b^n) = \varnothing$, and we deduce that

$$B(x_B, r - 2c_1' b^n) \subset \bigcup_{Q\in\mathcal{Q}_n(B)} Q \cup \bigcup_{\substack{Q\notin\mathcal{Q}_n(B)\\ Q\cap B\neq\varnothing}} Q = \bigcup_{Q\in\mathcal{Q}_n(B)} Q.$$

Hence

$$\begin{aligned} C^{-1}(r - 2c_1' b^n)^s &\leq \mu\big(B(x_B, r - 2c_1' b^n)\big) \leq \mu\Big(\bigcup_{Q\in\mathcal{Q}_n(B)} Q\Big) \\ &\leq \mu\Big(\bigcup_{Q\in\mathcal{Q}_n(B)} \overline{B}(x_Q, c_1' b^n)\Big) \leq \#\mathcal{Q}_n(B) C (c_1' b^n)^s, \end{aligned}$$

which derives that

$$\#\mathcal{Q}_n(B) \geq \frac{(r - 2c_1' b^n)^s}{C^2(c_1' b^n)^s} \quad (\forall n \geq N).$$

$\square$

Recall the notation $B^t(x,r) = B(x, r^{t/s})$.

Lemma 4.2 *Suppose that $\gamma_2 + \delta < s$. For $t > 0$ with $t < s - \gamma_2 - \delta$, let*

$$B^t(n) = \bigcup_{\substack{Q \in \mathcal{Q}_n \\ Z_n(Q)=1}} B^t_Q,$$

where $B_Q = B(x_Q, c_1 b^n) \subset Q$. Given $x \in X$, let $J_{n,x}$ be the unique dyadic cube in $\mathcal{Q}_n$ containing x. Then for $k \geq 1$, we have

$$\mathbb{P}\Big(\bigcup_{n \geq k} \{J_{n,x} \subset B^t(n)\}\Big) = 1.$$

Proof Given $x \in X$, there is a $x' \in J_{n,x}$ such that

$$B(x', c_1 b^n) \subset J_{n,x}.$$

Let

$$\mathcal{B}_n(x) = \Big\{J \in \mathcal{Q}_n \colon J \subset B\Big(x', \frac{1}{3}(c_1 b^n)^{t/s}\Big)\Big\}.$$

By Lemma 4.1, for n large enough, we have

$$\#\mathcal{B}_n(x) \geq \frac{1}{C^2}\Big(\frac{c_1^{t/s}}{3c_1'} b^{n(t/s-1)} - 2\Big)^s. \tag{4.4}$$

Now we show that for n large enough, we have

$$\big\{J_{n,x} \subset B^t(n)\big\} \supset \big\{Z_n(J) = 1 \text{ for some } J \in \mathcal{B}_n(x)\big\}.$$

If $Z_n(J) = 1$ for some $J \in \mathcal{B}_n(x)$, we have $\operatorname{dist}(x', x_J) \leq \frac{1}{3}(c_1 b^n)^{t/s} - c_1 b^n$, which follows that $B(x', c_1' b^n) \subset B(x_J, \frac{1}{3}(c_1 b^n)^{t/s} - c_1 b^n + c_1' b^n)$. Notice that $B(x_J, \frac{1}{3}(c_1 b^n)^{t/s} - c_1 b^n + c_1' b^n) \subset B(x_J, (c_1 b^n)^{t/s})$ holds for n large enough and $J_{n,x} \subset B(x', c_1' b^n)$, then we conclude that $J_{n,x} \subset B(x_J, (c_1 b^n)^{t/s})$ for some $J \in \mathcal{Q}_n$ with $Z_n(J) = 1$, that is $J_{n,x} \subset B^t(n)$.

Define $M_n(x) = \sum_{J \in \mathcal{B}_n(x)} Z_n(J)$, and note that

$$\big\{Z_n(J) = 1 \text{ for some } J \in \mathcal{B}_n(x)\big\} = \big\{M_n(x) > 0\big\}.$$

From Chebyshev's inequality, we obtain

$$\mathbb{P}(M_n(x) = 0) \leq \frac{\operatorname{Var}(M_n(x))}{(\mathbb{E}(M_n(x)))^2}.$$

Since $\operatorname{Cov}(Z_n(J), Z_n(J')) \leq \mathbb{E}(Z_n(J) Z_n(J')) \leq P_n(J)$, we have

$$\begin{aligned}
\operatorname{Var}(M_n(x)) &= \sum_{J \in \mathcal{B}_n(x)} \sum_{J' \in \mathcal{B}_n(x)} \operatorname{Cov}(Z_n(J), Z_n(J')) \\
&= \sum_{J \in \mathcal{B}_n(x)} \Big(\sum_{J' \in \mathcal{G}_n(J)} \operatorname{Cov}(Z_n(J), Z_n(J')) + \sum_{J' \in \mathcal{T}_n(J)} \operatorname{Cov}(Z_n(J), Z_n(J'))\Big) \\
&\leq \sum_{J \in \mathcal{B}_n(x)} \Big(\sum_{J' \in \mathcal{G}_n(J)} \epsilon P_n(J) P_n(J') + \sum_{J' \in \mathcal{T}_n(J)} P_n(J)\Big) \\
&\leq \epsilon \Big(\sum_{J \in \mathcal{B}_n(x)} P_n(J)\Big)\Big(\sum_{J' \in \mathcal{G}_n(J)} P_n(J')\Big) + \max_{J \in \mathcal{B}_n(x)} \#\mathcal{T}_n(J) \Big(\sum_{J \in \mathcal{B}_n(x)} P_n(J)\Big),
\end{aligned}$$

where

$$\begin{aligned}\mathcal{G}_n(J) &= \{J' \in \mathcal{B}_n(x) \colon \mathrm{Cov}(Z_n(J), Z_n(J')) < \epsilon P_n(J) P_n(J')\},\\ \mathcal{T}_n(J) &= \mathcal{B}_n(x) \setminus \mathcal{G}_n(J).\end{aligned}$$

Recalling the notation of the Correlation Condition, we obtain

$$\mathrm{Var}(M_n(x)) \le \epsilon \Big(\sum_{J \in \mathcal{B}_n(x)} P_n(J) \Big)^2 + f(n,\epsilon) \Big(\sum_{J \in \mathcal{B}_n(x)} P_n(J) \Big).$$

Note that $\mathbb{E}(M_n(x)) = \sum_{J \in \mathcal{B}_n(x)} P_n(J) \ge \#\mathcal{B}_n(x) \big(\min_{Q \in \mathcal{Q}_n} P_n(Q) \big)$, we derive that

$$\begin{aligned}\mathbb{P}(M_n(x) = 0) &\le \frac{\epsilon (\mathbb{E}(M_n(x)))^2 + f(n,\epsilon)(\mathbb{E}(M_n(x)))}{(\mathbb{E}(M_n(x)))^2}\\ &= \epsilon + \frac{f(n,\epsilon)}{\mathbb{E}(M_n(x))}\\ &\le \epsilon + \frac{f(n,\epsilon)}{\#\mathcal{B}_n(x) \big(\min_{Q \in \mathcal{Q}_n} P_n(Q) \big)}.\end{aligned}$$

From (4.3) and (4.4), for $\theta > 0$ with $2\theta < s - \delta - \gamma_2 - t$, we get

$$f(n,\epsilon) \le b^{-n(\delta+\theta)} \quad \text{and} \quad \#\mathcal{B}_n(x) \ge c_5 b^{n(t-s)}$$

for n large enough, where $c_5 > 0$ is an absolute constant, and from (4.2), we have

$$\min_{Q \in \mathcal{Q}_n} P_n(Q) \ge b^{n(\gamma_2+\theta)}$$

for infinitely many n, denoted by $\mathcal{N}$. Then

$$\limsup_{\substack{n \in \mathcal{N}\\ n \to \infty}} \frac{f(n,\epsilon)}{\#\mathcal{B}_n(x) \min_{Q \in \mathcal{Q}_n} P_n(Q)} \le \limsup_{\substack{n \in \mathcal{N}\\ n \to \infty}} c_5^{-1} b^{-n(2\theta+\delta+\gamma_2-s+t)} = 0,$$

which implies $\limsup\limits_{\substack{n \in \mathcal{N}\\ n \to \infty}} \mathbb{P}(M_n(x) = 0) = 0$. Therefore

$$\limsup_{\substack{n \in \mathcal{N}\\ n \to \infty}} \mathbb{P}(M_n(x) > 0) = 1.$$

So we show that

$$\limsup_{n \to \infty} \mathbb{P}\big(J_{n,x} \subset B^t(n) \big) = 1.$$

Hence for $k \ge 1$,

$$\mathbb{P}\Big(\bigcup_{n=k}^{\infty} \{J_{n,x} \subset B^t(n)\} \Big) \ge \limsup_{n \to \infty} \mathbb{P}\big(J_{n,x} \subset B^t(n) \big) = 1.$$

□

Proof of Theorem 4.1 Let $x \in X$ and $t > 0$ with $t < s - \gamma_2 - \delta$. By Lemma 4.2, we have $\mathbb{P}\Big(\bigcup_{n=k}^{\infty} \{J_{n,x} \subset B^t(n)\} \Big) = 1$ for $k \ge 1$. Note that

$$\Big\{ x \in \bigcup_{n=k}^{\infty} B^t(n) \Big\} \supset \big\{ \exists\, n \ge k \text{ such that } J_{n,x} \subset B^t(n) \big\}.$$

This yields that $\mathbb{P}\{x \in \bigcup_{n=k}^{\infty} B^t(n)\} = 1$. Write the set $F_k = \bigcup_{n=k}^{\infty} B^t(n)$, then

$$\int d\mu(x) \int \chi_{\{x \in F_k\}} d\mathbb{P}(\omega) = 1.$$

Using Fubini's Theorem, we obtain

$$\int d\mathbb{P}(\omega) \int \chi_{F_k}(x) d\mu(x) = 1,$$

implying that $\mu(F_k) = 1$ a.s. for any $k \geq 1$. Hence $\mu\Big(\limsup_{n\to\infty} B^t(n)\Big) = 1$ a.s.

By Corollary 1.2, for $t > 0$ with $t < s - \gamma_2 - \delta$, we have $\limsup_{n\to\infty} B(n) \in \mathcal{G}^t(X)$ a.s., which implies $\limsup_{n\to\infty} B(n) \in \mathcal{G}^{s-\gamma_2-\delta}(X)$ a.s. Hence by Proposition 2.2 (2), $\limsup_{n\to\infty} A(n) \in \mathcal{G}^{s-\gamma_2-\delta}(X)$ a.s. □

Proof of Corollary 4.1 If $s \geq \gamma_1$, let $t > s - \gamma_1$. For $\delta \in (0, 2c_1')$, there exists $k_0 \geq 1$ such that $2c_1' b^{k_0} \leq \delta < 2c_1' b^{k_0-1}$. Since

$$A \subset \bigcup_{n=k}^{\infty} \bigcup_{\substack{Q \in \mathcal{Q}_n \\ Z_n(Q)=1}} Q^o$$

for all $k \geq 1$, we have

$$\mathcal{H}_\delta^t(A) \leq \sum_{n \geq k_0} \sum_{Q \in \mathcal{Q}_n} (\text{diam } Q)^t Z_n(Q).$$

Then

$$\mathbb{E}(\mathcal{H}_\delta^t(A)) \leq \sum_{n \geq k_0} \sum_{Q \in \mathcal{Q}_n} (\text{diam } Q)^t P_n(Q).$$

Considering $\theta > 0$ with $\gamma_1 > \theta > s - t$, by (4.1), for n large enough,

$$\max_{Q \in \mathcal{Q}_n} P_n(Q) \leq b^{n\theta},$$

which implies that

$$\mathbb{E}(\mathcal{H}_\delta^t(A)) \leq \textstyle\sum_{n \geq k_0} \#\mathcal{Q}_n (2c_1' b^n)^t b^{n\theta} \leq C c_1^{-s} (2c_1')^t \sum_{n \geq k_0} b^{n(t-s+\theta)} < \infty, \tag{4.5}$$

giving $\dim_{\mathrm{H}} A \leq s - \gamma_1$ a.s. If $\gamma_1 > s$, taking $t > 0$ and $\theta = s$ in (4.5), we have $\mathbb{E}(\mathcal{H}_\delta^t(A)) \leq C c_1^{-s} (2c_1')^t \sum_{n \geq k_0} b^{nt} < \infty$ a.s., which implies $\dim_{\mathrm{H}} A \leq 0$ a.s.

When $s > \gamma_2 + \delta$, from Theorem 4.1, we have $A \in \mathcal{G}^{s-\gamma_2-\delta}(X)$ a.s. Then we obtain from Proposition 2.2 (4) that $\dim_{\mathrm{H}} A \geq s - \gamma_2 - \delta$ a.s. □

4.2 Random Covering Sets

Durand [8] showed that random covering sets in the one-dimensional torus $\mathbb{T}$, defined as limsup sets of a sequence of balls whose centers are independent and uniformly distributed random variables, have large intersection property. In this subsection, we consider random covering sets in $(X, \mathscr{B}, \mu, \rho)$ with weak dependence condition. Let $\{\xi_n\}_{n\geq 1}$ be a stationary process on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ having μ as probability law.

Definition 4.1 We say that $\{\xi_n\}_{n\geq 1}$ is exponentially mixing if for every $n \geq 1$, there exist two constants $c > 0$ and $0 < \gamma < 1$ such that

$$| \mathbb{P}(\xi_1 \in A \mid D) - \mathbb{P}(\xi_1 \in A) | \leq c\gamma^n$$

holds for every ball $A \subset X$ and $D \in \mathcal{B}_{n+1}$, where $\mathcal{B}_{n+1}$ is the sub-σ-field generated by $\{\xi_{n+i}\}_{i\geq 1}$.

Let $\{r_n\}_{n\geq 1}$ be a sequence of positive real numbers decreasing to zero. The *random covering set* is defined as

$$E := \limsup_{n\to\infty} B(\xi_n, r_n) = \{y \in X : y \in B(\xi_n, r_n) \text{ for infinitely many } n \geq 1\}.$$

The set E consists of the points which are covered by $\{B(\xi_n, r_n)\}_{n\geq 1}$ infinitely often.

Theorem 4.2 ([15]) *Let $\{\xi_n\}_{n\geq 1}$ be exponentially mixing and the probability law μ be Ahlfors s-regular. Then we have*

$$\mu(E) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} r_n^s < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} r_n^s = \infty \end{cases} \quad a.s.$$

Theorem 4.3 *Let $(X, \mathscr{B}, \mu, \rho)$ be a compact Ahlfors s-regular space $(0 < s < \infty)$. Let $\{\xi_n\}_{n\geq 1}$ be exponentially mixing and the probability law μ be Ahlfors s-regular. Then the random covering set $E \in \mathcal{G}^{s_0}(X)$ a.s., where $s_0 = \inf\{t \geq 0 : \sum_{n=1}^{\infty} r_n^t < \infty\}$.*

Proof For $t < s_0$, by Theorem 4.2, we have

$$\mu(\limsup_{n\to\infty} B(\xi_n, r_n^{t/s})) = 1 \quad a.s.,$$

since $\sum_{n=1}^{\infty} (r_n^{t/s})^s = \infty$. Applying Corollary 1.2, we finish the proof. □

4.3 Limsup Sets Generated by Rectangles

The mass transference principle from balls to limsup sets generated by rectangles was established by [24] in 2015. In 2021, Ding [4] extended the result to product compact metric spaces, and proved that such sets have large intersection property. In this subsection, we use a different method to show the large intersection property of limsup sets generated by rectangles.

For $1 \leq i \leq d$, let (X_i, ρ_i) be a compact metric spaces equipped with a Borel probability measure μ_i which is Ahlfors s_i-regular, that is, there exists a constant $\delta_i > 1$ such that $\delta_i^{-1} r^{s_i} \leq \mu_i(B_i(x, r)) \leq \delta_i r^{s_i}$, here $B_i(x, r)$ is the open ball in X_i. We consider the metric space $(\prod_{i=1}^{d} X_i, \rho)$ with the measure μ, where

$$\rho(x, y) = \max_{1\leq i\leq d} \{\rho_i(x_i, y_i)\}$$

for $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d) \in \prod_{i=1}^d X_i$ and μ satisfies that $\mu(\prod_{i=1}^d A_i) = \prod_{i=1}^d \mu_i(A_i)$ for A_i is a Borel set in X_i. The metric space $(\prod_{i=1}^d X_i, \rho)$ is compact.

Theorem 4.4 *For $1 \le i \le d$, let $\{B_i(x_{n,i}, r_n)\}_{n\ge1}$ be a sequence of balls in X_i with $r_n \to 0$ as $n \to \infty$. Let $\{a_i\}_{1\le i\le d}$ be a sequence of positive numbers with $1 \le a_1 \le \cdots \le a_d$. Assume that $\mu(\limsup_{n\to\infty} \prod_{i=1}^d B_i(x_{n,i}, r_n)) = 1$. Then $\limsup_{n\to\infty} \prod_{i=1}^d B_i(x_{n,i}, r_n^{a_i}) \in \mathcal{G}^s(\prod_{i=1}^d X_i)$, where*

$$s = \min_{1\le i\le d}\left\{\frac{\sum_{j=1}^d s_j + a_i\sum_{j=1}^i s_j - \sum_{j=1}^i a_j s_j}{a_i}\right\}.$$

Proof We assume that $r_n < 1$, for $n \ge 1$. Denote $B_n = \prod_{i=1}^d B_i(x_{n,i}, r_n)$ and $R_n = \prod_{i=1}^d B_i(x_{n,i}, r_n^{a_i})$. Let $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$. For $t < s$ and $y \in R_n$, we have

$$\phi_t(\mu, R_n, y) = \int_{R_n} \rho(x, y)^{-t} d\mu(x) \le \int_{\prod_{i=1}^d B_i(y_i, 2r_n^{a_i})} \rho(x, y)^{-t} d\mu(x).$$

Let j be the integer with $\sum_{i=1}^{j-1} s_i < t \le \sum_{i=1}^{j} s_i$. Denote

$$F_{1,j} = \left\{x \in \prod_{i=1}^d B_i(y_i, 2r_n^{a_i})\colon \max_{1\le i\le j} \rho_i(x_i, y_i) < 4r_n^{a_j}\right\},$$

$$F_{2,j} = \left\{x \in \prod_{i=1}^d B_i(y_i, 2r_n^{a_i})\colon \max_{1\le i\le j-1} \rho_i(x_i, y_i) \ge r_n^{a_j}\right\}.$$

Note that $R_n \subset F_{1,j} \cup F_{2,j}$. Write $f = (\max_{1\le i\le j-1} \rho_i(x_i, y_i))^{-t}$. Using Fubini's theorem, we derive that

$$\begin{aligned}
\phi_t(\mu, R_n, y) &\le \int_{F_{1,j}} \rho(x, y)^{-t} d\mu(x) + \int_{F_{2,j}} \rho(x, y)^{-t} d\mu(x)\\
&\le \int_{F_{1,j}} (\max_{1\le i\le j} \rho_i(x_i, y_i))^{-t} d\mu_1(x_1)\ldots d\mu_d(x_d) + \int_{F_{2,j}} f d\mu_1(x_1)\ldots d\mu_d(x_d)\\
&\le \beta_1 \prod_{i=j+1}^d r_n^{a_i s_i} \int_{\prod_{i=1}^j B_i(y_i, 4r_n^{a_j})} (\max_{1\le i\le j} \rho_i(x_i, y_i))^{-t} d\mu_1(x_1)\ldots d\mu_j(x_j)\\
&+\beta_1 \prod_{i=j}^d r_n^{a_i s_i} \int_{\prod_{i=1}^{j-1} B_i(y_i, 2r_n^{a_i})\setminus\prod_{i=1}^{j-1} B_i(y_i, r_n^{a_j})} f d\mu_1(x_1)\ldots d\mu_{j-1}(x_{j-1})\\
&\le \beta_1'\Big(\prod_{i=j+1}^d r_n^{a_i s_i} r_n^{-a_j t} \prod_{i=1}^j r_n^{a_j s_i} + \prod_{i=j}^d r_n^{a_i s_i} I_2\Big), \qquad (4.6)
\end{aligned}$$

where β_1, β_1' are absolute constants, β_1 depends on $\{\delta_i\}_{j\le i\le d}$ and $\{s_i\}_{j\le i\le d}$, and β_1' depends on t, $\{\delta_i\}_{j\le i\le d}$ and $\{s_i\}_{j\le i\le d}$. Then we have

$$
\begin{aligned}
I_2 &= \int_{\prod_{i=1}^{j-1} B_i(y_i,2r_n^{a_i})\setminus\prod_{i=1}^{j-1} B_i(y_i,r_n^{a_j})} \Big(\int_0^f du\Big) d\mu_1(x_1)\dots d\mu_{j-1}(x_{j-1})\\
&= \int_{\prod_{i=1}^{j-1} B_i(y_i,2r_n^{a_i})\setminus\prod_{i=1}^{j-1} B_i(y_i,r_n^{a_j})} r_n^{-a_j t} d\mu_1(x_1)\dots d\mu_{j-1}(x_{j-1})\\
&- \int_{\prod_{i=1}^{j-1} B_i(y_i,2r_n^{a_i})\setminus\prod_{i=1}^{j-1} B_i(y_i,r_n^{a_j})} \Big(\int_f^{r_n^{-a_j t}} du\Big) d\mu_1(x_1)\dots d\mu_{j-1}(x_{j-1})\\
&\le r_n^{-a_j t}\Big(\prod_{i=1}^{j-1}\mu_i(B_i(y_i,2r_n^{a_i})) - \prod_{i=1}^{j-1}\mu_i(B_i(y_i,r_n^{a_j}))\Big)\\
&- \int_0^{r_n^{-a_j t}} du \int_{\prod_{i=1}^{j-1} B_i(y_i,2r_n^{a_i})\setminus\prod_{i=1}^{j-1} B_i(y_i,u^{-1/t})} d\mu_1(x_1)\dots d\mu_{j-1}(x_{j-1})\\
&= -r_n^{-a_j t}\prod_{i=1}^{j-1}\mu_i(B_i(y_i,r_n^{a_j})) + \int_0^{r_n^{-a_j t}} \prod_{i=1}^{j-1}\mu_i(B_i(y_i,u^{-1/t}))du\\
&\le -r_n^{-a_j t}\prod_{i=1}^{j-1}\delta_i^{-1} r_n^{a_j s_i} + \prod_{i=1}^{j-1}\delta_i \int_0^{r_n^{-a_j t}} u^{-\sum_{i=1}^{j-1} s_i/t} du \le \alpha r_n^{-a_j t}\prod_{i=1}^{j-1} r_n^{a_j s_i},
\end{aligned}
\tag{4.7}
$$

where $\alpha > 0$ is an absolute constant which depends on t, $\{\delta_i\}_{1\le i\le j-1}$ and $\{s_i\}_{1\le i\le j-1}$. Therefore by (4.6) and (4.7),

$$
\phi_t(\mu, R_n, y) \le \beta_1'(1+\alpha)\prod_{i=j+1}^{d} r_n^{a_i s_i} r_n^{-a_j t}\prod_{i=1}^{j} r_n^{a_j s_i}.
$$

Hence there exists an absolute constant $\beta_2 > 0$ which depends on t, $\{\delta_i\}_{1\le i\le d}$ and $\{s_i\}_{1\le i\le d}$ such that

$$
\frac{I_t(\mu,R_n)\mu(B_n)}{\mu(R_n)^2} \le \beta_2 r_n^{-a_j t + a_j\sum_{i=1}^{j} s_i + \sum_{i=1}^{d} s_i - \sum_{i=1}^{j} a_i s_i} \le \beta_2. \tag{4.8}
$$

From the proof of Theorem 1.1, for every t such that $\sup_{n\ge 1}\frac{I_t(\mu,R_n)\mu(B_n)}{\mu(R_n)^2} < \infty$, we have $\limsup_{n\to\infty} R_n \in \mathcal{G}^t(\prod_{i=1}^d X_i)$. Note that (4.8) holds for any $t < s$, then $\limsup_{n\to\infty} R_n \in \mathcal{G}^s(\prod_{i=1}^d X_i)$. $\square$

Remark 4.2 In [4], Ding also showed that $\limsup_{n\to\infty}\prod_{i=1}^d B_i(x_{n,i}, r_n^{a_i})$ has large intersection property by showing the set satisfies (2.2) directly. Ding defined a new measure supported on $\limsup_{n\to\infty}\prod_{i=1}^d \overline{B}_i(x_{n,i}, r_n^{a_i})$. Then by applying the relationships between the measure of balls and their diameters and the version of Theorem 2.2 in $\prod_{i=1}^d X_i$, Ding proved the same conclusion as Theorem 4.4.

Acknowledgements We thank the referee for the helpful comments. The work was supported by NSFC 12271176, 11671151 and 11871208.

Data Availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of Interests The authors declared that they have no conflicts of interests to this work.

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Fractals, Vol. 30, No. 3 (2022) 2250055 (9 pages)

DOI: 10.1142/S0218348X22500554

ON THE HITTING PROBABILITIES OF LIMSUP RANDOM FRACTALS

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Received June 24, 2021
Accepted December 11, 2021
Published March 17, 2022

Abstract

Let A be a limsup random fractal with indices γ_1, γ_2 and δ on $[0,1]^d$. We determine the hitting probability $\mathbb{P}(A \cap G)$ for any analytic set G with the condition $(\star)$: $\dim_{\mathrm{H}}(G) > \gamma_2 + \delta$, where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension. This extends the correspondence of Khoshnevisan *et al.*[1] by relaxing the condition that the probability P_n of choosing each dyadic hyper-cube is homogeneous and $\lim_{n\to\infty} \frac{\log_2 P_n}{n}$ exists. We also present some counterexamples to show the Hausdorff dimension in condition $(\star)$ cannot be replaced by the packing dimension.

Keywords: Limsup Random Fractals; Hitting Probability; Hausdorff Dimension.

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1. INTRODUCTION

A limsup random fractal is a type of limsup set induced by a random model defined on the unit cube $[0,1]^d$ of Euclidean space. Let $\mathcal{Q}_n$ denote the collection of d-dimensional dyadic hyper-cubes in $[0,1]^d$ for $n \geq 0$, that is,

$$\mathcal{Q}_n = \{[k_1 2^{-n}, (k_1+1)2^{-n}] \times \cdots \times [k_d 2^{-n}, (k_d+1)2^{-n}] \colon 0 \leq k_i \leq 2^n - 1, 1 \leq i \leq d\}.$$

Let $\mathcal{Q} = \bigcup_{n\geq 0} \mathcal{Q}_n$. For $n \geq 1$, let $\{Z_n(Q), Q \in \mathcal{Q}_n\}$ be a collection of random variables defined on a probability space $(\Omega, \mathscr{B}, \mathbb{P})$, each taking values in $\{0,1\}$. We say that $Q \in \mathcal{Q}_n$ is chosen if $Z_n(Q) = 1$. Let

$$A(n) = \bigcup_{\substack{Q\in\mathcal{Q}_n,\\ Z_n(Q)=1}} Q.$$

That is, the union of chosen dyadic cubes of order n. The random set

$$A = \limsup_{n\to\infty} A(n)$$

is called a **limsup random fractal** associated to $\{Z_n(Q), n \geq 1, Q \in \mathcal{Q}_n\}$. For $n \geq 1$ and $Q \in \mathcal{Q}_n$, denote $P_n(Q) = \mathbb{P}(Z_n(Q) = 1)$, and write

$$\gamma_1 := -\limsup_{n\to\infty} \frac{\log_2(\max_{Q\in\mathcal{Q}_n} P_n(Q))}{n}, \qquad (1.1)$$

$$\gamma_2 := -\limsup_{n\to\infty} \frac{\log_2(\min_{Q\in\mathcal{Q}_n} P_n(Q))}{n}. \qquad (1.2)$$

We adopt the convention that $\log_2 0 = -\infty$.

We assume the following condition, which is a slight modification of Condition 5 in Ref. 1.

Correlation Condition: There is a constant $\delta \geq 0$ such that for all $\epsilon > 0$,

$$\limsup_{n\to\infty} \frac{1}{n} \log_2 f(n, \epsilon) \leq \delta, \qquad (1.3)$$

where

$$f(n,\epsilon) = \max_{Q\in\mathcal{Q}_n} \#\{Q' \in \mathcal{Q}_n \colon \mathrm{Cov}(Z_n(Q), Z_n(Q')) \geq \epsilon P_n(Q) P_n(Q')\}$$

and $\mathrm{Cov}(X, Y)$ is the covariance of random variables X and Y. We refer to γ_1, γ_2 and δ as the indices of the limsup random fractal A.

In 2000, Dembo *et al.*[2] established a lower bound for the Hausdorff dimension of limsup random fractals. Khoshnevisan *et al.*[1] studied the hitting probability of limsup random fractals with the conditions that the probability $P_n(Q)$ does not depend on Q, denoted by P_n and $\lim_{n\to\infty} \frac{\log_2 P_n}{n}$ exists, and they obtained an estimate of Hausdorff dimension of the intersection of the limsup random fractal and an analytic set. In 2013, Zhang[3] determined the Hausdorff dimension of limsup random fractals under the independence of $\{Z_n(Q), n \geq 1, Q \in \mathcal{Q}_n\}$. Hu *et al.*[4] extended the results in Ref. 1 on limsup random fractals to metric spaces. They proved that under a Correlation Condition, if the limits in (1.1) and (1.2) exist, that is,

$$\begin{aligned} \gamma_1 &= -\lim_{n\to\infty} \frac{\log_2(\max_{Q\in\mathcal{Q}_n} P_n(Q))}{n}, \\ \gamma_2 &= -\lim_{n\to\infty} \frac{\log_2(\min_{Q\in\mathcal{Q}_n} P_n(Q))}{n}, \end{aligned} \qquad (1.4)$$

then for an analytic set G,

$$\mathbb{P}(A \cap G \neq \varnothing) = 0 \quad \text{if } \dim_{\mathrm{P}}(G) < \gamma_1, \qquad (1.5)$$

$$\mathbb{P}(A \cap G \neq \varnothing) = 1 \quad \text{if } \dim_{\mathrm{P}}(G) > \gamma_2 + \delta, \qquad (1.6)$$

where $\dim_{\mathrm{P}}$ denotes the packing dimension.

There are many random sets which are also closely related to limsup random fractals, such as the fast points of Brownian motion,[5] thick points of Brownian motion,[2] random covering sets,[6] dynamical covering sets,[7] shrinking target sets[8] and so on. Li *et al.*[9] investigated the hitting probability of random covering sets in which the use of limsup random fractals is essential. Later Li and Suomala[10] studied the same problem under conditions different from those in Ref. 9. Wang *et al.*[11] considered the dynamical covering problems on the middle-third Cantor set. Hu *et al.*[4] applied the methods of limsup random fractals to investigate the intersection property of dynamical covering sets. Lyons[12] studied the percolation on trees, and gave the Hausdorff dimension of limsup random fractals. Fan[13] studied the sets of limsup deviation paths on trees similar to limsup random fractals, and gave their Hausdorff dimensions. For further results concerning stochastic process on trees, see Refs. 14 and 15.

In this paper, we are interested in whether both (1.5) and (1.6) hold if the limits in (1.4) do not exist. Our answer for this question is that (1.5) holds, but (1.6) does not. Hence if we replace $\dim_{\mathrm{P}}(G) > \gamma_2 + \delta$ by the stronger condition $\dim_{\mathrm{H}}(G) > \gamma_2 + \delta$, then (1.6) will hold, as is shown by the following theorem, which is the main theorem of this paper.

Throughout the paper, $\dim_{\mathrm{H}}$ and $\overline{\dim}_{\mathrm{B}}$ denote the Hausdorff dimension and upper box dimension, respectively.

Theorem 1.1. *Let $A = \limsup_{n\to\infty} A(n)$ be a limsup random fractal with indices γ_1, γ_2, and satisfying the Correlation Condition with $\delta \geq 0$. Then for any analytic set $G \subset [0,1]^d$,*

$$\mathbb{P}(A \cap G \neq \varnothing) = \begin{cases} 0 & \text{if } \dim_{\mathrm{P}}(G) < \gamma_1, \\ 1 & \text{if } \dim_{\mathrm{H}}(G) > \gamma_2 + \delta. \end{cases}$$

Remark 1.1. If $\gamma_2 + \delta < d$, by Theorem 1.1, we see that with probability one, for any hyper cube $Q \in \mathcal{Q}$, $A \cap Q \neq \varnothing$ almost surely (a.s. for short). It implies that A is dense in $[0,1]^d$ a.s. Hence under the condition $\gamma_2 + \delta < d$, $\dim_{\mathrm{P}}(A) = d$ a.s.

Hu *et al.*[4] provided an estimate for the Hausdorff dimension of $A \cap G$ requiring the existence of the limits in (1.4), and the following corollary of Theorem 1.1 indicates the existence of the limits can be relaxed.

Corollary 1.1. *Under the same conditions as Theorem* 1.1, *suppose $\delta = 0$. Then we have*

$$\mathbb{P}(A \cap G \neq \varnothing) = \begin{cases} 0 & \text{if } \dim_{\mathrm{P}}(G) < \gamma_1, \\ 1 & \text{if } \dim_{\mathrm{H}}(G) > \gamma_2 \end{cases}$$

and

$$\max\{0, \dim_{\mathrm{H}}(G) - \gamma_2\} \leq \dim_{\mathrm{H}}(A \cap G) \leq \max\{0, \dim_{\mathrm{P}}(G) - \gamma_1\} \quad a.s. \tag{1.7}$$

In particular, if $\gamma_1 = \gamma_2$, then $\dim_{\mathrm{H}}(A) = \max\{0, d - \gamma_1\}$ a.s.

In Theorem 1.1, the condition for probability one is that $\dim_{\mathrm{H}}(G) > \gamma_2 + \delta$, which can be weakened, by (1.6), to $\dim_{\mathrm{P}}(G) > \gamma_2 + \delta$ if (1.4) holds. A natural question is whether $\dim_{\mathrm{H}}(G)$ in Theorem 1.1 can be replaced by $\dim_{\mathrm{P}}(G)$ or not. The answer is negative.

Proposition 1.1. *There exists a closed set $G \subset [0,1]$ with $\dim_{\mathrm{P}}(G) = 1$ and a limsup random fractal A with $\gamma_1 = \gamma_2 = \delta = 0$ such that $\mathbb{P}(A \cap G \neq \varnothing) = 0$.*

Corollary 1.1 provides estimates for $\dim_{\mathrm{H}}(A \cap G)$. But both inequalities in (1.7) can be strict.

Proposition 1.2. *For $\gamma_0, t \in [0,1]$, there exists a closed set $G \subset [0,1]$ with $\dim_{\mathrm{H}}(G) = t$ and $\dim_{\mathrm{P}}(G) = 1$, and a limsup random fractal A with $\gamma_1 = \gamma_2 = \gamma_0$, $\delta = 0$ such that $\dim_{\mathrm{H}}(A \cap G) = \min\{t, 1 - \gamma_0\}$ a.s.*

Remark 1.2. We notice that the condition $\dim_{\mathrm{P}}(G) < \gamma_1$ for zero probability in Theorem 1.1 cannot be replaced by $\dim_{\mathrm{H}}(G) < \gamma_1$. Indeed given $\gamma_0 \in (0,1)$ and $t \in (0, \gamma_0)$, by Proposition 1.2, there is a limsup random fractal A with indices $\gamma_1 = \gamma_0$ and a set G with $\dim_{\mathrm{H}}(G) = t < \gamma_1$ such that $\dim_{\mathrm{H}}(A \cap G) > 0$ a.s., implying $A \cap G \neq \varnothing$ a.s.

2. PROOFS OF MAIN RESULTS

Before proving the results, we first fix some notation. Write $f_n \lesssim g_n$, $n \in \mathcal{I}$, if there is an absolute constant $0 < c < \infty$ such that for all $n \in \mathcal{I}$, $f_n \leq c g_n$. If $f_n \lesssim g_n$ and $g_n \lesssim f_n$ for $n \in \mathcal{I}$, then we denote $f_n \asymp g_n$. χ is the indicator function. Let $\mathcal{Q}'_n = \{[k_1 2^{-n}, (k_1+1)2^{-n}) \times \ldots \times [k_d 2^{-n}, (k_d+1)2^{-n}) \colon 0 \leq k_i \leq 2^n - 1,\ 1 \leq i \leq d\}$ and $\mathcal{Q}' = \bigcup_{n\geq 1} \mathcal{Q}'_n$.

We demonstrate Theorem 1.1 by using the following lemmas. For any $r > 0$, let $\mathcal{C}_r(G)$ be a collection of the smallest number of closed balls with radius r covering G and $N_r(G) = \#\mathcal{C}_r(G)$. From Ref. 16, the upper box dimension of G is defined as $\overline{\dim}_{\mathrm{B}}(G) = \limsup_{r\to 0} \frac{\log N_r(G)}{-\log r}$.

The following lemma is a slight modification of Lemma 3.2 in Ref. 4.

Lemma 2.1 (Ref. 4). *Let $G \subset [0,1]^d$ be an analytic set. If $\dim_{\mathrm{H}}(G) > t$, there is a nonempty Borel subset $G_\star \subset G$ such that*

$$\dim_{\mathrm{H}}(G_\star \cap V) > t$$

for all dyadic hyper-cubes $V \in \mathcal{Q}'$ with $G_\star \cap V \neq \varnothing$.

Proof. Since G is analytic, from Ref. 17, there is a closed set $K \subset G$ with $0 < \mathcal{H}^t(K) < \infty$ for some $t > s$. Let

$$U = \bigcup_{\substack{V \in \mathcal{Q} \\ \dim_{\mathrm{H}}(V \cap K) \leq s}} V.$$

Let $G_\star = K \setminus U$ which is a Borel set and $\dim_{\mathrm{H}}(V \cap G_\star) > s$ for all $V \in \mathcal{Q}'$ intersecting $G_\star$. Moreover

$$\begin{aligned} \mathcal{H}^t(K) &= \mathcal{H}^t(K \cap U \cup K \setminus U)) \\ &\leq \sum_{\substack{V \in \mathcal{Q}' \\ \dim_{\mathrm{H}}(V \cap K) \leq s}} \mathcal{H}^t(K \cap V) + \mathcal{H}^t(G_\star) \\ &= \mathcal{H}^t(G_\star). \end{aligned}$$

Hence $\mathcal{H}^t(G_\star) = \mathcal{H}^t(K) > 0$, implying $G_\star \neq \varnothing$. □

Lemma 2.2 (Ref. 10). *For G with $\dim_{\mathrm{H}}(G) > t$, there is $N \geq 1$ such that for $n \geq N$, there are at least 2^{nt} elements in $\mathcal{Q}'_n$ intersecting G, that is,*

$$\#\{Q \in \mathcal{Q}'_n \colon Q \cap G \neq \varnothing\} \geq 2^{nt}.$$

Proof of Theorem 1.1. (i) The proof of the zero probability is known (see Ref. 4) and is presented for the sake of completeness. It suffices to show that $\overline{\dim}_{\mathrm{B}}(G) < \gamma_1$ implies $A \cap G = \varnothing$ a.s. Indeed for $\dim_{\mathrm{P}}(G) < \gamma_1$, by

$$\dim_{\mathrm{P}}(G) = \inf\left\{\sup_n \overline{\dim}_{\mathrm{B}}(G_n)\colon G \subset \bigcup_{n=1}^{\infty} G_n\right\} \tag{2.1}$$

(see Ref. 16), there is a covering $\{G_n\}$ with $\overline{\dim}_{\mathrm{B}}(G_n) < \gamma_1$ for $n \geq 1$. Hence $\mathbb{P}(A \cap G \neq \varnothing) \leq \sum_n \mathbb{P}(A \cap G_n \neq \varnothing) = 0$.

When $\gamma_1 < \infty$, fix $\epsilon > 0$ with $\overline{\dim}_{\mathrm{B}}(G) < \gamma_1 - \epsilon$ and $\theta \in (\overline{\dim}_{\mathrm{B}}(G), \gamma_1 - \epsilon)$, then for n large enough, we have

$$\max_{Q \in \mathcal{Q}_n} P_n(Q) \leq 2^{-n(\gamma_1 - \epsilon)}, \quad N_{\sqrt{d}2^{-n-1}}(G) \lesssim 2^{n\theta},$$

$$\#\Gamma_n(B) = \#\{Q \in \mathcal{Q}_n\colon Q \cap B \neq \varnothing\} \leq M \quad (\forall\, B \in \mathcal{C}_{\sqrt{d}2^{-n-1}}(G)),$$

where $M > 0$ is an absolute constant, and these inequalities are immediate from their definitions. Hence for n large enough, we have

$$\begin{aligned}
&\mathbb{P}(G \cap A(n) \neq \varnothing)\\
&\quad \leq \mathbb{P}\left(\bigcup_{B \in \mathcal{C}_{\sqrt{d}2^{-n-1}}(G)} \bigcup_{Q \in \Gamma_n(B)} Q \cap A(n) \neq \varnothing\right)\\
&\quad \lesssim 2^{-n(\gamma_1 - \epsilon - \theta)}.
\end{aligned}$$

The Borel–Cantelli lemma implies $A \cap G = \varnothing$ a.s.

When $\gamma_1 = \infty$, fix $b > 0$ with $\overline{\dim}_{\mathrm{B}}(G) < b$. For $\theta \in (\overline{\dim}_{\mathrm{B}}(G), b)$, we have $N_{\sqrt{d}2^{-n-1}}(G) \lesssim 2^{n\theta}$ and $\max_{Q \in \mathcal{Q}_n} P_n(Q) \leq 2^{-nb}$ for n large enough, where the second inequality follows from (1.1). Similarly, we get $A \cap G = \varnothing$ a.s.

(ii) In the following, we will show $\mathbb{P}(A \cap G \neq \varnothing) = 1$ provided that $\dim_{\mathrm{H}}(G) > \gamma_2 + \delta$. By Lemma 2.1, there exists a Borel subset $G_\star$ and a closed subset K satisfying $G_\star \subset K \subset G$ such that for all $V \in \mathcal{Q}'$, whenever $G_\star \cap V \neq \varnothing$, $\dim_{\mathrm{H}}(G_\star \cap V) > \gamma_2 + \delta$. Fix an open set V such that $V \cap G_\star \neq \varnothing$. Denote

$$\mathcal{A}_n = \mathcal{A}_n(V \cap G_\star) = \{Q \in \mathcal{Q}'_n\colon Q \cap V \cap G_\star \neq \varnothing\}$$

and $N_n = \#\mathcal{A}_n$. Then by Lemma 2.2, we have that for any $\eta \in (\gamma_2 + \delta, \dim_{\mathrm{H}}(G_\star \cap V))$, there is $n(\eta) \geq 1$ such that

$$N_n \geq 2^{n\eta} \quad (\forall\, n \geq n(\eta)).$$

Define

$$S_n = \sum_{Q \in \mathcal{A}_n} Z_n(\overline{Q}),$$

where $\overline{Q}$ is the closure of Q and $\overline{Q} \in \mathcal{Q}$. We need only show that $\mathbb{P}(S_n > 0 \text{ i.o.}) = 1$, where i.o. stands for infinitely often. First, we estimate

$$\mathrm{Var}(S_n) = \sum_{Q \in \mathcal{A}_n} \sum_{Q' \in \mathcal{A}_n} \mathrm{Cov}(Z_n(\overline{Q}), Z_n(\overline{Q'})).$$

Fix $\epsilon > 0$ and for each $Q \in \mathcal{A}_n$, let $\mathcal{G}_n(Q)$ denote the collection of all $Q' \in \mathcal{A}_n$ such that

$$\mathrm{Cov}(Z_n(\overline{Q}), Z_n(\overline{Q'})) \leq \epsilon P_n(\overline{Q}) P_n(\overline{Q'}) \tag{2.2}$$

and we define $\mathcal{B}_n(Q) = \mathcal{A}_n \setminus \mathcal{G}_n(Q)$.

From the fact that $\mathrm{Cov}(Z_n(\overline{Q}), Z_n(\overline{Q'})) \leq \mathbb{E}(Z_n(\overline{Q})) = P_n(\overline{Q})$, we get

$$\begin{aligned}
\mathrm{Var}(S_n) &= \sum_{Q \in \mathcal{A}_n}\left(\sum_{Q' \in \mathcal{G}_n(Q)} \mathrm{Cov}\big(Z_n(\overline{Q}), Z_n(\overline{Q'})\big) + \sum_{Q' \in \mathcal{B}_n(Q)} \mathrm{Cov}\big(Z_n(\overline{Q}), Z_n(\overline{Q'})\big)\right)\\
&\leq \sum_{Q \in \mathcal{A}_n}\left(\sum_{Q' \in \mathcal{G}_n(Q)} \epsilon P_n(\overline{Q}) P_n(\overline{Q'}) + \sum_{Q' \in \mathcal{B}_n(Q)} P_n(\overline{Q})\right)\\
&= \sum_{Q \in \mathcal{A}_n} \sum_{Q' \in \mathcal{G}_n(Q)} \epsilon P_n(\overline{Q}) P_n(\overline{Q'}) + \sum_{Q \in \mathcal{A}_n} \big(\#\mathcal{B}_n(Q)\big) P_n(\overline{Q})\\
&\leq \epsilon \left(\sum_{Q \in \mathcal{A}_n} P_n(\overline{Q})\right)\left(\sum_{Q' \in \mathcal{A}_n} P_n(\overline{Q'})\right) + \left(\max_{Q \in \mathcal{A}_n} \#\mathcal{B}_n(Q)\right)\left(\sum_{Q \in \mathcal{A}_n} P_n(\overline{Q})\right).
\end{aligned}$$

Recalling the notation of the Correlation Condition, we have

$$\begin{aligned}
\mathrm{Var}(S_n) &\leq \epsilon \left(\sum_{Q \in \mathcal{A}_n} P_n(\overline{Q})\right)^2 + f(n, \epsilon) \sum_{Q \in \mathcal{A}_n} P_n(\overline{Q})\\
&= \epsilon\big(\mathbb{E}(S_n)\big)^2 + f(n, \epsilon)\mathbb{E}(S_n).
\end{aligned} \tag{2.3}$$

Combining (2.3) and the Paley–Zygmund inequality,[18] we obtain

$$\mathbb{P}(S_n > 0) \geq \frac{(\mathbb{E}(S_n))^2}{\mathbb{E}(S_n^2)} = \frac{(\mathbb{E}(S_n))^2}{(\mathbb{E}(S_n))^2 + \mathrm{Var}(S_n)}$$
$$\geq \frac{\mathbb{E}(S_n)}{(1+\epsilon)\mathbb{E}(S_n) + f(n,\epsilon)}.$$

That is,

$$\mathbb{P}(S_n > 0) \geq \frac{1}{1+\epsilon+\frac{f(n,\epsilon)}{\mathbb{E}(S_n)}}. \tag{2.4}$$

Since $\mathbb{E}(S_n) = \sum_{Q\in\mathcal{A}_n} P_n(\overline{Q}) \geq N_n(\min_{\overline{Q}\in\mathcal{Q}_n} P_n(\overline{Q}))$, recalling that $N_n = \#\mathcal{A}_n$, we have

$$\frac{f(n,\epsilon)}{\mathbb{E}(S_n)} \leq \frac{f(n,\epsilon)}{N_n(\min_{\overline{Q}\in\mathcal{Q}_n} P_n(\overline{Q}))}. \tag{2.5}$$

By the Correlation Condition, for any $\theta > 0$ with $2\theta < \eta - \delta - \gamma_2$, we have

$$f(n,\epsilon) \leq 2^{(\delta+\theta)n}$$

for n large enough, and from (1.2), we have

$$\min_{\overline{Q}\in\mathcal{Q}_n} P_n(\overline{Q}) \geq 2^{-(\gamma_2+\theta)n}$$

for infinitely many n, denoted by $\mathcal{N}$. From these inequalities and the arbitrariness of θ, we have

$$\limsup_{\substack{n\in\mathcal{N}\\ n\to\infty}} \frac{f(n,\epsilon)}{N_n(\min_{\overline{Q}\in\mathcal{Q}_n} P_n(\overline{Q}))}$$
$$\leq \limsup_{\substack{n\in\mathcal{N}\\ n\to\infty}} 2^{n(2\theta+\delta+\gamma_2-\eta)} = 0. \tag{2.6}$$

Then combining inequalities (2.4)–(2.6), by Fatou's lemma and arbitrariness of ϵ, we conclude that

$$\mathbb{P}(S_n > 0 \text{ i.o.}) \geq \limsup_{\substack{n\in\mathcal{N}\\ n\to\infty}} \mathbb{P}(S_n > 0) = 1.$$

It follows that

$$\mathbb{P}\left(\bigcup_{n=k}^{\infty}\bigcup_{Q\in\mathcal{A}_n}\{Z_n(\overline{Q}) = 1\}, \forall\, k \geq 1, \forall\, V \in \mathcal{Q}'\right) = 1.$$

Then given ω in the above event, for $Q_0 = [0,1)^d$, there is some $k_1 \geq 1$ such that

$$\exists\, Q_1 \in \mathcal{Q}'_{k_1}, \quad Q_1 \cap G_\star \cap Q_0 \neq \varnothing \quad \text{and}$$
$$Z_{k_1}(\overline{Q_1})(\omega) = 1.$$

Since $Q_1 \subset Q_0$ and $G_\star \cap Q_1 \neq \varnothing$, there is some $k_2 \geq k_1$ such that

$$\exists\, Q_2 \in \mathcal{Q}'_{k_2} \quad Q_2 \cap G_\star \cap Q_1 \neq \varnothing \quad \text{and}$$
$$Z_{k_2}(\overline{Q_2})(\omega) = 1.$$

We continue inductively, then get a sequence $\{Q_i\}$ such that for all $i \geq 1$,

$$Q_i \in \mathcal{Q}'_{k_i}, \quad Q_i \cap G_\star \cap Q_{i-1} \neq \varnothing \quad \text{and}$$
$$Z_{k_i}(\overline{Q_i})(\omega) = 1.$$

Notice that $\{Q_i\}$ are decreasing hyper cubes. Recall that $K \supset G_\star$ is closed. Hence $\varnothing \neq \bigcap_{i=1}^{\infty}\overline{Q_i} \subset K \cap A(\omega)$. Thus, $A \cap G \neq \varnothing$ a.s. □

We use the following lemma to prove Corollary 1.1.

Lemma 2.3 (Ref. 1). *Suppose $A = A(\omega)$ is a random set in $[0,1]^d$ (i.e. $\chi_{A(\omega)}(x)$ is jointly measurable) such that for any compact set $E \subset [0,1]^d$ with $\dim_{\mathrm{H}}(E) > \gamma$, we have $\mathbb{P}(A \cap E) = 1$. Then for any analytic set $E \subset [0,1]^d$,*

$$\dim_{\mathrm{H}}(E) - \gamma \leq \dim_{\mathrm{H}}(A \cap E) \quad \textit{a.s.}$$

Proof of Corollary 1.1. The proof of the upper bound is similar to that in Ref. 1, and is included for completeness. Since $\dim_{\mathrm{P}}(G) < \gamma_1$ implies $A \cap G = \varnothing$ a.s., we consider the case of $\dim_{\mathrm{P}}(G) \geq \gamma_1$. By (2.1), it suffices to prove $\dim_{\mathrm{H}}(A\cap G) \leq \overline{\dim}_{\mathrm{B}}(G) - \gamma_1$. Recall the notation in Theorem 1.1 and the start of Sec. 2, and define

$$H_n = \sum_{B\in\mathcal{C}_{\sqrt{d}2^{-n-1}}(G)} \sum_{Q\in\Gamma_n(B)} Z_n(Q).$$

For $\eta = \overline{\dim}_{\mathrm{B}}(G) - \gamma_1$ and any $\epsilon > 0$, for large enough n we have

$$\mathbb{E}(H_n) = \sum_{B\in\mathcal{C}_{\sqrt{d}2^{-n-1}}(G)} \sum_{Q\in\Gamma_n(B)} P_n(Q) \lesssim 2^{n(\eta+2\epsilon)}.$$

Due to the arbitrariness of ϵ, for any $\theta > \eta$, we have $\mathbb{E}(\sum_{n=1}^{\infty} H_n 2^{-n\theta}) < \infty$. Therefore,

$$\mathcal{H}^{\theta}(A\cap G) \leq \liminf_{m\to\infty} \sum_{n=m}^{\infty} H_n 2^{-n\theta} = 0 \quad \text{a.s.},$$

giving $\dim_{\mathrm{H}}(A\cap G) \leq \overline{\dim}_{\mathrm{B}}(G) - \gamma_1$ a.s.

The left-hand inequality in Corollary 1.1 follows from Lemma 2.3 and Theorem 1.1. □

Proof of Proposition 1.1. Let $(t_k)_{k\geq 1}$ be an increasing sequence with $t_k \in (0,1)$ and $\lim_{k\to\infty} t_k = 1$. Let $(n_k)_{k\geq 1}$ and $(m_k)_{k\geq 1}$ be increasing sequences of positive integers with $m_k < n_k$ (to be determined later). We first construct a homogeneous Cantor set G as follows. We divide $[0,1]$ into $N_1 := 2^{m_1}$ closed intervals, and inside each interval, choose a closed subinterval with length

$l_1 := 2^{-n_1}$. Denote this collection of intervals by $\mathcal{G}_1$. Let $I_1^1, I_1^2, \ldots, I_1^{N_1}$ be the intervals in $\mathcal{G}_1$, arranged from left to right, such that I_1^1 has the same left endpoint as $[0,1]$ and $I_1^{N_1}$ has the same right endpoint as $[0,1]$.

Suppose $\mathcal{G}_k$ has been chosen, which is comprised of some closed intervals, each with length l_k. For any $I \in \mathcal{G}_k$, we divide I into $2^{m_{k+1}}$ intervals of length $2^{-m_{k+1}}|I|$. Then, inside of each subinterval of length $2^{-m_k}l_k$, we choose one closed interval of length $l_{k+1} := 2^{-n_{k+1}}l_k$ such that the first subinterval has the same left endpoint as I, and the last subinterval has the same right endpoint as I. Denote these $N_{k+1} = 2^{m_{k+1}}N_k$ intervals with length l_{k+1} by $\mathcal{G}_{k+1}$. Let $G_k = \bigcup_{I\in\mathcal{G}_k} I$ and $G = \bigcap_{k\geq 1} G_k$. Choosing m_{k+1} large enough (depending on the choices of $(n_i)_{1\leq i\leq k}$ and $(m_i)_{1\leq i\leq k}$) such that $2^{m_{k+1}(1-t_k)}N_k l_k^{t_k} \geq 1$, from Ref. 19, we have

$$\begin{aligned}\dim_{\mathrm{P}}(G) &= \limsup_{k\to\infty} \frac{\log N_{k+1}}{-\log(l_k)+\log(N_{k+1}/N_k)}\\ &= \limsup_{k\to\infty} \frac{\log(2^{m_{k+1}}N_k)}{\log(l_k^{-1}2^{m_{k+1}})}\\ &\geq \limsup_{k\to\infty} t_k = 1.\end{aligned}$$

Before constructing a limsup random fractal, we fix some notation. Let $(\xi_m)_{m\geq 1}$ be a sequence of independent random variables on $(\Omega, \mathscr{B}, \mathbb{P})$ which are uniformly distributed on $[0,1]$. Let $M_k = \sum_{i=1}^k n_i$ and $b_n = 2^{M_k t_k}$ if $n \in [M_k, M_{k+1})$. Define the set $B = \{\exists\, m \in [b_{n-1}, b_n)$ such that $\xi_m \in Q\}$ and set $Z_n(Q) = \chi_B$. Let $A(n) = \bigcup_{\substack{Q\in\mathcal{Q}_n\\ Z_n(Q)=1}} Q$, and $A = \limsup_{n\to\infty} A(n)$.

For every $Q \in \mathcal{Q}_n$, it follows from our assumption of $(\xi_m)_{m\geq1}$ and the definition of $\mathcal{Q}_n$ that $\mathbb{P}(Z_n(Q) = 1)$ does not depend on Q, denoted by P_n. Therefore,

$$\begin{aligned}P_n = \mathbb{P}\left(\bigcup_{m=b_{n-1}}^{b_n-1} \xi_m \in Q\right) &\leq \sum_{m=b_{n-1}}^{b_n-1} \mathbb{P}(\xi_m \in Q)\\ &= 2^{-n}(b_n - b_{n-1}) \qquad (2.7)\end{aligned}$$

and

$$\begin{aligned}P_n &\geq \sum_{m=b_{n-1}}^{b_n-1} \mathbb{P}(\xi_m \in Q)\\ &\quad - \sum_{m=b_{n-1}}^{b_n-1} \sum_{\substack{m'=b_{n-1}\\ m'\neq m}}^{b_n-1} \mathbb{P}(\xi_m \in Q, \xi_{m'} \in Q)\\ &= \sum_{m=b_{n-1}}^{b_n-1} \mathbb{P}(\xi_m \in Q)\left(1 - \sum_{\substack{m'=b_{n-1}\\ m'\neq m}}^{b_n-1} \mathbb{P}(\xi_m \in Q)\right)\\ &= 2^{-n}(b_n - b_{n-1})(1 - 2^{-n}(b_n - b_{n-1})). \qquad (2.8)\end{aligned}$$

Write $x_n = 2^{-n}(b_n - b_{n-1})$. Note that for $k \geq 1$, if $M_k < n < M_{k+1}$, $x_n = 0$ and $x_{M_k} = \frac{2^{M_k t_k} - 2^{M_{k-1}t_{k-1}}}{2^{M_k}}$. From this and inequalities (2.7) and (2.8), we have

$$\limsup_{n\to\infty} \frac{\log_2 P_n}{n} = 0. \qquad (2.9)$$

Next we prove that A satisfies the Correlation Condition with $\delta = 0$. First we estimate $\mathrm{Cov}(Z_n(Q), Z_n(Q'))$ for $Q, Q' \in \mathcal{Q}_n$ with the distance $\mathrm{dist}(Q, Q') \geq 2^{-n}$,

$$\begin{aligned}&\mathrm{Cov}(Z_n(Q), Z_n(Q'))\\ &\quad= \mathbb{E}(Z_n(Q)Z_n(Q')) - \mathbb{E}(Z_n(Q))\mathbb{E}(Z_n(Q'))\\ &\quad\leq \mathbb{P}(Z_n(Q) = 1, Z_n(Q') = 1) - P_n^2\\ &\quad\leq \sum_{m=b_{n-1}}^{b_n-1} \mathbb{P}(\xi_m \in Q) \sum_{\substack{m'=b_{n-1}\\ m'\neq m}}^{b_n-1} \mathbb{P}(\xi_m \in Q) - P_n^2\\ &\quad\leq 2\left(\sum_{m=b_{n-1}}^{b_n-1} \mathbb{P}(\xi_m \in Q)\right)\\ &\qquad\times \sum_{m=b_{n-1}}^{b_n-1} \sum_{\substack{m'=b_{n-1}\\ m'\neq m}}^{b_n-1} \mathbb{P}(\xi_m \in Q, \xi_{m'} \in Q')\\ &\quad\lesssim 2^{-n+1}(b_n - b_{n-1})\mathbb{E}(Z_n(Q))\mathbb{E}(Z_n(Q')).\end{aligned} \qquad (2.10)$$

From (2.9), for $\epsilon > 0$, there is $n(\epsilon)$ such that for any $n \geq n(\epsilon)$, $2^{-n+1}(b_n - b_{n-1}) < \epsilon$. It implies that for n large enough, we have $f(n, \epsilon) \leq 3$, hence A satisfies the Correlation Condition with $\delta = 0$.

Denote $A_k = \bigcup_{n=M_{k-1}+1}^{M_k} A(n)$, then

$$\begin{aligned}&\mathbb{P}(A_k \cap G_k \neq \varnothing)\\ &\quad\leq \sum_{I\in\mathcal{G}_k} \mathbb{P}(I \cap A_k \neq \varnothing)\\ &\quad\leq N_k n_k \max_{M_{k-1}+1\leq n\leq M_k} \mathbb{P}(I \cap A(n) \neq \varnothing)\\ &\quad\leq 2N_k n_k \max_{M_{k-1}+1\leq n\leq M_k} P_n \lesssim N_k n_k 2^{-M_k} 2^{M_k t_k}\\ &\quad= n_k N_k l_k 2^{M_k t_k} \end{aligned} \qquad (2.11)$$

and by choosing n_k large enough (depending on t_k, the choice of $(n_i)_{1\le i<k}$ and $(m_i)_{1\le i\le k}$), we have $n_k N_k l_k 2^{M_k t_k} \le 2^{-k}$. It then follows from the Borel–Cantelli lemma that

$$\mathbb{P}(A_k \cap G_k \neq \varnothing \text{ i.o.}) = 0.$$

This means that there is some $N \ge 1$ such that for any $k \ge N$, $A_k \cap G_k = \varnothing$ a.s. Since $G \subset G_k$, $k \ge 1$, it yields that $G \cap A(k) \neq \varnothing$ for only finitely many k. Therefore, $\mathbb{P}(A \cap G \neq \varnothing) = 0$. □

To show Proposition 1.2, we make use of the following lemma. Here we suppose that $(\xi_m)_{m\ge 1}$ is a sequence of independent and uniformly distributed random variables on $[0,1]$, which are defined on $(\Omega, \mathscr{B}, \mathbb{P})$.

Lemma 2.4. *Let $0 \le \gamma_0 < 1$ and $a_n = 2^{n(1-\gamma_0)}$. For any $Q \in \mathcal{Q}_n$, write $B = \{\exists\, m \in [a_{n-1}, a_n)$ s.t. $\xi_m \in Q\}$ and set $Z_n(Q) = \chi_B$. For $0 < \beta < 1-\gamma_0$, let*

$$A^\beta(n) = \bigcup_{\substack{Q\in\mathcal{Q}_n\\ Z_n(Q)=1}} Q^\beta,$$

where Q^β is the interval with length $|Q|^\beta$ concentric with Q. Then

$$\lim_{n\to\infty} \mathbb{P}(A^\beta(n) = [0,1]) = 1.$$

Proof. For $n \ge 1$, $Q \in \mathcal{Q}_n$, write

$$\mathcal{Q}_n(Q) = \big\{Q' \in \mathcal{Q}_n \colon \operatorname{dist}(Q, Q') \le 2^{-(n+1)}(2^{(1-\beta)n} - 3)\big\},$$

hence $\#\mathcal{Q}_n(Q) = \lfloor 2^{(1-\beta)n}\rfloor$. Note that

$$\{Q \not\subseteq A^\beta(n)\} \subset \{Z_n(Q') = 0 \text{ for all } Q' \in \mathcal{Q}_n(Q)\}.$$

Then for n large enough, we have

$$\begin{aligned}\mathbb{P}(Q \not\subseteq A^\beta(n)) &\le \mathbb{P}(Z_n(Q') = 0 \text{ for all } Q' \in \mathcal{Q}_n(Q))\\ &\le \prod_{m=a_{n-1}}^{a_n} \mathbb{P}\left(\xi_m \notin \bigcup_{Q'\in\mathcal{Q}_n(Q)} Q'\right)\\ &= (1 - \lfloor 2^{(1-\beta)n}\rfloor 2^{-n})^{2^{n(1-\gamma_0)}}\\ &\le (1 - 2^{-n\beta-1})^{2^{n(1-\gamma_0)}}. \end{aligned} \tag{2.12}$$

Note that $\{[0,1] \not\subseteq A^\beta(n)\} = \{\exists\, Q \in \mathcal{Q}_n$ such that $Q \not\subseteq A^\beta(n)\}$, and hence

$$\begin{aligned}\mathbb{P}([0,1] \not\subseteq A^\beta(n)) &= \mathbb{P}(\exists\, Q \in \mathcal{Q}_n \text{ such that } Q \not\subseteq A^\beta(n))\\ &\le \sum_{Q\in\mathcal{Q}_n} \mathbb{P}(Q \not\subseteq A^\beta(n))\\ &\le \sum_{Q\in\mathcal{Q}_n} \mathbb{P}(Z_n(Q') = 0 \text{ for all } Q' \in \mathcal{Q}_n(Q))\\ &\le \sum_{Q\in\mathcal{Q}_n} \prod_{m=a_{n-1}}^{a_n} \mathbb{P}\left(\xi_m \notin \bigcup_{Q'\in\mathcal{Q}_n(Q)} Q'\right)\\ &\le 2^n (1 - 2^{-n\beta-1})^{2^{n(1-\gamma_0)}}, \end{aligned} \tag{2.13}$$

which tends to 0 as $n \to \infty$. □

Proof of Proposition 1.2. Let $(n_k)_{k\ge 1}$ be an increasing sequence of positive integers (to be determined later). We divide $[0,1]$ into $N_1 := \lfloor 2^{n_1 t}\rfloor$ closed intervals, and inside each interval, choose a closed subinterval with length $l_1 := 2^{-n_1}$, with the first subinterval on the left having the same left endpoint as $[0,1]$, and the right endpoint of the last subinterval on the right having the same as that of $[0,1]$. We denote this collection of intervals by $\mathcal{G}_1$.

Suppose that $\mathcal{G}_k$, consisting of some closed intervals with length l_k, has been chosen. Then for any $I \in \mathcal{G}_k$, we divide I into $\lfloor 2^{n_{k+1}t}\rfloor$ closed intervals of length $2^{-\lfloor n_{k+1}t\rfloor} l_k$ and inside each of these, choose one closed interval of length $l_{k+1} := 2^{-n_{k+1}} l_k$, such that the left endpoint of the first subinterval is the same as that of I, and the right endpoint of the last subinterval is the same as that of I. Denote these $N_{k+1} = \lfloor 2^{n_{k+1}t}\rfloor N_k$ intervals with length l_{k+1} by $\mathcal{G}_{k+1}$. Let $G_k = \bigcup_{I\in\mathcal{G}_k} I$ and $G = \bigcap_{k\ge 1} G_k$. G is then a homogeneous Cantor set. From Ref. 19, we have

$$\begin{aligned}\dim_{\mathrm{H}}(G) &= \liminf_{k\to\infty} \frac{\log N_{k+1}}{-\log l_k}\\ &= \liminf_{k\to\infty} \frac{\log \prod_{i=1}^k \lfloor 2^{n_i t}\rfloor}{\log \prod_{i=1}^k 2^{n_i}} \le t\end{aligned}$$

and for k large enough we also have

$$\begin{aligned}\frac{\log \prod_{i=1}^k \lfloor 2^{n_i t}\rfloor}{\log \prod_{i=1}^k 2^{n_i}} &\ge \frac{\log \prod_{i=1}^k (2^{n_i t} - 1)}{\log \prod_{i=1}^k 2^{n_i}}\\ &= t + \frac{\log \prod_{i=1}^k (1 - 2^{-n_i t})}{\sum_{i=1}^k n_i}.\end{aligned}$$

Since $\sum_{i=1}^\infty 2^{-n_i t} < \infty$, it follows that $\prod_{i=1}^\infty (1 - 2^{-n_i t}) > 0$, implying $\frac{\log \prod_{i=1}^k (1-2^{-n_i t})}{\sum_{i=1}^k n_i} \to 0$ as $k \to \infty$. Therefore, $\dim_{\mathrm{H}}(G) = t$. From the proof of Proposition 1.1, we choose an appropriate $(n_k)_{k\ge 1}$ such that $\dim_{\mathrm{P}}(G) = 1$.

Using the same notation as Lemma 2.4, let $A(n) = \bigcup_{\substack{Q\in\mathcal{Q}\\ Z_n(Q)=1}} Q$ and $A = \limsup_{n\to\infty} A(n)$. We assume that $\gamma_0 > 0$, since we can replace $1-\gamma_0$ by an increasing sequence of positive real numbers with limit $1-\gamma_0$ if $\gamma_0 = 0$. For every $Q \in \mathcal{Q}_n$, the probability $\mathbb{P}(Z_n(Q) = 1)$, denoted by P_n, does not depend on Q. Then

$$P_n = \mathbb{P}\left(\bigcup_{m=a_{n-1}}^{a_n-1} \xi_m \in Q\right) \le \sum_{m=a_{n-1}}^{a_n-1} \mathbb{P}(\xi_m \in Q)$$
$$= 2^{-n}(a_n - a_{n-1}) \asymp 2^{-n\gamma_0} \tag{2.14}$$

and

$$\begin{aligned} P_n &\ge \sum_{m=a_{n-1}}^{a_n-1} \mathbb{P}(\xi_m \in Q) \\ &\quad - \sum_{m=a_{n-1}}^{a_n-1} \sum_{\substack{m'=a_{n-1}\\ m'\neq m}}^{a_n-1} \mathbb{P}(\xi_m \in Q, \xi_{m'} \in Q) \\ &= \sum_{m=a_{n-1}}^{a_n-1} \mathbb{P}(\xi_m \in Q)\left(1 - \sum_{\substack{m'=a_{n-1}\\ m'\neq m}}^{a_n-1} \mathbb{P}(\xi_m \in Q)\right) \\ &\asymp 2^{-n\gamma_0}. \end{aligned} \tag{2.15}$$

We can conclude from (2.14) and (2.15) that

$$\lim_{n\to\infty} \frac{\log_2 P_n}{n} = -\gamma_0.$$

For $Q, Q' \in \mathcal{Q}_n$ with the distance $\mathrm{dist}(Q, Q') \ge 2^{-n}$, since

$$\begin{aligned} &\mathrm{Cov}(Z_n(Q), Z_n(Q')) \\ &\quad = \mathbb{E}(Z_n(Q)Z_n(Q')) - \mathbb{E}(Z_n(Q))\mathbb{E}(Z_n(Q')) \\ &\quad \le \mathbb{P}(Z_n(Q) = 1, Z_n(Q') = 1) - P_n^2 \\ &\quad \le \sum_{m=a_{n-1}}^{a_n-1} \mathbb{P}(\xi_m \in Q) \sum_{\substack{m'=a_{n-1}\\ m'\neq m}}^{a_n-1} \mathbb{P}(\xi_m \in Q) - P_n^2 \\ &\quad \le 2\left(\sum_{m=a_{n-1}}^{a_n-1} \mathbb{P}(\xi_m \in Q)\right) \\ &\qquad \times \sum_{m=a_{n-1}}^{a_n-1} \sum_{\substack{m'=a_{n-1}\\ m'\neq m}}^{a_n-1} \mathbb{P}(\xi_m \in Q, \xi_{m'} \in Q') \\ &\quad \asymp 2^{-n\gamma_0}\mathbb{E}(Z_n(Q))\mathbb{E}(Z_n(Q')) \end{aligned} \tag{2.16}$$

for $\epsilon > 0$, we have $f(n,\epsilon) \le 3$ for n large enough. Then we have proved that A satisfies the Correlation Condition with $\delta = 0$.

If $\gamma_0 = 1$, then $\dim_{\mathrm{H}}(A) = 0$, which implies that $\dim_{\mathrm{H}}(A \cap G) = 0$. We therefore consider the case $\gamma_0 < 1$. Since $\dim_{\mathrm{H}}(F \cap A) \le \min\{t, 1-\gamma_0\}$, we prove that $\dim_{\mathrm{H}}(F \cap A) \ge \min\{t, 1-\gamma_0\}$ a.s. Let $(\epsilon_n)_{n\ge1}$ be a sequence with $\epsilon_n > 0$ for all n and such that $\sum_{n=1}^{\infty} \epsilon_n < 1$, and let $(t_n)_{n\ge1}$ be an increasing sequence with $\lim_{n\to\infty} t_n = 1-\gamma_0$. By Lemma 2.4, there is $C \ge 1$ such that for all $n \ge C$, we have

$$\mathbb{P}(\mathbb{T} = A^{t_n}(n)) > 1 - \epsilon_n,$$

then using the independence of $(\xi_m)_{n\ge1}$, we derive

$$\begin{aligned} &\mathbb{P}(\mathbb{T} = A^{t_n}(n) \text{ for all } n \ge C) \\ &\qquad = \prod_{n=C}^{\infty} \mathbb{P}(\mathbb{T} = A^{t_n}(n)) > 1 - \sum_{n=C}^{\infty} \epsilon_n > 0. \end{aligned}$$

Fix $\omega \in \{\mathbb{T} = A^{t_n}(n) \text{ for all } n\}$, and without losing generality, we suppose $n_1 \ge C$. Let $(m_k)_{k\ge1}$ be an increasing sequence of positive integers satisfying $m_k > \sum_{i=1}^{k} n_i$ for $k \ge 1$(to be determined later). Let $\mathcal{G}_1' = \mathcal{G}_1$. For each $I \in \mathcal{G}_1'$, by Lemma 2.4, we can choose a subfamily $\mathcal{A}_1(I) := \{Q \in \mathcal{Q}_{m_1} \colon Q^{t_{m_1}} \subset I\}$ such that $\#\mathcal{A}_1(I) = \lfloor \frac{l_1}{2^{-m_1 t_{m_1}}} \rfloor - 2$; here we can choose $m_1 > \frac{n_1+1}{t_{m_1}}$ such that $\mathcal{A}_1(I)$ exists. Let $\mathcal{F}_1 = \{\mathcal{A}_1(I) \colon I \in \mathcal{G}_1'\}$, then we have $\bigcup_{Q\in\mathcal{G}_1} Q \subset A(m_1)(\omega)$ and $\#\mathcal{F}_1 = \#\mathcal{G}_1'(\lfloor \frac{l_1}{2^{-m_1 t_{m_1}}} \rfloor - 2)$.

For $Q \in \mathcal{F}_1$, choose $\lfloor \frac{2^{-m_1}\lfloor 2^{n_2 t}\rfloor}{l_1} \rfloor - 2$ elements in $\mathcal{G}_2$ which are contained in Q. We denote the chosen elements by $\mathcal{G}_2'$, that is $\mathcal{G}_2' = \{J \in \mathcal{G}_2 \colon J \subset Q,\ Q \in \mathcal{F}_1\}$ and $\#\mathcal{G}_2' = (\lfloor \frac{2^{-m_1}\lfloor 2^{n_2 t}\rfloor}{l_1} \rfloor - 2)\#\mathcal{F}_1$. For $J \in \mathcal{G}_2'$, choose a subfamily $\mathcal{A}_2(J) := \{Q \in \mathcal{Q}_{m_2} \colon Q^{t_{m_2}} \subset J\}$ such that $\#\mathcal{A}_2(J) = \lfloor \frac{l_2}{2^{-m_2 t_{m_2}}} \rfloor - 2$, here we can select $m_2 > \frac{n_1+n_2+1}{t_{m_2}}$ to guarantee it. Write $\mathcal{F}_2 = \{\mathcal{A}_2(J) \colon J \in \mathcal{G}_2'\}$, then $\#\mathcal{F}_2 = \#\mathcal{G}_2'(\lfloor \frac{l_2}{2^{-m_2 t_{m_2}}} \rfloor - 2)$.

Continuing inductively, we have families $\mathcal{G}_k'$ and $\mathcal{F}_k$ with

$$\begin{aligned} \#\mathcal{F}_k &= \left(\left\lfloor \frac{l_k}{2^{-m_k t_{m_k}}} \right\rfloor - 2\right)\#\mathcal{F}_k' \\ &= N_1 \prod_{i=1}^{k} \left(\left\lfloor \frac{l_i}{2^{-m_i t_{m_i}}} \right\rfloor - 2\right) \\ &\quad \times \prod_{i=1}^{k-1} \left(\left\lfloor \frac{2^{-m_i}\lfloor 2^{n_{i+1} t}\rfloor}{l_i} \right\rfloor - 2\right) \end{aligned}$$

and

$$\begin{aligned}\#\mathcal{G}'_{k+1} &= \left(\left\lfloor \frac{2^{-m_k}\lfloor 2^{n_{k+1}t}\rfloor}{l_k}\right\rfloor - 2\right)\#\mathcal{G}_k \\ &= N_1\prod_{i=1}^{k}\left(\left\lfloor \frac{l_i}{2^{-m_i t_{m_i}}}\right\rfloor - 2\right) \\ &\quad\times\left(\left\lfloor \frac{2^{-m_i}\lfloor 2^{n_{i+1}t}\rfloor}{l_i}\right\rfloor - 2\right).\end{aligned}$$

We choose m_k and n_{k+1} large enough such that

$$\#\mathcal{F}_k \asymp \frac{2^{(1-t)(\sum_{i=1}^{k} n_i)}}{2^{(\sum_{i=1}^{k} m_i(1-t_{m_i}))-m_k}} \quad \text{and}$$

$$\lim_{k\to\infty}\frac{\log\#\mathcal{F}_k}{-\log 2^{-m_k}} = 1-\gamma_0 \tag{2.17}$$

and

$$\#\mathcal{G}'_{k+1} \asymp \frac{2^{t(\sum_{i=1}^{k+1} n_i)}}{2^{\sum_{i=1}^{k} m_i(1-t_{m_i})}} \quad \text{and}$$

$$\lim_{k\to\infty}\frac{\log\#\mathcal{G}'_k}{-\log l_k} = t. \tag{2.18}$$

Let $F=\bigcap_{k\geq 1}\bigcup_{Q\in\mathcal{F}_k} Q=\bigcap_{k\geq 1}\bigcup_{I\in\mathcal{G}'_k} I$. Then $F\subset G\cap A(\omega)$.

By (2.17), (2.18) and Lemma 2 in Ref. 19, we conclude that

$$\dim_{\mathrm{H}}(F)=\min\{1-\gamma_0,t\}.$$

Hence $\dim_{\mathrm{H}}(G\cap A)\geq\min\{t,1-\gamma_0\}$ with positive probability. Since $\dim_{\mathrm{H}}(G\cap A)\geq\min\{t,1-\gamma_0\}$ is a tail event, by the Kolmogorov zero-one law, we have $\dim_{\mathrm{H}}(G\cap A)=\min\{t,1-\gamma_0\}$ a.s. □

ACKNOWLEDGMENTS

We are very grateful to the referees for their valuable comments. The research of Bing Li and Zhang-Nan Hu is supported by NSFC 11671151, 11871227 and Guangdong Natural Science Foundation 2018B0303110005. Wen-Chiao Cheng is partially supported by NSC Grant (109-2115-M-034-001).

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Mathematische Zeitschrift (2022) 301:485–513
https://doi.org/10.1007/s00209-021-02924-2

Mathematische Zeitschrift

On the intersection of dynamical covering sets with fractals

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Received: 19 May 2021 / Accepted: 22 October 2021 / Published online: 18 January 2022

Abstract

Let $(X, \mathscr{B}, \mu, T, d)$ be a measure-preserving dynamical system with exponentially mixing property, and let μ be an Ahlfors s-regular probability measure. The dynamical covering problem concerns the set $E(x)$ of points which are covered by the orbits of $x \in X$ infinitely many times. We prove that the Hausdorff dimension of the intersection of $E(x)$ and any regular fractal G with $\dim_{\mathrm{H}} G > s-\alpha$ equals $\dim_{\mathrm{H}} G+\alpha-s$, where $\alpha = \dim_{\mathrm{H}} E(x)$ μ–a.e. Moreover, we obtain the packing dimension of $E(x)\cap G$ and an estimate for $\dim_{\mathrm{H}}(E(x)\cap G)$ for any analytic set G.

Keywords Dynamical covering sets · Exponentially mixing · Hausdorff dimension

Mathematics Subject Classification Primary 37A50; Secondary 28A80 · 60A10

1 Introduction

Let (X, d) be a compact metric space and let $(X, \mathscr{B}, T, \mu, d)$ be a metric measure preserving system (m.m.p.s. for short). The distribution of the orbit of a point in X is an important topic in ergodic theory and has been studied by many authors. See, for example [1,3,5,10,11,13,18]. The well-known Poincaré recurrence theorem shows that μ-a.e. $x \in X$ is recurrent, that is

$$\liminf_{n\to\infty} d(T^n x, x) = 0.$$

This research of Bing Li and Zhangnan Hu was supported by NSFC 11671151 and Guangdong Natural Science Foundation 2018B0303110005. The research of Yimin Xiao is partially supported by the NSF Grant DMS-1855185.

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Boshernitzan [3] proved that if there is some $\tau > 0$ such that the τ-dimensional Hausdorff measure $\mathcal{H}^\tau$ of X is σ-finite, then for μ-a.e. $x \in X$,

$$\liminf_{n\to\infty} n^{\frac{1}{\tau}} d(T^n x, x) < \infty.$$

If μ is ergodic, then for every fixed point $y \in X$, we have, for μ-a.e. $x \in X$,

$$\liminf_{n\to\infty} d(T^n x, y) = 0.$$

For an exponentially mixing metric measure preserving system, Fan, Langlet and Li [10] proved that if $t < 1/\alpha_{max}$, then for μ-a.e. $x \in X$, we have $\liminf_{n\to\infty} n^t d(T^n x, y) = 0$ for uniformly for all $y \in X$, where α_{max} is the maximal local dimension of μ.

Hill and Velani [15] introduced the shrinking targets theory, which concerns the following set of points whose orbits are close to a given point, that is for any given $y \in X$,

$$S(y) = \{x \in X\colon T^n x \in B(y, \ell_n) \text{ i.o.}\}, \tag{1.1}$$

where $\{\ell_n\}_{n\geq 1}$ is a sequence of positive real numbers tending to 0 and i.o. stands for infinitely often. Li, Wang, Wu and Xu [28] studied the shrinking target problem in the case when T is the Gauss map and determined the Hausdorff dimension of the set $S(y)$ in (1.1) for certain choices of $\{\ell_n\}$. Bugeaud and Wang [5] studied the problem for the case when T is the β-transformation. Aspenberg and Persson [1] extended their results to piecewise expanding maps.

Motivated by the Diophantine approximation, Fan, Schmeling and Troubetzkoy [11] proposed the dynamical covering set defined by

$$E(x) = \{y \in X\colon T^n x \in B(y, \ell_n) \text{ i.o.}\}, \tag{1.2}$$

which is the set of points y that are well approximated by the orbit of x. Among other interesting results, they considered the case when X is the unit interval and $T\colon x \mapsto 2x$ (mod 1) and computed $\dim_{\mathrm{H}} E(x)$, where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension. In [29], Liao and Seuret determined $\dim_{\mathrm{H}} E(x)$ when T is an expanding Markov map. Later, Persson and Rams [34] considered more general piecewise expanding maps than the Markov maps.

In 2017, Wang, Wu and Xu [40] considered the case when X is the middle-third Cantor set, $Tx = 3x$ (mod 1), and μ is the standard Cantor measure. They gave a complete characterization of the size $E(x)$ for μ-almost all x. In [17], Hu and Li investigated the dynamical covering sets in $(X, \mathscr{B}, T, \mu, d)$ with exponentially mixing property, where μ is an Ahlfors s-regular Borel probability measure. They showed that the measure $\mu(E(x))$ is 0 or 1 for μ-a.e. x according to the convergence or divergence of the series $\sum_{n=1}^{\infty} \ell_n^s$, and for μ-a.e. x,

$$\dim_{\mathrm{H}} E(x) = \alpha,$$

where α is the *upper Besicovitch–Taylor index* of $\{\ell_n\}_{n\geq 1}$ defined by

$$\alpha := \inf\left\{t \leq s\colon \sum_{n=1}^{\infty} \ell_n^t < \infty\right\} = \sup\left\{t \leq s\colon \sum_{n=1}^{\infty} \ell_n^t = \infty\right\}. \tag{1.3}$$

In particular, these results hold when X is the middle-third Cantor set and μ is the standard Cantor measure.

Motivated by the aforementioned research, we are interested in following natural questions for the dynamical covering set.

Question 1.1 *For a given set $G \subset X$, are there points in G which can be well approximated by the orbit of $x \in X$? Equivalently, when is $E(x) \cap G \neq \varnothing$ for $x \in X$?*

Question 1.2 *If $E(x) \cap G \neq \varnothing$, then how large is the intersection?*

Question 1.1 is also closely related to Mahler's question [31] which is concerned with approximating the numbers in the middle-third Cantor set $C_{1/3}$ by rational numbers and bears some analogy with the dynamical covering set. Several authors have investigated Mahler's question by measuring the size of the intersection set $\mathcal{W}_{\mathcal{A}}(\psi) \cap C_{1/3}$, where $\mathcal{A}$ is an infinite subset of $\mathbb{N}$ and

$$\mathcal{W}_{\mathcal{A}}(\psi) = \left\{ x \in \mathbb{R} \colon \left| x - \frac{p}{q} \right| \leq \psi(q) \text{ for infinitely many } (p,q) \in \mathbb{Z} \times \mathcal{A} \right\},$$

and ψ is the approximation speed. For example, when $\mathcal{A} := \{3^n : n = 0, 1, 2 \ldots\}$, Levesley, Slap and Velani [25] studied the f-Hausdorff measure $\mathcal{H}^f$ of the set $\mathcal{W}_{\mathcal{A}}(\psi) \cap C_{1/3}$, where f is a measure function, and provided a criterion for $\mathcal{H}^f(\mathcal{W}_{\mathcal{A}}(\psi) \cap C_{1/3})$ to be 0 or $\mathcal{H}^f(C_{1/3})$. As for the case when $\mathcal{A} = \mathbb{N}$, the problem is still open. Levesley, Salp and Velani [25] conjectured that if $\psi(q) = q^{-\tau}$ with $\tau \geq 2$ then

$$\dim_{\mathrm{H}}(\mathcal{W}_{\mathbb{N}}(q^{-\tau}) \cap C_{1/3}) = \frac{2}{\tau} \dim_{\mathrm{H}} C_{1/3}. \tag{1.4}$$

However, Bugeaud and Durand [4] disagreed with (1.4) and proposed another conjecture:

$$\dim_{\mathrm{H}}(\mathcal{W}_{\mathbb{N}}(q^{-\tau}) \cap C_{1/3}) = \max\left\{ \dim_{\mathrm{H}}(\mathcal{W}_{\mathbb{N}}(q^{-\tau})) + \dim_{\mathrm{H}} C_{1/3} - 1, \ \frac{1}{\tau} \dim_{\mathrm{H}} C_{1/3} \right\}. \tag{1.5}$$

Bugeaud and Durand [4] provided some results to support their conjecture. In particular, they showed that (1.5) holds for a natural probabilistic model which is a random covering set on the unit circle $\mathbb{T}$ generated by intervals whose centers are uniformly distributed independent random variables in $\mathbb{T}$ (see [4, Section 2]). Recently, Yu [41] proved that the conjecture (1.5) holds for the middle-pth Cantor set when $p > 10^7$ is odd and $\tau \in (1, 1+c)$ for some number $c > 0$. Here, for each odd integer $p > 2$, the middle-pth Cantor set is the set of numbers whose base p expansions do not have digit $(p-1)/2$.

Before stating the main results of this paper, we recall some definitions that will be used throughout this paper.

Definition 1 A Borel measure μ on $(X, \mathscr{B})$ is called Ahlfors s-regular ($0 < s < \infty$) if there exists a constant $1 \leq c_1 < \infty$ such that

$${c_1}^{-1} r^s \leq \mu\big(B(x,r)\big) \leq c_1 r^s \tag{1.6}$$

for all $x \in X$ and $0 < r \leq \operatorname{diam} X$, where $B(x,r)$ is the closed ball in metric d whose center is x with radius r and $\operatorname{diam} X$ is the diameter of X.

Definition 2 A m.m.p.s. $(X, \mathscr{B}, \mu, T, d)$ is *exponentially mixing* if there exist two constants $C > 0$ and $0 < \rho < 1$ such that

$$|\mu(E|T^{-n}F) - \mu(E)| \leq C\rho^n$$

for all $n \geq 1$, balls $E \subseteq X$, and measurable sets $F \in \mathscr{B}$ with $\mu(F) > 0$. Here $\mu(A|B)$ denotes the conditional measure $\frac{\mu(A \cap B)}{\mu(B)}$. Sometimes we say μ is exponentially mixing.

Throughout we denote packing dimension and upper box dimension in the metric space (X, d) by $\dim_{\mathrm{P}}$ and $\overline{\dim}_{\mathrm{B}}$, respectively. We adopt the convention that the Hausdorff dimension and packing dimension of empty sets are equal to $-\infty$ as in [4] to distinguish the empty set from a non-empty set with dimension 0.

Theorem 1 provides a criterion for Question 1.1. The condition (C) in Theorem 1 is stated in the Sect. 2.

Theorem 1 *Let $(X, \mathscr{B}, \mu, T, d)$ be an exponentially mixing m.m.p.s. and the measure μ be Ahlfors s-regular with $0 < s < \infty$. Let $\{\ell_n\}_{n\geq 1}$ be a sequence of positive numbers tending to 0 with the upper Besicovitch–Taylor index $\alpha < s$ and, for any $x \in X$, let $E(x)$ be the dynamical covering set defined in (1.2). Then for any analytic set $G \subset X$ we have, for μ-almost every $x \in X$*

$$E(x) \cap G \begin{cases} = \varnothing & \text{if } \dim_{\mathrm{P}}(G) < s - \alpha, \\ \neq \varnothing & \text{if } \dim_{\mathrm{H}}(G) > s - \alpha. \end{cases}$$

If, in addition, the condition (C) holds, then

$$E(x) \cap G \neq \varnothing \quad \text{if } \dim_{\mathrm{P}}(G) > s - \alpha.$$

Remark 1 Theorem 1 does not provide complete answers in the critical case of $\dim_{\mathrm{P}}(G) \geq s - \alpha \geq \dim_{\mathrm{H}}(G)$. Even if the condition (C) holds, it is not clear for the case of $\dim_{\mathrm{P}}(G) = s - \alpha$.

Our Theorem 2 is concerned with Question 1.2 and measures the size of $E(x) \cap G$.

Theorem 2 *Let $E(x)$ be the dynamical covering set as in Theorem 1. For any analytic set $G \subset X$ we have, for μ-almost every $x \in X$*

$$\dim_{\mathrm{H}}(E(x) \cap G) \begin{cases} \leq \dim_{\mathrm{P}}(G) + \alpha - s & \text{if } \dim_{\mathrm{P}}(G) \geq s - \alpha, \\ = -\infty & \text{if } \dim_{\mathrm{P}}(G) < s - \alpha, \\ \geq \dim_{\mathrm{H}}(G) + \alpha - s & \text{if } \dim_{\mathrm{H}}(G) > s - \alpha. \end{cases}$$

Moreover, if $\dim_{\mathrm{P}}(G) > s - \alpha$ and the condition (C) holds, then

$$\dim_{\mathrm{P}}(E(x) \cap G) = \dim_{\mathrm{P}}(G), \quad \text{a.e.}$$

Remark 2 When $\dim_{\mathrm{P}}(G) > s - \alpha \geq \dim_{\mathrm{H}}(G)$, the Hausdorff dimension $E(x) \cap G$ is not explicit. When $\dim_{\mathrm{P}}(G) = s - \alpha$, $E(x) \cap G$ is an empty set or a set with Hausdorff dimension 0.

As an immediate consequence of Theorem 2, we have

Corollary 1 *For any regular analytic set $G \subset X$ in the sense that $\dim_{\mathrm{H}}(G) = \dim_{\mathrm{P}}(G)$, we have for μ-almost every $x \in X$*

$$\dim_{\mathrm{H}}(E(x) \cap G) = \begin{cases} \dim_{\mathrm{P}}(G) + \alpha - s & \text{if } \dim_{\mathrm{P}}(G) > s - \alpha, \\ -\infty & \text{if } \dim_{\mathrm{P}}(G) < s - \alpha. \end{cases}$$

The rest of this article is organized as follows. In Sect. 2, we describe a more general probabilistic setting and prove results on the hitting probabilities of the random covering set. Then Theorems 1–2 follow from Theorems 3–5. This probabilistic setting is closely related to the classical Dvoretzky covering problem concerning the set $\limsup_{n\to\infty} B(x_n, \ell_n)$, where the centers $\{x_n\}$ are independent and uniformly distributed. Järvenpää et al. [20] studied the hitting probability of the Dvoretzky covering set in a general metric space. We extend their results from the independence setting to a stationary process which is exponentially mixing, and also the uniform distribution is generalized to Ahlfors regular distribution. These results will give the lower and upper bounds for $\dim_{\mathrm{H}}(E(x) \cap G)$ in Theorem 4. In Sect. 4, we derive Theorem 5 by extending the general method of Khoshnevisan, Peres, and Xiao [24] to limsup random fractals in metric spaces. Moreover, we obtain the packing dimension of $E(x) \cap G$ in Theorem 5. The proofs of our main results in Sect. 2 are given in Sects. 3 and 5, respectively. Finally, in Sect. 6, we provide some examples of dynamical systems that satisfy our assumptions so that the main theorems in the paper are applicable to them.

2 General results for stationary processes

In this section, we investigate Questions 1.1 and 1.2 in a general probabilistic setting, which extends the results in [4] and [20].

Let $\{\xi_n\}_{n\geq 1}$ be a stationary process defined on a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ and take values in a compact metric space (X, d). Let μ be the the distribution of ξ_1 which is the probability measure defined by

$$\mu(A) = \mathbb{P}(\xi_1 \in A) \tag{2.1}$$

for all Borel sets $A \subset X$.

Definition 3 We say that $\{\xi_n\}_{n\geq 1}$ is *exponentially mixing* if there exist two constants $C > 0$ and $0 < \rho < 1$ such that

$$\left|\mathbb{P}(\xi_1 \in A|D) - \mathbb{P}(\xi_1 \in A)\right| \leq C\rho^n$$

for all $n \geq 1$, balls $A \subset X$ and $D \in \mathscr{A}^{n+1}$, where $\mathscr{A}^{n+1}$ is the sub-σ-field generated by $\{\xi_{n+i}\}_{i\geq 1}$.

Remark 3 If $(X, \mathscr{B}, \mu, T, d)$ is an exponentially mixing m.m.p.s., define $\xi_n := T^{n-1}x$ for every $x \in X$, then the process $\{\xi_n\}_{n\geq 1}$ is an exponentially mixing process on probability space $(X, \mathscr{B}, \mu)$. Hence the probabilistic model described above covers the dynamical case.

Let $\{\ell_n\}_{n\geq 1}$ be a sequence of positive numbers decreasing to zero. For every $n \geq 1$, denote $I_n := B(\xi_n, \ell_n)$. Define

$$E := \limsup_{n\to\infty} I_n = \{y \in X : y \in I_n \text{ i.o.}\}.$$

The set E is a *random covering set* and consists of the points which are covered by $\{I_n\}_{n\geq 1}$ infinitely often.

The following theorem is concerned with the hitting probabilities of the random covering set E.

Theorem 3 *Let $\{\xi_n\}_{n\geq 1}$ be an exponentially mixing stationary process taking values in X with probability distribution μ. We assume that μ is Ahlfors s-regular with $0 < s < \infty$. Let α be the upper Besicovitch–Taylor index of $\{\ell_n\}_{n\geq 1}$ with $\alpha < s$. Then for every analytic set $G \subset X$, we have*

$$\mathbb{P}(E \cap G \neq \varnothing) = \begin{cases} 0 & \text{if } \dim_{\mathrm{P}}(G) < s - \alpha, \\ 1 & \text{if } \dim_{\mathrm{H}}(G) > s - \alpha. \end{cases}$$

The theorem below provides an estimate on the Hausdorff dimension of the intersection $E \cap G$.

Theorem 4 *Under the setting of Theorem 3, for every analytic set $G \subset X$, with probability one we have*

$$\dim_{\mathrm{H}}(E \cap G) \begin{cases} \leq \dim_{\mathrm{P}}(G) + \alpha - s & \text{if } \dim_{\mathrm{P}}(G) \geq s - \alpha, \\ = -\infty & \text{if } \dim_{\mathrm{P}}(G) < s - \alpha, \\ \geq \dim_{\mathrm{H}}(G) + \alpha - s & \text{if } \dim_{\mathrm{H}}(G) > s - \alpha. \end{cases}$$

The following corollary is an immediate consequence of Theorem 4.

Corollary 2 *Under the setting of Theorem 3, for any analytic set $G \subset X$ with $\dim_{\mathrm{H}}(G) = \dim_{\mathrm{P}}(G) = \gamma$ we have, with probability one*

$$\dim_{\mathrm{H}}(E \cap G) = \begin{cases} \gamma + \alpha - s & \text{if } \gamma > s - \alpha, \\ -\infty & \text{if } \gamma < s - \alpha. \end{cases}$$

In general, in Theorem 3, $\mathbb{P}(E \cap G \neq \varnothing) = 1$ may not hold if $\dim_{\mathrm{H}} G > s - \alpha$ is replaced by the weaker condition $\dim_{\mathrm{P}} G > s - \alpha$. A counter-example was given by Li and Suomala [27] when $\{\xi_n\}$ is a sequence of independent and uniformly distributed random variables on the circle. Therefore in the result below, we will make use of the following condition on the sequence $\{\ell_n\}$:

(C) Let $b \in (0, \frac{1}{3})$ be a constant. There exists an increasing sequence of positive integers $\{k_i\}$ such that $k_i \to \infty$ as $i \to \infty$,

$$\lim_{i \to \infty} \frac{k_{i+1}}{k_i} = 1 \tag{2.2}$$

and

$$\lim_{i \to \infty} \frac{\log_{b^{-1}} n_{k_i}}{k_i} = \alpha, \tag{2.3}$$

where

$$n_k = \#\{n \geq 1 \colon \ell_n \in [b^{k-1}, b^{k-2})\}.$$

Remark 4 By Li, Shieh and Xiao [26], the upper Besicovitch–Taylor index α of $\{\ell_n\}_{n \geq 1}$ can be expressed as

$$\alpha = \limsup_{k \to \infty} \frac{\log_{b^{-1}} n_k}{k}. \tag{2.4}$$

Under Condition (C), we are able to improve Theorem 3 by showing that the hitting probability of the random covering set E with an arbitrary analytic set $G \subseteq X$ is determined by the packing dimension of G.

Theorem 5 *Under the setting of Theorem 3, if the condition (C) holds, then for every analytic set $G \subset X$ with $\dim_{\mathrm{P}}(G) > s - \alpha$, we have*

$$\mathbb{P}(E \cap G \neq \varnothing) = 1.$$

Moreover, if $\dim_{\mathrm{P}}(G) > s - \alpha$, then $\dim_{\mathrm{P}}(E \cap G) = \dim_{\mathrm{P}}(G)$ a.s.

We end this section with some remarks about studies of random covering sets. The random covering problem goes back to 1897 when Borel investigated questions related to random placement of circular arcs in the unit circle [2]. Let $\xi = \{\xi_n\}$ be a sequence of independent and uniformly distributed random variables on the circle $\mathbb{T}$ and $\{\ell_n\}$ be a sequence of positive numbers decreasing to 0, define the random covering set

$$E(\xi) = \limsup_{n\to\infty} B(\xi_n, \ell_n).$$

In 1956, Dvoretzky [8] called the attention on the study of $E(\xi)$. He asked the question when $E(\xi) = \mathbb{T}$ a.s. or not. In 1971, Shepp [38] gave a sufficient and necessary condition: $E(\xi) = \mathbb{T}$ a.s. if and only if $\sum_{n=1}^{\infty} \frac{1}{n^2} \exp(\ell_1 + \cdots + \ell_n) = \infty$. For the high dimensional case, the Dvoretzky covering problem for balls is still open (more details see [23])

The Hausdorff dimensions of random covering sets were firstly investigated by Fan and Wu [9]. Järvenpää et al. [19] introduced the self-affine covering sets and obtained the dimension formula in terms of the singular value function, which was generalized to any Lebesgue measurable set covering by Feng et al. [12].

The random covering problem is related to many other fields such as number theory. For example, in 2017, Haynes and Koivusalo [14] used a covering argument in Dvoretzky [8] to prove the randomized version of the Littlewood Conjecture.

3 Proofs of Theorem 3 and Theorem 4

3.1 A nesting family in a metric space

We start this section by recalling Theorem 2.1 of Käenmäki, Rajala and Suomala [22], which provides a nesting family of "cubes" in a metric space with the finite doubling property [i.e., every ball $B(x, 2r) \subset X$ may be covered by finitely many balls of radius r.] This family shares most of the good properties of dyadic cubes of Euclidean spaces.

Theorem 6 [22] *Let (X, d) be a metric space with the finite doubling property and let $0 < b < \frac{1}{3}$ be a constant. Then there exists a collection $\{Q_{k,i} : k \in \mathbb{Z}, i \in \mathbb{N}_k \subset \mathbb{N}\}$ of Borel sets that have the following properties:*

1. *$X = \bigcup_{i\in\mathbb{N}_k} Q_{k,i}$ for every $k \in \mathbb{Z}$.*
2. *$Q_{k,i} \cap Q_{m,j} = \varnothing$ or $Q_{k,i} \subset Q_{m,j}$, where $k, m \in \mathbb{Z}$, $k \geq m$, $i \in \mathbb{N}_k$ and $j \in \mathbb{N}_m$.*
3. *For every $k \in \mathbb{Z}$ and $i \in \mathbb{N}_k$, there exists a point $x_{k,i} \in X$ such that*
$$U(x_{k,i}, c_2 b^k) \subset Q_{k,i} \subset B(x_{k,i}, c_2' b^k), \tag{3.1}$$
where $c_2 = \frac{1}{2} - \frac{b}{1-b}$, $c_2' = \frac{1}{1-b}$ and $U(x_{k,i}, c_2 b^k)$ is the open ball with center $x_{k,i}$ and radius $c_2 b^k$.
4. *There exists a point $x_0 \in X$ so that for every $k \in \mathbb{Z}$, there is an index $i \in \mathbb{N}_k$ with $U(x_0, c_2 b^k) \subset Q_{k,i}$.*
5. *$\{x_{k,i} : i \in \mathbb{N}_k\} \subset \{x_{k+1,i} : i \in \mathbb{N}_{k+1}\}$ for all $k \in \mathbb{Z}$.*

Remark 5 From the construction of $\{Q_{k,i} : k \in \mathbb{Z}, i \in \mathbb{N}_k \subset \mathbb{N}\}$ in [22] we see that for any $k \in \mathbb{Z}$, $Q_{k,i} \cap Q_{k,j} = \varnothing$ for $i \neq j \in \mathbb{N}_k$. When μ is Ahlfors s-regular, then by (1) and (3) in Theorem 6 and the equalities (1.6), we have

$$c_1^{-1} {c'}_2^{-s} b^{-ks} \leq \#\mathbb{N}_k = \#\{Q_{k,i} : i \in \mathbb{N}_k\} \leq c_1 c_2^{-s} b^{-ks}. \tag{3.2}$$

We will use this fact in our proofs of Theorems 3, 5 and 7 below.

If (X, d) is a compact metric space endowed with an Ahlfors s-regular measure μ, then it can be verified that X has the finite doubling property. Let $b \in (0, \frac{1}{3})$ be the constant in the condition (C) and let $\{Q_{k,i} : k \in \mathbb{Z}, i \in \mathbb{N}_k\}$ be the nesting family as in Theorem 6 which we call "generalized dyadic cubes" of (X, d). For convenience, we write $\mathcal{Q}_0 = \{X\}$ and $\mathcal{Q}_k = \{Q_{k,i} : i \in \mathbb{N}_k\}$ for $k \geq 1$, and $\mathcal{B}_k = \{B(x_{k,i}, c_2' b^k) : i \in \mathbb{N}_k\}$, where $x_{k,i}$ $(i \in \mathbb{N}_k)$ are the points in Part (3) of Theorem 6.

Lemma 1 *Let (X, d) be a compact metric space endowed with an Ahlfors s-regular measure μ, where $0 < s < \infty$ is a constant. Then, for any constants $a_0 > 0$ and $k \geq 1$, a ball B of radius $a_0 b^k$ may intersect at most $\frac{c_1^2(2c_2'+a_0)^s}{c_2^s}$ elements in $\mathcal{Q}_k$, where c_1, c_2 and c_2' are the constants given in (1.6) and Theorem 6, respectively.*

Proof Write $B = B(x_B, a_0 b^k)$ and

$$\mathcal{A} = \Big\{ Q_{k,i} \in \mathcal{Q}_k : Q_{k,i} \cap B \neq \varnothing,\ i \in \mathbb{N}_k \Big\}.$$

For $Q_{k,i} \in \mathcal{A}$, from Theorem 6 (3), there exists one point $x_{k,i} \in Q_{k,i}$ so that (3.1) holds. We denote the collection of such points by Υ, and so $\#\mathcal{A} = \#\Upsilon$. Notice that

$$B \subset \bigcup_{Q_{k,i} \in \mathcal{A}} Q_{k,i} \subset B(x_B, (2c_2' + a_0) b^k),$$

and

$$\bigcup_{x_{k,i} \in \Upsilon} U(x_{k,i}, c_2 b^k) \subset \bigcup_{Q_{k,i} \in \mathcal{A}} Q_{k,i}.$$

Therefore

$$\sum_{x_{k,i} \in \Upsilon} \mu\Big(U(x_{k,i}, c_2 b^k)\Big) \leq \mu\Big(\bigcup_{Q_{k,i} \in \mathcal{A}} Q_{k,i}\Big)\, le\mu\Big(B(x_B, (2c_2' + a_0) b^k)\Big).$$

Since μ is Ahlfors s-regular, we have

$$\sum_{x_{k,i} \in \Upsilon} \mu(U(x_{k,i}, c_2 b^k)) \geq (\#\mathcal{A})(c_2 b^k)^s c_1^{-1},$$

and

$$\mu(B(x_B, (2c_2' + a_0) b^k)) \leq c_1 (2c_2' + a_0)^s b^{ks}.$$

Hence $\#\mathcal{A} \leq \frac{c_1^2(2c_2'+a_0)^s}{c_2^s}$. This proves Lemma 1. □

3.2 Proofs of Theorem 3 and Theorem 4

For $k \geq 1$, denote by $\mathfrak{I}_k = \{j \geq 1 : \ell_j \in [b^{k-1}, b^{k-2})\}$ and $n_k = \#\mathfrak{I}_k$. Given a constant $c > \frac{s-\alpha}{\log_b \rho}$, where ρ is the constant in Definition 3, let $\mathfrak{I}_k'$ be a maximal collection of $\mathfrak{I}_k$ having mutual distances at least ck. One important property of $\mathfrak{I}_k'$ is that any pair of integers $n, m \in \mathfrak{I}_k'$ are at least of distance ck from each other. We will use this fact to prove Theorems 3 and 5. See Remark 6 below for more details.

Write $m_k = \#\mathfrak{I}_k'$. Since the sequence $\{\ell_n\}_{n \geq 1}$ is decreasing, the elements in $\mathfrak{I}_k$ are consecutive to each other. Then $m_k = \lceil (ck)^{-1} n_k \rceil$, where $\lceil \cdot \rceil$ stands for the ceiling function.

Lemma 2 *Let (X, d) be a compact metric space, and $G \subset X$ be an analytic set.*

(1) *If* $\dim_H G > t$*, there is a nonempty compact subset $G^\star \subset G$ such that*

$$\dim_H(G^\star \cap V) > t$$

for all open sets $V \subset X$ with $G^\star \cap V \neq \varnothing$.

(2) *If* $\dim_P G > t$*, there is a nonempty compact subset $G_\star \subset G$ such that*

$$\dim_P(G_\star \cap V) > t$$

for all open sets $V \subset X$ with $G_\star \cap V \neq \varnothing$.

The conclusions in Lemma 2 are known. We include a proof for completeness.

Proof Since G is analytic, from [16], there exists a compact set $K \subset G$ with $0 < \mathcal{H}^{t'}(K) < \infty$ for some $t' > t$. Let $\{V_i\}_{i \geq 1}$ be a countable basis of X. Denote

$$\mathcal{V} = \{i : \dim_H(K \cap V_i) \leq t\}, \tag{3.3}$$

Then $\bigcup_{i \in \mathcal{V}} V_i$ is relatively open in K, and hence $G^\star = K \backslash \bigcup_{i \in \mathcal{V}} V_i$ is compact. Note that

$$\begin{aligned} \mathcal{H}^{t'}(K) &= \mathcal{H}^{t'}\Big(K \cap \bigcup_{i \in \mathcal{V}} V_i\Big) + \mathcal{H}^{t'}\Big(K \backslash \bigcup_{i \in \mathcal{V}} V_i\Big) \\ &\leq \mathcal{H}^{t'}\Big(K \cap \bigcup_{i \in \mathcal{V}} V_i\Big) + \mathcal{H}^{t'}(G^\star) \\ &\leq \sum_{i \in \mathcal{V}} \mathcal{H}^{t'}(K \cap V_i) + \mathcal{H}^{t'}(G^\star). \end{aligned}$$

From (3.3) and $t' > t$, we get $\mathcal{H}^{t'}(K \cap V_i) = 0$. Therefore

$$\mathcal{H}^{t'}(G^\star) = \mathcal{H}^{t'}(K) > 0.$$

In particular, $G^\star \neq \varnothing$. Since $\{V_i\}$ is a basis of X, for any open set $V \subset X$ with $V \cap G^\star \neq \varnothing$, we have $\dim_H(G^\star \cap V) > t$.

The statement (2) can be obtained similarly with (1) by applying Corollary 1 of Joyce and Preiss [21]. We omit the details. □

The following lemma is directly derived from the definitions of Hausdorff dimension and Hausdorff measure.

Lemma 3 *If $G \subset X$ with $\dim_H(G) > t$, then there is $k_0 \geq 1$ such that for $k \geq k_0$, there are at least b^{-kt} elements in $\mathcal{Q}_k$ intersecting G, that is*

$$\#\{Q \in \mathcal{Q}_k : Q \cap G \neq \varnothing\} \geq b^{-kt}.$$

For $k \geq 1$, and $Q \in \mathcal{Q}_k$, denote by $B_Q(\in \mathcal{B}_k)$ the closed ball containing Q given in Sect. 3.1. Write X_Q for the indicator function of the event $\{\xi_n \in B_Q$ for some $n \in \mathfrak{I}'_k\}$. Then

$$\mathbb{E}(X_Q) = \mathbb{P}\Big(\bigcup_{n \in \mathfrak{I}'_k} \{\xi_n \in B_Q\}\Big).$$

Remark 6 In the lemma below, we estimate $\mathrm{Cov}(X_Q, X_{Q'})$ for some $Q', Q \in \mathcal{Q}_k$. Here we use $\mathfrak{I}'_k$ instead of $\mathfrak{I}_k$. Hence for any pair $n, m \in \mathfrak{I}'_k$, we have $\mathrm{dist}(n, m) \geq ck$ so that we can apply the exponential mixing property of $\{\xi_n\}$ to derive that

$$\frac{\sum_{n\in\mathfrak{I}'_k} \sum_{\substack{m\in\mathfrak{I}'_k \\ n>m}} \rho^{n-m}\mathbb{P}(\xi_n \in B_Q)}{\left(\sum_{n\in\mathfrak{I}'_k} \mathbb{P}(\xi_n \in B_Q)\right)^2} \to 0,$$

as $k \to \infty$, which is crucial in our proofs.

Lemma 4 *Suppose $\alpha < s$ and $\epsilon > 0$. There exists a constant $k_0 \geq 1$ such that for $k \geq k_0$, the following statements hold.*

(1) *If $B_Q \cap B_{Q'} = \varnothing$, where $Q, Q' \in \mathcal{Q}_k$, then*

$$\mathrm{Cov}(X_Q, X_{Q'}) < \epsilon\mathbb{E}(X_Q)\mathbb{E}(X_{Q'}).$$

(2) *There is a constant $0 < M_0 < \infty$ such that*

$$\max_{Q\in\mathcal{Q}_k} \#\{Q' \in \mathcal{Q}_k : B_Q \cap B_{Q'} \neq \varnothing\} \leq M_0.$$

Proof Let $Q,\ Q' \in \mathcal{Q}_k$ such that $B_Q \cap B_{Q'} = \varnothing$. Then

$$\begin{aligned}
&\mathrm{Cov}(X_Q, X_{Q'}) = \mathbb{E}(X_Q X_{Q'}) - \mathbb{E}(X_Q)\mathbb{E}(X_{Q'}) \\
&\quad = \mathbb{P}\Bigg(\bigcup_{n\in\mathfrak{I}'_k}\bigcup_{m\in\mathfrak{I}'_k} \{\xi_n \in B_Q, \xi_m \in B_{Q'}\}\Bigg) - \mathbb{P}\Bigg(\bigcup_{n\in\mathfrak{I}'_k} \{\xi_n \in B_Q\}\Bigg)\mathbb{P}\Bigg(\bigcup_{m\in\mathfrak{I}'_k} \{\xi_m \in B_{Q'}\}\Bigg).
\end{aligned} \tag{3.4}$$

Observe that

$$\begin{aligned}
&\mathbb{P}\Bigg(\bigcup_{n\in\mathfrak{I}'_k}\bigcup_{m\in\mathfrak{I}'_k} \{\xi_n \in B_Q, \xi_m \in B_{Q'}\}\Bigg) \\
&\quad = \mathbb{P}\Bigg(\bigcup_{n\in\mathfrak{I}'_k} \{\xi_n \in B_Q \cap B_{Q'}\} \cup \bigcup_{n\in\mathfrak{I}'_k}\bigcup_{\substack{m\in\mathfrak{I}'_k \\ m\neq n}} \{\xi_n \in B_Q, \xi_m \in B_{Q'}\}\Bigg) \\
&\quad = \mathbb{P}\Bigg(\bigcup_{n\in\mathfrak{I}'_k}\bigcup_{\substack{m\in\mathfrak{I}'_k \\ m\neq n}} \{\xi_n \in B_Q, \xi_m \in B_{Q'}\}\Bigg),
\end{aligned} \tag{3.5}$$

where the last equality follows from the fact that $B_Q \cap B_{Q'} = \varnothing$.

Since $\{\xi_j\}_{j\geq 1}$ is an exponentially mixing stationary process, if $n > m$, we obtain

$$\mathbb{P}(\xi_n \in B_Q, \xi_m \in B_{Q'}) \leq \mathbb{P}(\xi_n \in B_Q)\mathbb{P}(\xi_m \in B_{Q'}) + C\rho^{n-m}\mathbb{P}(\xi_n \in B_Q).$$

Whence we get an upper bound for (3.5):

$$\begin{aligned}
&\mathbb{P}\Bigg(\bigcup_{n\in\mathfrak{I}'_k}\bigcup_{\substack{m\in\mathfrak{I}'_k \\ m\neq n}} \{\xi_n \in B_Q, \xi_m \in B_{Q'}\}\Bigg) \\
&\quad \leq \Bigg(\sum_{n\in\mathfrak{I}'_k} \mathbb{P}(\xi_n \in B_Q)\Bigg)\Bigg(\sum_{m\in\mathfrak{I}'_k} \mathbb{P}(\xi_m \in B_{Q'})\Bigg) + \frac{C\rho^{ck}}{1-\rho^{ck}} \sum_{n\in\mathfrak{I}'_k} \mathbb{P}(\xi_n \in B_Q),
\end{aligned} \tag{3.6}$$

and a lower bound for

$$\begin{aligned}\mathbb{P}\Big(\bigcup_{n\in\mathfrak{I}'_k}\{\xi_n\in B_Q\}\Big)&\geq\sum_{n\in\mathfrak{I}'_k}\mathbb{P}(\xi_n\in B_Q)-\sum_{n\in\mathfrak{I}'_k}\sum_{\substack{n'\in\mathfrak{I}'_k\\ n'\neq n}}\mathbb{P}(\xi_n\in B_Q,\xi_{n'}\in B_Q)\\&\geq\Big(\sum_{n\in\mathfrak{I}'_k}\mathbb{P}(\xi_n\in B_Q)\Big)\Big(1-\sum_{n'\in\mathfrak{I}'_k}\mathbb{P}(\xi_{n'}\in B_Q)-\frac{C\rho^{ck}}{1-\rho^{ck}}\Big).\end{aligned}\tag{3.7}$$

Since $\alpha<s$, we can pick $q\in(\alpha,s)$. It follows from equality (2.4) that there exists $k_1\geq 1$ such that $n_k\leq b^{-kq}$ for all $k\geq k_1$. Hence for $k\geq k_1$,

$$\sum_{n\in\mathfrak{I}'_k}\mathbb{P}(\xi_n\in B_Q)\leq c_1{c'_2}^s b^{ks}m_k\leq c_1{c'_2}^s\Big(b^{ks}+(ck)^{-1}b^{k(s-q)}\Big),$$

which yields $\sum_{n\in\mathfrak{I}'_k}\mathbb{P}(\xi_n\in B_Q)\to 0$ as $k\to\infty$. From (3.7), there is a $k_2\geq 1$ such that for $k\geq k_2$,

$$\mathbb{E}(X_Q)\geq M_1\sum_{n\in\mathfrak{I}'_k}\mathbb{P}(\xi_n\in B_Q),\tag{3.8}$$

where $0<M_1<1$ is a constant. By combining (3.4)–(3.8), we see that

$$\begin{aligned}\mathrm{Cov}(X_Q,X_{Q'})\leq&\Big(\sum_{n\in\mathfrak{I}'_k}\mathbb{P}(\xi_n\in B_Q)\Big)\Big(\sum_{m\in\mathfrak{I}'_k}\mathbb{P}(\xi_m\in B_{Q'})\Big)\Big(\sum_{\substack{n\in\mathfrak{I}'_k\\ n\neq m}}\mathbb{P}(\xi_n\in B_Q)\\&+\sum_{\substack{m\in\mathfrak{I}'_k\\ m\neq n}}\mathbb{P}(\xi_m\in B_{Q'})+\frac{4C\rho^{ck}}{1-\rho^{ck}}\Big)+\frac{2C\rho^{ck}}{1-\rho^{ck}}\Big(\sum_{n\in\mathfrak{I}'_k}\mathbb{P}(\xi_n\in B_Q)\Big)\\\leq&\Big(\sum_{n\in\mathfrak{I}'_k}\mathbb{P}(\xi_n\in B_Q)\Big)\Big(\sum_{m\in\mathfrak{I}'_k}\mathbb{P}(\xi_m\in B_{Q'})\Big)\Big\{\sum_{n\in\mathfrak{I}'_k}\mathbb{P}(\xi_n\in B_Q)\\&+\sum_{m\in\mathfrak{I}'_k}\mathbb{P}(\xi_m\in B_{Q'})+\frac{2C\rho^{ck}}{1-\rho^{ck}}\Big(2+\Big(\sum_{m\in\mathfrak{I}'_k}\mathbb{P}(\xi_m\in B_{Q'})\Big)^{-1}\Big)\Big\}.\end{aligned}$$

The inequality above together with (3.8) implies that for $k\geq k_2$, we have

$$\begin{aligned}\frac{\mathrm{Cov}(X_Q,X_{Q'})}{\mathbb{E}(X_Q)\mathbb{E}(X_{Q'})}\leq&\frac{1}{M_1^2}\Big\{\sum_{n\in\mathfrak{I}'_k}\mathbb{P}(\xi_n\in B_Q)+\sum_{m\in\mathfrak{I}'_k}\mathbb{P}(\xi_m\in B_{Q'})\\&+\frac{2C\rho^{ck}}{1-\rho^{ck}}\Big(2+\Big(\sum_{m\in\mathfrak{I}'_k}\mathbb{P}(\xi_m\in B_{Q'})\Big)^{-1}\Big)\Big\}.\end{aligned}\tag{3.9}$$

Since $c>\frac{s-\alpha}{\log_b\rho}$, we derive that the right-hand side of (3.9) tends to 0 as $k\to\infty$.

Thereby for any $\epsilon>0$, there is a $k_0\geq 1$ satisfying for all $k\geq k_0$, if $Q,Q'\in\mathcal{Q}_k$ with $B_Q\cap B_{Q'}=\varnothing$, we always have

$$\mathrm{Cov}(X_Q,X_{Q'})<\epsilon\mathbb{E}(X_Q)\mathbb{E}(X_{Q'}).$$

Notice that if the distance $\mathrm{dist}(Q, Q') \geq 2(c_2' - c_2)b^k$ for $Q, Q' \in \mathcal{Q}_k$, we have $B_Q \cap B_{Q'} = \varnothing$. Then for $Q \in \mathcal{Q}_k$,

$$\#\{Q' \colon B_Q \cap B_{Q'} \neq \varnothing\} \leq \#\{Q' \colon \mathrm{dist}(Q, Q') \leq 2(c_2' - c_2)b^k\}.$$

From Lemma 1, there exists a constant $0 < M_0 < \infty$ independent of k such that

$$\max_{Q \in \mathcal{Q}_k} \#\{Q' \colon B_Q \cap B_{Q'} \neq \varnothing\} \leq M_0$$

holds for $k \geq k_0$. □

Now we are ready to prove Theorems 3 and 4.

Proof of Theorem 3 Firstly we show that $\dim_{\mathrm{P}}(G) < s - \alpha$ implies $\mathbb{P}(E \cap G \neq \varnothing) = 0$. By Tricot [39], it suffices to show that whenever $\overline{\dim}_{\mathrm{B}}(G) < s - \alpha$, then $E \cap G = \varnothing$ a.s.

We denote by $C_{\ell_n} = C_{\ell_n}(G)$ a collection of the smallest number of the closed balls with radius ℓ_n that cover the set G. Let $N_{\ell_n}(G) = \#C_{\ell_n}$. Fixing an arbitrary $\eta > 0$ such that $\eta \in (\overline{\dim}_{\mathrm{B}}(G), s - \alpha)$, we have

$$\limsup_{n \to \infty} \frac{\log N_{\ell_n}(G)}{-\log(\ell_n)} \leq \overline{\dim}_{\mathrm{B}}(G) < \eta,$$

so there exists an integer $n_0 \in \mathbb{N}$ such that

$$N_{\ell_n}(G) < \ell_n^{-\eta} \tag{3.10}$$

for all $n \geq n_0$. For any ball B in X with radius ℓ_n, since $\{\xi_n\}_{n \geq 1}$ is a stationary process, we have

$$\mathbb{P}\{I_n \cap B \neq \varnothing\} = \mathbb{P}(\xi_n \in B(x_B, 2\ell_n)) = \mu(B(x_B, 2\ell_n)) \leq c_1 2^s \ell_n^s,$$

where $I_n = B(\xi_n, \ell_n)$ and x_B is the center of B. Note that the event

$$\{I_n \cap G \neq \varnothing\} \subset \bigcup_{B \in C_{\ell_n}} \{I_n \cap B \neq \varnothing\}.$$

We derive from this and (3.10) that

$$\mathbb{P}\{I_n \cap G \neq \varnothing\} \leq \sum_{B \in C_{\ell_n}} \mathbb{P}\{I_n \cap B \neq \varnothing\} \leq N_{\ell_n}(G) c_1 2^s \ell_n^s < c_1 2^s \ell_n^{s-\eta}$$

for all $n \geq n_0$. Hence the series $\sum_{n=1}^{\infty} \mathbb{P}\{I_n \cap G \neq \varnothing\}$ converges by the definition of α and $\eta < s - \alpha$. By the Borel–Cantelli Lemma, we have

$$\mathbb{P}\{I_n \cap G \neq \varnothing \quad \text{i.o.}\} = 0.$$

That is, $E \cap G = \varnothing$ a.s.

Now we prove that if $\dim_{\mathrm{H}}(G) > s - \alpha$, then $\mathbb{P}(E \cap G \neq \varnothing) = 1$. By Lemma 2, we may assume that G is compact and satisfies $\dim_{\mathrm{H}}(G \cap V) > s - \alpha$ whenever $V \subset X$ is an open set with $G \cap V \neq \varnothing$. We choose constants β and t such that $\dim_{\mathrm{H}}(G \cap V) > t > s - \beta > s - \alpha$.

Denote by $U(\xi_n, \ell_n)$ the open ball with center ξ_n and radius ℓ_n. It suffices to show that

$$\mathbb{P}\big\{U(\xi_n, \ell_n) \cap G \cap V \neq \varnothing \quad \text{i.o.}\big\} = 1. \tag{3.11}$$

Letting V run over a countable basis of X, we have $G \cap \bigcup_{n=k}^{\infty} B(\xi_n, \ell_n)$ is dense a.s. and relatively open in G for any $k \geq 1$. Then from Baire's category theorem we derive $G \cap \limsup_{n \to \infty} U(\xi_n, \ell_n) \neq \varnothing$ a.s., which implies that $E \cap G \neq \varnothing$ a.s.

Now we prove that the equality (3.11) holds. By (2.4) we have

$$\alpha = \limsup_{k\to\infty} \frac{\log_{b^{-1}} n_k}{k} > \beta,$$

then there are infinitely many k such that $n_k \geq b^{-k\beta}$. This implies that the set defined as

$$\mathcal{N} = \{k \geq 1 \colon n_k \geq b^{-k\beta}\}$$

satisfies $\#\mathcal{N} = \infty$.

Fix an open set V with $G \cap V \neq \varnothing$. Define

$$\mathcal{Z}_k = \{Q \in \mathcal{Q}_k \colon Q \cap G \cap V \neq \varnothing\}.$$

From Lemma 3, we have $N_k = \#\mathcal{Z}_k \geq b^{-tk}$ for all k large enough. For $Q \in \mathcal{Q}_k$, there exists $B(x_Q, c_2' b^k) \in \mathcal{B}_k$, denoted by B_Q. For $k \in \mathcal{N}$, define

$$S_k = \#\{Q \in \mathcal{Z}_k \colon \xi_n \in B_Q \text{ for some } n \in \mathfrak{I}_k'\},$$

that is $S_k = \sum_{Q\in\mathcal{Z}_k} X_Q$. For $\epsilon > 0$ and $Q \in \mathcal{Z}_k$, write

$$\mathcal{D}_k(Q) = \left\{Q' \in \mathcal{Z}_k \colon \mathrm{Cov}(X_Q, X_{Q'}) \geq \epsilon \mathbb{E}(X_Q)\mathbb{E}(X_{Q'})\right\},$$

then from Lemma 4, there is a constant $0 < M_0' < \infty$ such that $\max_{Q\in\mathcal{Z}_k} \#\mathcal{D}_k(Q) \leq M_0'$ for k large enough. Since

$$\mathrm{Var}(S_k) = \sum_{Q\in\mathcal{Z}_k} \sum_{Q'\in\mathcal{Z}_k} \mathrm{Cov}(X_Q, X_{Q'}),$$

and

$$\mathrm{Cov}(X_Q, X_{Q'}) = \mathbb{E}(X_Q X_{Q'}) - \mathbb{E}(X_Q)\mathbb{E}(X_{Q'}) \leq \mathbb{E}(X_Q),$$

it follows that

$$\begin{aligned}
\mathrm{Var}(S_k) &\leq \sum_{Q\in\mathcal{Z}_k} \Bigg(\sum_{Q'\in\mathcal{Z}_k\setminus\mathcal{D}_k(Q)} \epsilon \mathbb{E}(X_Q)\mathbb{E}(X_{Q'}) + M_0'\mathbb{E}(X_Q) \Bigg) \\
&\leq \epsilon \Bigg(\sum_{Q\in\mathcal{Z}_k} \mathbb{E}(X_Q) \Bigg)^2 + M_0' \sum_{Q\in\mathcal{Z}_k} \mathbb{E}(X_Q).
\end{aligned}$$

From (3.7), we have

$$\mathbb{E}(S_k) = \sum_{Q\in\mathcal{Z}_k} \mathbb{E}(X_Q) \geq \sum_{Q\in\mathcal{Z}_k} \Bigg(\sum_{n\in\mathfrak{I}_k'} \mathbb{P}(\xi_n \in B_Q) \Bigg) \Bigg(1 - \sum_{n\in\mathfrak{I}_k'} \mathbb{P}(\xi_n \in B_Q) - \frac{2C\rho^{ck}}{1-\rho^{ck}} \Bigg).$$

It yields that there is a constant $0 < M_2 < \infty$ such that for all $k \in \mathcal{N}$ large enough, we have

$$\mathbb{E}(S_k) \geq M_2 N_k m_k b^{ks} \geq M_2 (ck)^{-1} b^{k(s-t-\beta)}.$$

Recall $t + \beta > s$, if $k \in \mathcal{N}$ and $k \to \infty$, we get $\mathbb{E}(S_k) \to \infty$.

Combining these and Chebyshev's inequality, we obtain

$$\mathbb{P}(S_k = 0) \leq \frac{\mathrm{Var}(S_k)}{\mathbb{E}^2(S_k)} \leq \epsilon + \frac{M_0'}{\mathbb{E}(S_k)}.$$

Therefore

$$\limsup_{\substack{k\in\mathcal{N}\\ k\to\infty}} \mathbb{P}(S_k = 0) = 0.$$

We observe that

$$\{U(\xi_n, \ell_n) \cap G \cap V \neq \varnothing \quad \text{i.o.}\} \supset \{S_k > 0 \quad \text{i.o.}\}.$$

Finally this together with Fatou's lemma implies that

$$\mathbb{P}(U(\xi_n, \ell_n) \cap G \cap V \neq \varnothing \quad \text{i.o.}) \geq \limsup_{k\to\infty} \mathbb{P}(S_k > 0) = 1.$$

This finishes the proof of Theorem 3. □

For proving Theorem 4, we will make use of the following lemma, which is an analogue of Lemma 3.4 in [24], where $X = [0, 1]^N$ was considered. Here (X, d) is an Ahlfors regular metric space.

Lemma 5 *Equip X with the Borel σ-algebra. Suppose that $A = A(\omega)$ is a random set in X (i.e., the indicator function $\chi_{A(\omega)}(x)$ is jointly measurable) such that for any analytic set $G \subset X$ with $\dim_{\mathrm{H}} G > \gamma$, we have*

$$\mathbb{P}(A \cap G \neq \varnothing) = 1.$$

Then

$$\dim_{\mathrm{H}}(A \cap G) \geq \dim_{\mathrm{H}} G - \gamma, \quad \text{a.s.}$$

if $\dim_{\mathrm{H}} G > \gamma$.

We postpone the proof of Lemma 5 to the end of this section. Let us first prove Theorem 4.

Proof of Theorem 4 Firstly we prove that $\dim_{\mathrm{H}}(E\cap G) \leq \dim_{\mathrm{P}} G+\alpha-s$ a.s. when $\dim_{\mathrm{P}} G+\alpha - s \geq 0$. It suffices to show that

$$\dim_{\mathrm{H}}(E \cap G) \leq \overline{\dim}_{\mathrm{B}} G + \alpha - s \quad \text{a.s.}$$

Let $\mathcal{C}_{\ell_n}$ be a collection of the closed balls with radius ℓ_n whose union covers G such that $\mathcal{N}_{\ell_n}(G) = \#\mathcal{C}_{\ell_n}$ is the smallest. Let $\xi := \overline{\dim}_{\mathrm{B}} G + \alpha - s$, then for any $\epsilon > 0$, we have

$$\xi + s - \alpha + \epsilon > \overline{\dim}_{\mathrm{B}} G \geq \limsup_{n\to\infty} \frac{\log \mathcal{N}_{\ell_n}(G)}{-\log(\ell_n)},$$

which implies that

$$\mathcal{N}_{\ell_n}(G) < \ell_n^{\alpha-s-\xi-\epsilon} \tag{3.12}$$

for all n large enough. For any $\delta > 0$, we choose $N \geq 1$ such that for all $n \geq N$, we have $\ell_n < \delta$ and inequality (3.12) holds.

Let

$$\mathcal{A}_n = \{B \in \mathcal{C}_{\ell_n} : B \cap I_n \neq \varnothing\}.$$

Notice that

$$\mathbb{E}(\#\mathcal{A}_n) \leq \sum_{B\in\mathcal{A}_n} \mathbb{P}(I_n \cap B \neq \varnothing) \leq c_1 2^s \ell_n^{\alpha-\xi-\epsilon},$$

and

$$E \cap G \subset \bigcup_{n=N}^{\infty} I_n \cap G \subset \bigcup_{n=N}^{\infty} \bigcup_{B \in \mathcal{A}_n} B.$$

Hence for any $\theta > \xi$, by choosing $\epsilon > 0$ with $2\epsilon < \theta - \xi$, we have

$$\mathbb{E}\big(\mathcal{H}^{\theta}_{\delta}(E \cap G)\big) \leq \sum_{n=n_1}^{\infty} \mathbb{E}(\#\mathcal{A}_n)(2\ell_n)^{\theta} \leq \sum_{n=n_1}^{\infty} c_1 2^{\theta+s} \ell_n^{\alpha+s}.$$

By the definition of α, we obtain that

$$\mathbb{E}\big(\mathcal{H}^{\theta}_{\delta}(E \cap G)\big) < \infty.$$

Therefore $\mathcal{H}^{\theta}(E \cap G) < \infty$ a.s. which implies $\dim_{\mathrm{H}}(E \cap G) \leq \theta$ a.s. Hence $\dim_{\mathrm{H}}(G \cap E) \leq \dim_{\mathrm{P}} G + \alpha - s$ a.s.

When $\dim_{\mathrm{P}} G + \alpha - s < 0$, by Theorem 3, $\mathbb{P}(E \cap G = \varnothing) = 1$. Hence $\dim_{\mathrm{H}}(E \cap G) = -\infty$ a.s.

The lower bound of $\dim_{\mathrm{H}}(E \cap G)$ in Theorem 4 follows from Lemma 5. This finishes the proof of Theorem 4. □

It remains to prove Lemma 5. The fractal percolation Γ_t used in Lemma 5 can be constructed as follows.

Let $0 < p < 1$. For each $Q \in \mathcal{Q}$, let $Z(Q)$ be a random variable taking value 1 with probability p and value 0 with probability $1 - p$. We assume that these random variables are independent for different $Q \in \mathcal{Q}$. We define the random fractal percolation set as

$$\Gamma(p) = \bigcap_{n \in \mathbb{N}} \bigcup_{\substack{Q \in \mathcal{Q}_n \\ Z(Q)=1}} \overline{Q},$$

where $\overline{Q}$ is the closure of the set Q. When $p = 2^{-t} < 1$, for convenience, we denote $\Gamma_t = \Gamma(2^{-t})$.

Notice that there is a nature tree structure behind the definition of $\mathcal{Q}$, which we describe now.

Label each $Q \in \mathcal{Q}$ with a vertex v_Q and let T be a graph with vertex set $\{v_Q\}_{Q \in \mathcal{Q}}$. There exists an edge between vertices $v_{Q_{n,i}}$ and $v_{Q_{m,j}}$ if and only if $|n - m| = 1$ and $Q_{n,i} \cap Q_{m,j} \neq \varnothing$. Then T is a tree with v_X as its root. The boundary of the tree ∂T consists of all infinite paths $(v_{0,i_0} v_{1,i_1} v_{2,i_2} \ldots)$, where $v_{m,i_m} = v_{Q_{m,i_m}}$ for $Q_{m,i_m} \in \mathcal{Q}_m$ and $Q_{m,i_m} \subset Q_{n,i_n}$, if $m \geq n$. We call these infinite paths *rays*. Then we can define a projection $\Pi : \partial T \to X$ as

$$\Pi : (v_{0,i_0} v_{1,i_1} v_{2,i_2} \ldots) \mapsto \bigcap_{n=0}^{\infty} \overline{Q_{n,i_n}}. \tag{3.13}$$

Note that $\Pi(\partial T) = X$. For $v = (v_{0,i_0} v_{1,i_1} v_{2,i_2} \ldots)$, $u = (u_{0,i_0} u_{1,i_1} u_{2,i_2} \ldots) \in \partial T$, we define

$$\kappa(v, u) = \begin{cases} 0 & \text{if } u = v, \\ b^{\min\{j : v_{j,i_j} \neq u_{j,i_j}\}} & \text{if } u \neq v. \end{cases}$$

Then $(\partial T, \kappa)$ is a metric space. We claim that for every $G \subset X$, we have

$$\dim_{\mathrm{H}} G = \dim_{\mathrm{H}}^{\kappa}(\Pi^{-1} G), \tag{3.14}$$

where $\dim_{\mathrm{H}}^{\kappa}$ is the Hausdorff dimension in the metric space $(\partial T, \kappa)$.

Now we prove this claim. For any $x, y \in X$, let n be the maximal integer with $x, y \in Q_n \in \mathcal{Q}_n$. By Theorem 6, we have $\mathrm{diam}\,(Q_n) \le 2c_2' b^n$. Then we get

$$d(x, y) \le \mathrm{diam}\,(Q_n) \le 2c_2' b^n = 2c_2' b^{-1} \kappa(\Pi^{-1}x, \Pi^{-1}y),$$

which derives $\dim_{\mathrm{H}} G \le \dim_{\mathrm{H}}^{\kappa}(\Pi^{-1}G)$.

Further, let $t > \dim_{\mathrm{H}} G$. For $\epsilon > 0$, there exists a δ-covering $\{U_i\}_{i\ge 1}$ of G which satisfies $\sum_{i\ge 1}(\mathrm{diam}\, U_i)^t < \epsilon$. For $i \ge 1$, let $n(i)$ be the integer with

$$2c_2' b^{n(i)} \le \mathrm{diam}\,(U_i) < 2c_2' b^{n(i)-1}. \tag{3.15}$$

Write

$$\mathcal{A}(U_i) = \{Q \in \mathcal{Q}_{n(i)} \colon Q \cap U_i \ne \varnothing\}.$$

Then using (3.15), there is a constant $0 < M_3 < \infty$ such that for all $i \ge 1$, $\#\mathcal{A}(U_i) \le M_3$. Note that

$$\Pi^{-1}G \subset \Pi^{-1}\Big(\bigcup_{i\ge 1} U_i\Big) \subset \Pi^{-1}\Big(\bigcup_{i\ge 1}\bigcup_{Q\in\mathcal{A}(U_i)} Q\Big).$$

Hence

$$\mathcal{H}_\delta^t(\Pi^{-1}G) \le \sum_{i\ge 1}\sum_{Q\in\mathcal{A}(U_i)} \kappa\Big((\Pi^{-1}(Q)\Big)^t \le \frac{M_3 b^t}{(2c_2')^t}\sum_{i\ge 1}(\mathrm{diam}\, U_i)^t < \epsilon. \tag{3.16}$$

The second inequality in (3.16) holds due to (3.15). Then we have $\mathcal{H}^t(\Pi^{-1}G) = 0$ which derives that $t \ge \dim_{\mathrm{H}}^{\kappa}(\Pi^{-1}G)$. Therefore $\dim_{\mathrm{H}} G \ge \dim_{\mathrm{H}}^{\kappa}(\Pi^{-1}G)$. This proves (3.14).

For $0 < p < 1$, let T be the tree defined above. *Percolation* at level p on T is obtained by removing each edge of T with probability $1-p$ and retaining it with probability p, with mutual independence among edges. The random graph connecting the root which is left will be denoted by $\widetilde{\Gamma}(p)$. Then the law of $\Gamma(p)$ given by $\mathbb{P}$ is the same as that of $\Pi(\widetilde{\Gamma}(p))$. By combining this with (3.14), we obtain the following analogue of the result of [30, p. 957] in our metric space setting. See also Lemma 5.1 in [33] for the case of $X = [0, 1]^N$.

Lemma 6 [30] *Let $p = 2^{-t} < 1$. For any analytic set $G \subset X$, the following statements hold.*

1. *If $\dim_{\mathrm{H}}(G) < t$, then $\Gamma_t \cap G = \varnothing$ almost surely.*
2. *If $\dim_{\mathrm{H}}(G) > t$, then $\Gamma_t \cap G \ne \varnothing$ with positive probability.*
3. *If $\dim_{\mathrm{H}}(G) > t$, then $\|\dim_{\mathrm{H}}(G \cap \Gamma_t)\|_\infty = \dim_{\mathrm{H}} G - t$, where the L^∞ norm is the essential supremum in the underlying probability space.*

We end this section with a proof of Lemma 5, by extending the method in [24] to the Ahlfors regular metric spaces.

Proof of Lemma 5 For $t < \dim_{\mathrm{H}}(G) - \gamma$, let Γ_t be a fractal percolation at level 2^{-t} in X [30,33], which is independent of A, and $\dim_{\mathrm{H}}(\Gamma_t) = s - t$ a.s. By Lemma 6, we have $\mathbb{P}(\Gamma_t \cap G \ne \varnothing) > 0$ whenever $\dim_{\mathrm{H}}(G) > t$, whereas $\mathbb{P}(\Gamma_t \cap G \ne \varnothing) = 0$ with $\dim_{\mathrm{H}}(G) < t$, and

$$\|\dim_{\mathrm{H}}(G \cap \Gamma_t)\|_\infty = \dim_{\mathrm{H}}(G) - t.$$

Let $\widehat{\Gamma}_t$ be a union of countably many independent and identically distributed copies of Γ_t. The Borel-Cantelli lemma implies that

$$\mathbb{P}(\widehat{\Gamma}_t \cap G \neq \varnothing) = \begin{cases} 0 & \text{if } \dim_{\mathrm{H}}(G) < t, \\ 1 & \text{if } \dim_{\mathrm{H}}(G) > t. \end{cases} \tag{3.17}$$

Also we have

$$\dim_{\mathrm{H}}(\widehat{\Gamma}_t \cap G) = \dim_{\mathrm{H}}(G) - t > \gamma, \quad \text{a.s.}$$

By the condition given in the lemma, we have $A \cap G \cap \widehat{\Gamma}_t \neq \varnothing$ a.s. in the product space. Using (3.17), we get $\dim_{\mathrm{H}}(A \cap G) \geq t$ a.s. Letting t tend to $\dim_{\mathrm{H}} G - \gamma$ along rational numbers, we complete our proof. □

4 Limsup random fractals in metric space

In order to prove Theorem 5, we first extend the results on hitting probabilities of limsup random fractals in [24] to metric spaces.

Let $\mathcal{Q} = \{\mathcal{Q}_n\}_{n \geq 0}$ be the collection of generalized dyadic cubes in (X, d) given in Sect. 3.1. To make sure the boundaries of sets in Q_n are covered, we will use 2Q, $Q \in Q_n$ instead of generalized dyadic cubes in this section. However, we still denote them by Q and Q_n respectively for simplicity of notation. For each $n \geq 1$, let $\{Z_n(Q), Q \in \mathcal{Q}_n\}$ be a collection of random variables, each taking values in $\{0, 1\}$.

Let

$$A(n) = \bigcup_{\substack{Q \in \mathcal{Q}_n, \\ Z_n(Q)=1}} Q^o,$$

where Q^o is the interior of Q. The random set

$$A = \limsup_{n \to \infty} A(n)$$

is called a limsup random fractal associated to $\{Z_n(Q), n \geq 1, Q \in \mathcal{Q}_n\}$.

We assume the following conditions (H1)–(H2), where (H1) is more general than Condition 4 in [24]. We allow the probability $P_n(Q)$ to depend not only on the level n, but also on the cubes $Q \in \mathcal{Q}_n$.

(H1) Suppose that for every $n \geq 1$, and $Q \in \mathcal{Q}_n$ the probability $P_n(Q) := \mathbb{P}(Z_n(Q) = 1)$ satisfies

$$\lim_{n \to \infty} \frac{\min_{Q \in \mathcal{Q}_n} \log_{b^{-1}} P_n(Q)}{n} = -\gamma_1,$$

and

$$\lim_{n \to \infty} \frac{\max_{Q \in \mathcal{Q}_n} \log_{b^{-1}} P_n(Q)}{n} = -\gamma_2,$$

where $\gamma_1, \gamma_2 > 0$ are constants.

Remark 7 From the proofs of Theorems 7, 8 and Corollary 3, we see that they still hold if the condition (H1) is replaced by the weaker condition (H1′):

(H1′) For some constants $\gamma_1, \gamma_2 > 0$,

$$\limsup_{n\to\infty} \frac{\min_{Q\in\mathcal{Q}_n} \log_{b^{-1}} P_n(Q)}{n} = -\gamma_1,$$
$$\limsup_{n\to\infty} \frac{\max_{Q\in\mathcal{Q}_n} \log_{b^{-1}} P_n(Q)}{n} = -\gamma_2,$$

and there exists an increasing sequence of positive integers $\{n_i\}$ with $\lim_{i\to\infty} \frac{n_{i+1}}{n_i} = 1$ such that

$$\lim_{i\to\infty} \frac{\min_{Q\in\mathcal{Q}_{n_i}} \log_{b^{-1}} P_{n_i}(Q)}{n_i} = -\gamma_1,$$

and

$$\lim_{i\to\infty} \frac{\max_{Q\in\mathcal{Q}_{n_i}} \log_{b^{-1}} P_{n_i}(Q)}{n_i} = -\gamma_2.$$

The next condition is concerned with the strength of dependence among the random variables $\{Z_n(Q),\ n \geq 1,\ Q \in \mathcal{Q}_n\}$. It is a slight modification of Condition 5 in [24].

(H2) A bound on correlation length: for any $\epsilon > 0$, define

$$f(n,\epsilon) = \max_{Q\in\mathcal{Q}_n} \#\{Q' \in \mathcal{Q}_n : \mathrm{Cov}(Z_n(Q), Z_n(Q')) \geq \epsilon P_n(Q) P_n(Q')\}.$$

Suppose that there is a constant $\delta \geq 0$ such that for all $\epsilon > 0$,

$$\limsup_{n\to\infty} \frac{1}{n} \log_{b^{-1}} f(n,\epsilon) \leq \delta. \tag{*}$$

The following theorem characterizes the hitting probability of the limsup random set A. It extends Theorem 3.1 in [24].

Theorem 7 *Assume that* $A = \limsup_{n\to\infty} A(n)$ *is a limsup random set that satisfies the conditions* $(H1)$ *and* $(H2)$. *Then for any analytic set* $G \subset X$,

$$\mathbb{P}(A \cap G \neq \varnothing) = \begin{cases} 0 & \textit{if } \dim_{\mathrm{P}}(G) < \gamma_2, \\ 1 & \textit{if } \dim_{\mathrm{P}}(G) > \gamma_1 + \delta. \end{cases}$$

For proving Theorem 7, we will use the following lemma on upper box dimension. From [32], for any $r > 0$ and any bounded set $G \subset X$, let $N_r(G)$ be the smallest number of closed balls with radius r covering G. Then

$$\overline{\dim}_{\mathrm{B}}(G) = \limsup_{r\to 0} \frac{\log N_r(G)}{-\log r}.$$

From it, we have the following lemma whose proof is standard and thus omitted.

Lemma 7 *Let* $\{k_i\}_{i\geq 1}$ *be an increasing sequence of positive integers satisfying* (2.2). *Then for any bounded set* $G \subset X$,

$$\overline{\dim}_{\mathrm{B}}(G) = \limsup_{i\to\infty} \frac{\log N_{a_1 r^{k_i}}(G)}{-\log(a_1 r^{k_i})}, \tag{4.1}$$

where $a_1 > 0$ *is a constant.*

Remark 8 For any bounded set $G \subset X$, consider $\mathcal{A}' = \{Q \in \mathcal{Q}_n : Q \cap G \neq \varnothing\}$, then $\#\mathcal{A}' \geq N_{2c_2' b^n}(G)$ which derives from

$$G \subset \bigcup_{Q \in \mathcal{A}'} Q \subset \bigcup_{x_Q \in Q \in \mathcal{A}'} B(x_Q, 2c_2' b^n),$$

where $x_Q \in Q$.

Proof of Theorem 7 The proof is a modification of that of Theorem 3.1 in [24]. We include it for the sake of completeness. Firstly, we show that $\dim_{\mathrm{P}}(G) < \gamma_2$ implies that $A \cap G = \varnothing$, a.s. It suffices to show that whenever $\overline{\dim}_{\mathrm{B}} G < \gamma_2$, then $A \cap G = \varnothing$ a.s. Fix an arbitrary but small $\eta > 0$ so that $\overline{\dim}_{\mathrm{B}}(G) < \gamma_2 - \eta$.

We denote by $\mathcal{C}_{b^n} = \mathcal{C}_{b^n}(G)$ a collection of the smallest number of the closed balls with radius b^n that cover the set G. Let $N_{b^n}(G) = \#\mathcal{C}_{b^n}$. For any $\theta \in (\overline{\dim}_{\mathrm{B}}(G), \gamma_2 - \eta)$, by Lemma 7, we have

$$\limsup_{n \to \infty} \frac{\log N_{b^n}(G)}{-\log(b^n)} = \overline{\dim}_{\mathrm{B}}(G) < \theta,$$

then there exists an integer $n(\theta) \geq 1$ such that

$$N_{b^n}(G) < b^{-n\theta} \tag{4.2}$$

for all $n \geq n(\theta)$. For any $W \in \mathcal{C}_{b^n}$, denote

$$\mathcal{A}(W) = \{Q \in \mathcal{Q}_n : Q \cap W \neq \varnothing\}.$$

Then we have

$$G \subset \bigcup_{W \in \mathcal{C}_{b^n}} \bigcup_{Q \in \mathcal{A}(W)} Q.$$

In the meanwhile, by Lemma 1, there is a constant $0 < M < \infty$, which does not depend on n, such that for all $W \in \mathcal{C}_{b^n}$, we have $\#\mathcal{A}(W) \leq M$.

On the other hand, by condition (H1), for any $\eta > 0$, there exists $n(\eta)$ such that for all $n \geq n(\eta)$,

$$\max_{Q \in \mathcal{Q}_n} P_n(Q) \leq b^{n(\gamma_2 - \eta)}. \tag{4.3}$$

It follows from (4.2) and (4.3) that for any $n \geq \max\{n(\theta), n(\eta)\}$,

$$\begin{aligned}
\mathbb{P}\big(G \cap A(n) \neq \varnothing\big) &\leq \mathbb{P}\Bigg(\bigcup_{W \in \mathcal{C}_{b^n}} \bigcup_{Q \in \mathcal{A}(W)} Q \cap A(n) \neq \varnothing\Bigg) \\
&\leq M b^{-n\theta} \max_{Q \in \mathcal{Q}_n} \mathbb{P}\big(Q \cap A(n) \neq \varnothing\big) \\
&= M b^{-n\theta} \max_{Q \in \mathcal{Q}_n} P_n(Q) \leq M b^{n(\gamma_2 - \eta - \theta)}.
\end{aligned}$$

Since $\theta < \gamma_2 - \eta$, then the series $\sum_{n \geq 1} \mathbb{P}\big(G \cap A(n) \neq \varnothing\big)$ is convergent. The Borel-Cantelli lemma implies that $G \cap A(n) = \varnothing$ a.s. for all n large enough. This shows that $A \cap G = \varnothing$ a.s.

In the following, we prove that if $\dim_{\mathrm{P}} G > \gamma_1 + \delta$, then $\mathbb{P}(A \cap G \neq \varnothing) = 1$. From Lemma 2 (2), we can find a compact subset $G_\star \subset G$ such that for all open sets V, whenever $G_\star \cap V \neq \varnothing$, then $\overline{\dim}_{\mathrm{B}}(G_\star \cap V) > \gamma_1 + \delta$.

Fix an open set V such that $V \cap G_\star \neq \varnothing$. Denote

$$\widetilde{\mathcal{A}}_n = \{Q \in \mathcal{Q}_n : Q^o \cap V \cap G_\star \neq \varnothing\}.$$

Let $\mathcal{N}_n$ be the total number of $\widetilde{\mathcal{A}}_n$. Then

$$G_\star \cap V \subset \bigcup_{Q \in \widetilde{\mathcal{A}}_n} Q \subset \bigcup_{Q \in \widetilde{\mathcal{A}}_n} B\big(x_Q, 2c_2' b^n\big),$$

where $x_Q \in Q$. Using Lemma 7, we have

$$\limsup_{n\to\infty} \frac{\log N_{2c_2' b^n}(V \cap G_\star)}{-\log(2c_2' b^n)} = \overline{\dim}_{\mathrm{B}}(V \cap G_\star) > \gamma_1 + \delta.$$

For any $\eta \in (\gamma_1 + \delta, \overline{\dim}_{\mathrm{B}}(V \cap G_\star))$, we derive that

$$N_{2c_2' b^n}(V \cap G_\star) \geq (2c_2')^{-\eta} b^{-n\eta}$$

holds for infinitely many n. Hence $\mathcal{N}_n \geq (2c_2')^{-\eta} b^{-n\eta}$ for infinitely many n. This implies the set

$$\mathfrak{N} := \{i \geq 1 : \mathcal{N}_{n_i} \geq (2c_2')^{-\eta} b^{-n_i\eta}\} \tag{4.4}$$

satisfies $\#\mathfrak{N} = \infty$.We define

$$S_i := \sum_{Q \in \widetilde{\mathcal{A}}_{n_i}} Z_{n_i}(Q),$$

where S_i is the total number of sets $Q \in \mathcal{Q}_{n_i}$ with $Q \cap V \cap G_\star \cap A(n_i) \neq \varnothing$. Observe that

$$\{A(n) \cap G_\star \cap V \neq \varnothing \ \text{ i.o.}\} \supset \{S_i > 0 \ \text{ i.o.}\}.$$

We need only show that $\mathbb{P}(S_i > 0 \ \text{ i.o.}) = 1$. Firstly, we estimate

$$\mathrm{Var}(S_i) = \sum_{Q \in \widetilde{\mathcal{A}}_{n_i}} \sum_{Q' \in \widetilde{\mathcal{A}}_{n_i}} \mathrm{Cov}(Z_{n_i}(Q), Z_{n_i}(Q')).$$

Fix $\epsilon > 0$ and for each $Q \in \mathcal{Q}_{n_i}$, let $\mathcal{G}_{n_i}(Q)$ denote the collection of all $Q' \in \mathcal{Q}_{n_i}$ such that

1. $Q' \cap V \cap G_\star \neq \varnothing$;
2. $\mathrm{Cov}(Z_{n_i}(Q), Z_{n_i}(Q')) \leq \epsilon P_{n_i}(Q) P_{n_i}(Q')$.

That is

$$\mathcal{G}_{n_i}(Q) = \{Q' \in \widetilde{\mathcal{A}}_{n_i} : \mathrm{Cov}(Z_{n_i}(Q), Z_{n_i}(Q')) \leq \epsilon P_{n_i}(Q) P_{n_i}(Q')\}.$$

If $Q' \in \mathcal{Q}_{n_i}$ satisfies (1) but not (2), then we say $\mathcal{B}_{n_i}(Q)$,

$$\mathcal{B}_{n_i}(Q) = \{Q' \in \widetilde{\mathcal{A}}_{n_i} : \mathrm{Cov}(Z_{n_i}(Q), Z_{n_i}(Q')) > \epsilon P_{n_i}(Q) P_{n_i}(Q')\}.$$

Then

$$\mathrm{Var}(S_i) \leq \epsilon \Bigg(\sum_{Q \in \widetilde{\mathcal{A}}_{n_i}} P_{n_i}(Q) \Bigg)^2 + \Bigg(\max_{Q \in \mathcal{Q}_{n_i}} \#\mathcal{B}_{n_i}(Q) \Bigg) \Bigg(\sum_{Q \in \widetilde{\mathcal{A}}_{n_i}} P_{n_i}(Q) \Bigg). \tag{4.5}$$

The last term follows from the fact that $\mathrm{Cov}(Z_n(Q), Z_n(Q')) \leq \mathbb{E}(Z_n(Q)) = P_n(Q)$. Recalling the notation of (H2), we have

$$\mathrm{Var}(S_i) \leq \epsilon \Bigg(\sum_{Q \in \widetilde{\mathcal{A}}_{n_i}} P_{n_i}(Q) \Bigg)^2 + f(n_i, \epsilon) \Bigg(\sum_{Q \in \widetilde{\mathcal{A}}_{n_i}} P_{n_i}(Q) \Bigg).$$

Combining this with the Paley–Zygmund inequality, we obtain

$$\begin{aligned}\mathbb{P}(S_i > 0) &\geq \frac{\big(\mathbb{E}(S_i)\big)^2}{\mathbb{E}(S_i^2)} = \frac{1}{1 + \frac{\mathrm{Var}(S_i)}{(\mathbb{E}(S_i))^2}} \\ &\geq \frac{1}{1 + \epsilon + \frac{f(n_i,\epsilon)}{\sum_{Q\in\widetilde{\mathcal{A}}_{n_i}} P_{n_i}(Q)}},\end{aligned} \tag{4.6}$$

since $\mathbb{E}(S_i) = \sum_{Q\in\widetilde{\mathcal{A}}_{n_i}} P_{n_i}(Q)$. By the conditions (H1) and (H2), for any $\theta > 0$ with $2\theta < \eta - \delta - \gamma_1$, there exists N such that for all $n \geq N$, we have

$$f(n,\epsilon) \leq b^{-(\delta+\theta)n} \quad \text{and} \quad \min_{Q\in\mathcal{Q}_n} P_n(Q) \geq b^{(\gamma_1+\theta)n}.$$

Thus from (4.4) and arbitrariness of θ, we have

$$\limsup_{\substack{i\,\in\,\mathfrak{N} \\ i\,\to\,\infty}} \frac{f(n_i,\epsilon)}{\sum_{Q\in\widetilde{\mathcal{A}}_{n_i}} P_{n_i}(Q)} \leq \limsup_{\substack{i\,\in\,\mathfrak{N} \\ i\,\to\,\infty}} (2c_2')^{-\eta} b^{-n_i(2\theta+\delta+\gamma_1-\eta)} = 0. \tag{4.7}$$

By inequalities (4.6), (4.7), and Fatou's lemma, we derive that

$$\mathbb{P}(S_i > 0 \ \text{ i.o.}) \geq \limsup_{\substack{i\in\mathfrak{N} \\ i\to\infty}} \mathbb{P}(S_i > 0) = 1.$$

Define the open set $B(n) := \bigcup_{k=n}^{\infty} A(k)$ for $n \geq 1$. It follows that

$$\mathbb{P}(B(n) \cap G_\star \cap V \neq \varnothing,\ \forall n \geq 1) = 1$$

for every open set V with $G_\star \cap V \neq \varnothing$. Since compact metric spaces are separated, then there exists a countable basis for open sets of (X, d). Letting V run over the countable basis, we obtain that for all $n \geq 1$, the set $B(n) \cap G_\star$ is a.s. dense in $G_\star$. Since $G_\star$ is a complete metric space, by Baire's category theorem, we have $\cap_{n=1}^{\infty} B(n) \cap G_\star$ is a.s. dense in $G_\star$. In particular, $A \cap G_\star \neq \varnothing$ with probability one. □

For $n \geq 1$, let $\mathcal{B}_n = \{B(x_{n,i}, 2c_2' b^n) : i \in \mathbb{N}_n\}$ where $x_{n,i}, c_2'$ are given in Sect. 3.1, and $\mathcal{B} = \{\mathcal{B}_n\}_{n\geq 0}$. For each $n \geq 1$, let $\{Z_n(B), B \in \mathcal{B}_n\}$ be a collection of random variables, each taking values in $\{0, 1\}$. Let

$$F(n) = \bigcup_{\substack{B\in\mathcal{B}_n, \\ Z_n(B)=1}} B^o.$$

Define the random set

$$F = \limsup_{n\to\infty} F(n).$$

Theorem 8 *Assume that F satisfies the corresponding conditions $(H1)$ and $(H2)$. Then for any analytic set $G \subset X$, if $\dim_{\mathrm{P}}(G) > \gamma_1 + \delta$, we have*

$$\mathbb{P}(F \cap G \neq \varnothing) = 1.$$

Proof By replacing the generalized dyadic cubes $\mathcal{Q}$ by balls in $\mathcal{B}$, we obtain the theorem directly from the proof of Theorem 7. □

Corollary 3 *Suppose A is a discrete limsup random fractal satisfying conditions (H1) and (H2) with $\delta = 0$. Then for any analytic set $G \subset X$, with probability one,*

$$\dim_{\mathrm{H}}(A \cap G) \begin{cases} \leq \dim_{\mathrm{P}}(G) - \gamma_2 & \textit{if } \dim_{\mathrm{P}}(G) \geq \gamma_2, \\ = -\infty & \textit{if } \dim_{\mathrm{P}}(G) < \gamma_2, \\ \geq \dim_{\mathrm{H}}(G) - \gamma_1 & \textit{if } \dim_{\mathrm{H}}(G) > \gamma_1. \end{cases}$$

In particular, if $\gamma_1 = \gamma_2 < s$, then $\dim_{\mathrm{H}}(A) = s - \gamma_1$, a.s.

Proof It suffices to prove $\dim_{\mathrm{H}}(A \cap G) \leq \overline{\dim}_{\mathrm{B}} G - \gamma_2$ a.s., if $\dim_{\mathrm{P}}(G) \geq \gamma_2$. Let $\mathcal{C}_{b^n}$ and $\mathcal{A}(W)$, $W \in \mathcal{C}_{b^n}$ be the same as described in the proof of Theorem 7. Define

$$S_n = \sum_{W \in \mathcal{C}_{b^n}} \sum_{Q \in \mathcal{A}(W)} Z_n(Q).$$

Let $\xi = \overline{\dim}_{\mathrm{B}} G - \gamma_2$, then for n large enough, we have

$$\mathbb{E}(S_n) = \sum_{W \in \mathcal{C}_{b^n}} \sum_{Q \in \mathcal{A}(W)} P_n(Q) \leq M b^{-n(\xi + 2\epsilon)},$$

where $0 < M < \infty$ is a constant independent of n. Hence from the arbitrariness of ϵ, it follows that for any $\theta > \xi$, $\mathbb{E}(\sum_n S_n b^{\theta n}) < \infty$.

Hence

$$\mathcal{H}^{\theta}(A \cap G) \leq \sum_{n=m}^{\infty} S_n b^{n\theta} < \infty \quad \text{a.s.}$$

which derives that $\dim_{\mathrm{H}}(A \cap G) \leq \dim_{\mathrm{P}} G - \gamma_2$ a.s.

Finally, the the lower bound of $\dim_{\mathrm{H}}(A \cap G)$ follows from Lemma 5. The proof is complete. □

5 Proof of Theorem 5

Proof of Theorem 5 The proof is divided into three parts. In order to show that $\dim_{\mathrm{P}}(G) > s - \alpha$ implies $\mathbb{P}(E \cap G \neq \varnothing) = 1$, we will use the hitting probability of limsup random fractals in Sect. 4. This is done in Parts (i) and (ii). Part (iii) determines the packing dimension of $E \cap G$.

(i) Construction of a limsup random fractal $E_\star \subset E$.

Firstly, we recall some notations from Sect. 4. For any $k \geq 2$, $\mathcal{B}_k = \{B(x_{k,i}, 2c_2' b^k) : i \in \mathbb{N}_k \subset \mathbb{N}\}$, $\mathfrak{I}_k = \{n \geq 1 : \ell_n \in [b^{k-1}, b^{k-2})\}$, $n_k = \#\mathfrak{I}_k$ and $\mathfrak{I}_k'$ is a maximal collection of points in $\mathfrak{I}_k$ having mutual distances at least ck, where c is a given constant with $c > \frac{\alpha - s}{\log_{b^{-1}} \rho}$, and ρ appears in Definition 3. In this way, any pair of integers $n, m \in \mathfrak{I}_k'$ are at least of distance ck from each other. Also, recall that $m_k = \#\mathfrak{I}_k' = \lceil (ck)^{-1} n_k \rceil$.

For every $J \in \mathcal{B}_k$, define

$$Z_k(J) = \begin{cases} 1 & \text{if } \exists\, n \in \mathfrak{I}_k' \text{ such that } J \subset I_n = B(\xi_n, \ell_n), \\ 0 & \text{otherwise.} \end{cases}$$

Let $A(k)$ be the union of the interiors of sets in $\mathcal{B}_k$ that are contained in some I_n in $\mathfrak{I}_k'$ with length $\ell_n \in [b^{k-1}, b^{k-2})$, that is

$$A(k) = \bigcup_{\substack{J \in \mathcal{B}_k \\ Z_k(J) = 1}} J^o.$$

We observe that

$$A(k) \subset \bigcup_{n \in \mathfrak{I}'_k} I_n.$$

Define $E_\star := \limsup_{k \to \infty} A(k)$. From the above, we have $E_\star \subset E$.

(ii) Hitting probability of $E_\star$.

Now let $G \subset X$ be an analytic set such that $\dim_{\mathrm{P}}(G) > s - \alpha$. We show $\mathbb{P}(E_\star \cap G \neq \varnothing) = 1$.

For every $J \in \mathcal{B}_k$, the probability

$$\mathbb{P}(Z_k(J) = 1) = \mathbb{P}\{\exists\, n \in \mathfrak{I}'_k \text{ such that } J \subset I_n\}.$$

Denote the above probability by $P_k(J)$.

Write $J = B(x_J, 2c'_2 b^k)$. By using the stationarity on $\{\xi_n\}$, we derive that,

$$\begin{aligned} P_k(J) &\le \sum_{n \in \mathfrak{I}'_k} \mathbb{P}(J \subset I_n) = \sum_{n \in \mathfrak{I}'_k} \mathbb{P}(\xi_n \in B(x_J, \ell_n - 2c'_2 b^k)) \\ &\le c_1 \sum_{n \in \mathfrak{I}'_k} (\ell_n - 2c'_2 b^k)^s \le c_3 m_k b^{ks}, \end{aligned} \tag{5.1}$$

where $c_3 = c_1(\frac{1}{b^2} - 2c'_2)^s$ is a constant.

On the other hand,

$$P_k(J) \ge \sum_{n \in \mathfrak{I}'_k} \mathbb{P}(J \subset I_n) - \sum_{n \in \mathfrak{I}'_k} \sum_{\substack{m \in \mathfrak{I}'_k \\ m \neq n}} \mathbb{P}(J \subset I_n, J \subset I_m). \tag{5.2}$$

Since $\{\xi_n\}_{n \ge 1}$ is stationary and exponentially mixing, if $n > m$, we have

$$\mathbb{P}(J \subset I_n, J \subset I_m) \le \mathbb{P}(J \subset I_m)\mathbb{P}(J \subset I_n) + C\rho^{n-m}\mathbb{P}(J \subset I_n),$$

where C, ρ are constants. We notice that $n, m \in \mathfrak{I}'_k$ derives $n - m \ge ck$, then

$$\begin{aligned} &\sum_{n \in \mathfrak{I}'_k} \sum_{\substack{m \in \mathfrak{I}'_k \\ m \neq n}} \mathbb{P}(J \subset I_n, J \subset I_m) \\ &\le \Bigg(\sum_{n \in \mathfrak{I}'_k} \mathbb{P}(J \subset I_n)\Bigg)\Bigg(\sum_{\substack{m \in \mathfrak{I}'_k \\ m \neq n}} \mathbb{P}(J \subset I_m)\Bigg) + 2C \sum_{n \in \mathfrak{I}'_k} \sum_{\substack{m \in \mathfrak{I}'_k \\ m < n}} \rho^{n-m}\mathbb{P}(J \subset I_n) \\ &\le \Bigg(\sum_{n \in \mathfrak{I}'_k} \mathbb{P}(J \subset I_n)\Bigg)\Bigg(\sum_{\substack{m \in \mathfrak{I}'_k \\ m \neq n}} \mathbb{P}(J \subset I_m)\Bigg) + \frac{2C\rho^{ck}}{1 - \rho^{ck}}\Bigg(\sum_{n \in \mathfrak{I}'_k} \mathbb{P}(J \subset I_n)\Bigg). \end{aligned} \tag{5.3}$$

It follows from (5.2) and (5.3) that

$$\begin{aligned} P_k(J) &\ge \Bigg(\sum_{n \in \mathfrak{I}'_k} \mathbb{P}(J \subset I_n)\Bigg)\Bigg(1 - \sum_{\substack{m \in \mathfrak{I}'_k \\ m \neq n}} \mathbb{P}(J \subset I_m) - \frac{2C\rho^{ck}}{1 - \rho^{ck}}\Bigg) \\ &\ge c_4 m_k b^{ks}\Big(1 - c_3 m_k b^{ks} - \frac{2C\rho^{ck}}{1 - \rho^{ck}}\Big), \end{aligned} \tag{5.4}$$

where $c_4 = c_1^{-1}(\frac{1}{b} - 2c_2')^s$ is a constant. Combining (5.1) and (5.4), together with the condition (C), we derive that

$$\limsup_{k\to\infty} \frac{\max_{J\in\mathcal{Q}_k} \log_{b^{-1}} P_k(J)}{k} = \limsup_{k\to\infty} \frac{\min_{J\in\mathcal{Q}_k} \log_{b^{-1}} P_k(J)}{k} = \alpha - s, \tag{5.5}$$

and there is an increasing sequence of integers $\{k_i\}$ that satisfies (2.2) such that

$$\lim_{i\to\infty} \frac{\max_{J\in\mathcal{Q}_{k_i}} \log_{b^{-1}} P_{k_i}(J)}{k_i} = \lim_{i\to\infty} \frac{\min_{J\in\mathcal{Q}_{k_i}} \log_{b^{-1}} P_{k_i}(J)}{k_i} = \alpha - s. \tag{5.6}$$

Next we verify that there is a bound on correlation length.

First we estimate $\mathrm{Cov}(Z_k(J_1)Z_k(J_2))$, where $J_1, J_2 \in \mathcal{B}_k$ with $d(J_1, J_2) \geq b^{k-3}$. From (5.3), we have

$$\begin{aligned}
&\mathbb{E}(Z_k(J_1)Z_k(J_2)) = \mathbb{P}(Z_k(J_1) = 1, Z_k(J_2) = 1)\\
&\quad = \mathbb{P}\{\exists\, m, n \in \mathfrak{I}_k' \text{ such that } J_1 \subset I_n \text{ and } J_2 \subset I_m\}\\
&\quad \leq \sum_{n\in\mathfrak{I}_k'} \sum_{\substack{m\in\mathfrak{I}_k'\\ m\neq n}} \mathbb{P}(J_1 \subset I_n, J_2 \subset I_m)\\
&\quad \leq \sum_{n\in\mathfrak{I}_k'} \mathbb{P}(J_1 \subset I_n) \sum_{\substack{m\in\mathfrak{I}_k'\\ m\neq n}} \mathbb{P}(J_2 \subset I_m) + \frac{2C\rho^{ck}}{1-\rho^{ck}} \sum_{n\in\mathfrak{I}_k'} \mathbb{P}(J_1 \subset I_n).
\end{aligned} \tag{5.7}$$

Since

$$\begin{aligned}
\mathrm{Cov}(Z_k(J_1), Z_k(J_2)) &= \mathbb{E}(Z_k(J_1)Z_k(J_2)) - \mathbb{E}(Z_k(J_1))\mathbb{E}(Z_k(J_2))\\
&= \mathbb{E}(Z_k(J_1)Z_k(J_2)) - P_k(J_1)P_k(J_2),
\end{aligned} \tag{5.8}$$

from the first inequality in (5.4) and (5.7), we get

$$\begin{aligned}
&\mathrm{Cov}(Z_k(J_1), Z_k(J_2))\\
&\leq \Bigg(\sum_{n\in\mathfrak{I}_k'} \mathbb{P}(J_1 \subset I_n)\Bigg)\Bigg(\sum_{m\in\mathfrak{I}_k'} \mathbb{P}(J_2 \subset I_m)\Bigg)\Bigg(\sum_{\substack{n\in\mathfrak{I}_k'\\ n\neq m}} \mathbb{P}(J_1 \subset I_n) + \sum_{\substack{m\in\mathfrak{I}_k'\\ m\neq n}} \mathbb{P}(J_2 \subset I_m) + \frac{4C\rho^{ck}}{1-\rho^{ck}}\Bigg)\\
&\quad + \frac{2C\rho^{ck}}{1-\rho^{ck}}\Bigg(\sum_{n\in\mathfrak{I}_k'} \mathbb{P}(J_1 \subset I_n)\Bigg)\\
&\leq \Bigg(\sum_{n\in\mathfrak{I}_k'} \mathbb{P}(J_1 \subset I_n)\Bigg)\Bigg(\sum_{m\in\mathfrak{I}_k'} \mathbb{P}(J_2 \subset I_m)\Bigg)\Bigg(\sum_{\substack{n\in\mathfrak{I}_k'\\ n\neq m}} \mathbb{P}(J_1 \subset I_n) + \sum_{\substack{m\in\mathfrak{I}_k'\\ m\neq n}} \mathbb{P}(J_2 \subset I_m) + \frac{4C\rho^{ck}}{1-\rho^{ck}}\\
&\quad + \frac{2C\rho^{ck}}{(1-\rho^{ck})\sum_{m\in\mathfrak{I}_k'} \mathbb{P}(J_2 \subset I_m)}\Bigg).
\end{aligned} \tag{5.9}$$

We notice that from (5.4) there exists a constant $0 < M_4 < \infty$ such that

$$\mathbb{E}(Z_k(J)) = P_k(J) \geq M_4 \sum_{n\in\mathfrak{I}_k'} \mathbb{P}(J \subset I_n)$$

holds for all $J \in \mathcal{B}_k$ and all large enough k. Then the inequality (5.9) reads as follows

$$\begin{aligned}&\mathrm{Cov}(Z_k(J_1), Z_k(J_2))\\ &\leq \frac{1}{M_4^2}\mathbb{E}(Z_k(J_1))\mathbb{E}(Z_k(J_2))\left(2c_3 m_k b^{ks} + \frac{4C\rho^{ck}}{1-\rho^{ck}} + \frac{2C\rho^{ck}}{c_4(1-\rho^{ck})m_k b^{ks}}\right).\end{aligned}$$

Here choosing k_i in (5.6), then $\lim_{i\to\infty} m_{k_i} b^{k_i s} = 0$, and

$$\lim_{i\to\infty} \frac{\rho^{ck_i}}{m_{k_i} b^{k_i s}} \leq \lim_{i\to\infty} \frac{ck_i\rho^{ck_i}}{n_{k_i} b^{sk_i}} = \lim_{i\to\infty} \frac{ck_i \rho^{ck_i}}{b^{(s-\alpha)k_i}} = 0,$$

due to $\rho^c < b^{s-\alpha}$. We derive from the equalities above that for any $\epsilon > 0$,

$$\mathrm{Cov}\Big(Z_{k_i}(J_1), Z_{k_i}(J_2)\Big) < \epsilon \mathbb{E}(Z_{k_i}(J_1))\mathbb{E}(Z_{k_i}(J_2))$$

holds for i large enough. From Lemma 1, this implies that there is a constant $0 < M_5 < \infty$ independent of k_i such that $f(k_i, \epsilon) \leq M_5$, and recall

$$f(k,\epsilon) = \max_{J_2\in\mathcal{B}_k} \#\Big\{J_1 \in \mathcal{B}_k : \mathrm{Cov}\Big(Z_k(J_1), Z_k(J_2)\Big) \geq \epsilon\mathbb{E}(Z_k(J_1))\mathbb{E}(Z_k(J_2))\Big\}.$$

In particular,

$$\lim_{i\to\infty} \frac{\log_{b^{-1}} f(k_i,\epsilon)}{k_i} = 0.$$

Thus we have shown that the condition (H2) in Sect. 4 is satisfied with $\delta = 0$.

Now that we have verified that $E_\star$ satisfies the conditions (H1′) and (H2) in Sect. 4, we apply Theorem 8 to conclude $E_\star \cap G \neq \varnothing$ a.s., which yields $\mathbb{P}(E \cap G \neq \varnothing) = 1$.

(iii) The packing dimension of $E \cap G$.

On the same probability space, let E' be a random covering set that is independent of E and is associated with $\{\xi'_n\}$ and $\{\ell'_n\}$, where $\{\xi'_n\}$ is an exponentially mixing stationary process, and $\{\ell'_n\}$ satisfies the condition (C) with the Besicovitch–Taylor index $\alpha' < s$.

Let the compact set $G_\star$ and the open sets $A(k)$ be as described in the proofs of Theorems 7 and 5, respectively. Let $\{A'(k)\}$ be the sequence of open sets corresponding to E'. If $\dim_{\mathrm{P}} G > s - \min\{\alpha, \alpha'\}$, by the first part of Theorem 5, we have

$$\mathbb{P}\bigg(\Big(\bigcup_{k=n}^{\infty} A(k)\Big) \cap V \cap G_\star \neq \varnothing, \forall n \geq 1\bigg) = \mathbb{P}\bigg(\Big(\bigcup_{k=n}^{\infty} A'(k)\Big) \cap V \cap G_\star \neq \varnothing, \forall n \geq 1\bigg) = 1$$

for all open set V satisfying $V \cap G_\star \neq \varnothing$. By independence, we have

$$\mathbb{P}\bigg(\Big(\bigcup_{k=n}^{\infty} A(k)\Big) \cap V \cap G_\star \neq \varnothing,\ \Big(\bigcup_{k=n}^{\infty} A'(k)\Big) \cap V \cap G_\star \neq \varnothing,\ \forall n \geq 1\bigg) = 1.$$

Let V run over a countable basis of X, then $\big\{\bigcup_{k=n}^{\infty} A(k) \cap G_\star\big\}_{n\geq 1} \cup \big\{\bigcup_{k=n}^{\infty} A'(k) \cap G_\star\big\}_{n\geq 1}$ is a countable collection of open, dense subsets of the complete metric space $G_\star$. Baire's category theorem implies that

$$\mathbb{P}(E \cap E' \cap G_\star \text{ is dense in } G_\star) = 1.$$

In particular, $E \cap E' \cap G_\star \neq \varnothing$ a.s. Hence $\mathbb{P}(E \cap E' \cap G \neq \varnothing) = 1$.

Now we consider the random covering set E', and regard $E \cap G$ as the target set. By Theorem 3, we must have $\dim_{\mathrm{P}}(E \cap G) \geq s - \alpha'$ a.s. Hence we have proved that $\dim_{\mathrm{P}}(G) >$

$s - \min\{\alpha, \alpha'\}$ implies $\dim_P(E \cap G) \geq s - \alpha'$ a.s. Consequently, if $\dim_P(G) > s - \alpha$, then for $\alpha' \in (s - \dim_P G, \alpha)$ we have $\dim_P(E \cap G) \geq s - \alpha'$ a.s. Letting α' tend to $s - \dim_P G$ along rational numbers, we get

$$\dim_P(G \cap E) \geq \dim_P G, \quad \text{a.s.}$$

Hence $\dim_P(G \cap E) = \dim_P G$ a.s. This finishes the proof of Theorem 5. □

6 Applications

In this section we present some dynamical systems which satisfy all conditions of our results.

6.1 Continued fraction dynamical system

Let T_G be the Gauss map on $(0, 1]$. The Gauss measure μ on $(0, 1]$ are given by

$$d\mu = \frac{1}{\log 2} \frac{dx}{(1 + x)}.$$

From [37], T_G preserves the Gauss measure μ which is equivalent to the Lebesgue measure $\mathcal{L}$. By Philipp [35], the system $((0, 1], T_G)$ is exponentially mixing with respect to the Gauss measure μ. Hence our results are applicable to the continued fraction dynamical system. For a given point $x \in (0, 1]$, let

$$E_G(x) = \{y \in (0, 1] \colon y \in B(T_G^n x, \ell_n) \text{ i.o.}\}.$$

Theorem 9 *Let μ be the Gauss measure. For any analytic set $G \subset (0, 1]$ we have, for μ-a.e. $x \in (0, 1]$*

$$E_G(x) \cap G = \varnothing \text{ if } \dim_P(G) < 1 - \alpha,$$
$$E_G(x) \cap G \neq \varnothing \text{ if } \dim_H(G) > 1 - \alpha.$$

Furthermore under the condition (C), if $\dim_P(G) > 1 - \alpha$,

$$E_G(x) \cap G \neq \varnothing \quad \text{a.e.}$$

Theorem 10 *For any analytic set $G \subset (0, 1]$ we have a.e.*

$$\dim_H(E_G(x) \cap G) \begin{cases} \leq \dim_P(G) + \alpha - 1 & \text{if } \dim_P(G) \geq 1 - \alpha, \\ = -\infty & \text{if } \dim_P(G) < 1 - \alpha, \\ \geq \dim_H(G) + \alpha - 1 & \text{if } \dim_H(G) > 1 - \alpha. \end{cases}$$

Moreover, if $\dim_P(G) > 1 - \alpha$ *and the condition (C) is satisfied, then* $\dim_P(E_G(x) \cap G) = \dim_P(G)$ a.e.

6.2 The β-dynamical system

For a real number $\beta > 1$, define the transformation $T_\beta : [0, 1] \to [0, 1]$ by

$$T_\beta : x \mapsto \beta x \bmod 1.$$

Let μ be the Parry measure with the density

$$h(x) = \left(\int_0^1 \sum_{n:T^n 1 < x} \frac{1}{\beta^n} dx\right)^{-1} \sum_{n:T^n 1 < x} \frac{1}{\beta^n}.$$

It was shown by [36] that the Parry measure μ is invariant under T_β and equivalent to the Lebesgue measure $\mathcal{L}$. Hence μ is Ahlfors regular. From Philipp [35], T_β is exponentially mixing. Combining these, we obtain that the β-dynamical system $([0, 1], T_\beta)$ satisfies all the conditions stated in our results. For a given $x \in [0, 1]$, define the dynamical covering set

$$E_\beta(x) = \{y \in [0, 1] \colon y \in B(T_\beta^n x, \ell_n) \text{ i.o.}\}.$$

Theorem 11 *Let* $([0, 1], T_\beta)$ *be the* β*-dynamical system endowed with the Parry measure* μ*. Then for any analytic set* $G \subset [0, 1]$ *we have*

$$\begin{aligned} E_\beta(x) \cap G = \varnothing \quad &\text{if } \dim_{\mathrm{P}}(G) < 1 - \alpha, \\ E_\beta(x) \cap G \neq \varnothing \quad &\text{if } \dim_{\mathrm{H}}(G) > 1 - \alpha \end{aligned}$$

for μ-a.e. $x \in [0, 1]$. *Under the corresponding condition (C), if* $\dim_{\mathrm{P}}(G) > 1 - \alpha$*,*

$$E_\beta(x) \cap G \neq \varnothing \quad \text{a.e.}$$

Theorem 12 *Let* $([0, 1], T_\beta)$ *be the* β*-dynamical system endowed with the Parry measure* μ*. For any analytic set* $G \subset [0, 1]$ *we have a.e.*

$$\dim_{\mathrm{H}}(E_\beta(x) \cap G) \begin{cases} \le \dim_{\mathrm{P}}(G) + \alpha - 1 & \textit{if } \dim_{\mathrm{P}}(G) \ge 1 - \alpha, \\ = -\infty & \textit{if } \dim_{\mathrm{P}}(G) < 1 - \alpha, \\ \ge \dim_{\mathrm{H}}(G) + \alpha - 1 & \textit{if } \dim_{\mathrm{H}}(G) > 1 - \alpha. \end{cases}$$

Moreover, if $\dim_{\mathrm{P}}(G) > 1 - \alpha$ *and the condition (C) is satisfied, then* $\dim_{\mathrm{P}}(E_\beta(x) \cap G) = \dim_{\mathrm{P}}(G)$ a.e.

6.3 The middle-third Cantor set

Our results are applicable to the middle-third Cantor set $C_{1/3}$. In fact, the results also hold for a range of homogeneous self-similar sets satisfying the open set condition.

Let $T_3 x = 3x \pmod 1$ be the natural map on $C_{1/3}$, and μ be the standard Cantor measure. Let $\gamma = \log_3 2$ be the Hausdorff dimension of $C_{1/3}$. From Lemma 3.2 in [40], μ is exponentially mixing. Also μ is Ahlfors γ-regular. Then all the conditions are fulfilled for Theorem 1.1 and 1.2. Define the dynamical covering set

$$E_3(x) = \{y \in C_{1/3} \colon y \in B(T_3^n x, \ell_n) \ \text{ i.o.}\},$$

where $x \in C_{1/3}$ is a given point.

Theorem 13 *Let* μ *be the standard Cantor measure. Then for any analytic set* $G \subset C_{1/3}$*, we have*

$$\begin{aligned} E_3(x) \cap G = \varnothing \text{ if } \dim_{\mathrm{P}}(G) < \gamma - \alpha, \\ E_3(x) \cap G \neq \varnothing \text{ if } \dim_{\mathrm{H}}(G) > \gamma - \alpha \end{aligned}$$

for μ-a.e. $x \in C_{1/3}$. *Under the corresponding condition (C), if* $\dim_{\mathrm{P}}(G) > \gamma - \alpha$*,* $E_3(x) \cap G \neq \varnothing$ *holds for* μ-a.e. $x \in C_{1/3}$.

Theorem 14 *For any analytic set $G \subset C_{1/3}$ we have a.e.*

$$\dim_{\mathrm{H}}(E_3(x) \cap G) \begin{cases} \leq \dim_{\mathrm{P}}(G) + \alpha - \gamma & \text{if } \dim_{\mathrm{P}}(G) \geq \gamma - \alpha, \\ = -\infty & \text{if } \dim_{\mathrm{P}}(G) < \gamma - \alpha, \\ \geq \dim_{\mathrm{H}}(G) + \alpha - \gamma & \text{if } \dim_{\mathrm{H}}(G) > \gamma - \alpha. \end{cases}$$

Moreover, if $\dim_{\mathrm{P}}(G) > \gamma - \alpha$ *and the condition (C) is satisfied, then* $\dim_{\mathrm{P}}(E_3(x) \cap G) = \dim_{\mathrm{P}}(G)$ a.e.

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 Communications in Theoretical Physics

Commun. Theor. Phys. 73 (2021) 105001 (13pp) https://doi.org/10.1088/1572-9494/ac08fb

Bounded multi-soliton solutions and their asymptotic analysis for the reversal-time nonlocal nonlinear Schrödinger equation

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Received 6 April 2021, revised 13 May 2021
Accepted for publication 8 June 2021
Published 16 August 2021

Abstract

In this paper, we construct the Darboux transformation (DT) for the reverse-time integrable nonlocal nonlinear Schrödinger equation by loop group method. Then we utilize the DT to derive soliton solutions with zero seed. We investigate the dynamical properties for those solutions and present a sufficient condition for the non-singularity of multi-soliton solutions. Furthermore, the asymptotic analysis of bounded multi-solutions has also been established by the determinant formula.

Keywords: nonlocal nonlinear Schrödinger equation, multi-soliton solution, singularity, asymptotic analysis

(Some figures may appear in colour only in the online journal)

1. Introduction

Integrable nonlocal nonlinear Schrödinger equations (nNLSE) play a vital role in mathematical physics [1–8] and have been studied extensively [9–15]. This type of equation is symmetric because it is invariant under the joint transformation $x \to -x$, $t \to -t$ and complex conjugation. For a AKNS spectral problem, the reduction $r(x, t) = \sigma q(x, t)^* \sigma = \pm 1$ was thought to be the only interesting one until 2013. However, Ablowitz and Musslimani [9] discovered that there existed another interesting reduction $r(x, t) = \sigma q^*(-x, t)$ which results in a new form of integrable nNLSE:

$$\mathrm{i}q_t(x, t) = q_{xx}(x, t) - 2\sigma q^2(x, t) q^*(-x, t).$$

It was not long before they found two more reductions of the AKNS scattering problem: $r(x, t) = \sigma q(-x, -t)$ and $r(x, t) = \sigma q(x, -t)$, which gave rise to the so-called reverse-space–time NLS and reverse-time NLS equations, respectively reported by [16]

$$\mathrm{i}q_t(x, t) = q_{xx}(x, t) - 2\sigma q^2(x, t) q(-x, -t),$$
$$\mathrm{i}q_t(x, t) = q_{xx}(x, t) - 2\sigma q^2(x, t) q(x, -t).$$

In this paper, we focus on a nonlocal reverse-time NLS equation [16, 17]

$$\mathrm{i}p_t(x, t) = p_{xx}(x, t) + 2\sigma p^2(x, t) p(x, -t), \tag{1}$$

which was generalized by Ma [18] to a multi-component one. Yang [17] has derived general multi-solitons in three types of nNLSE, including the reverse-time, reverse-space and reverse-space–time nNLSE, and also presented a unified Riemann–Hilbert framework for them. Ma utilized inverse scattering transformation to construct N-soliton solutions to multi-component nNLS equations under the framework of Riemann–Hilbert problem [18]. There are also many other scholars who have made their own contributions to the study of reverse-time nNLSE. Lou [19] have established some new types of methods to solve nonlinear systems, as the full reversal invariant method and so on. Very recently, Ye and Zhang [20] constructed the general soliton solutions with zero and non-zero background to a reverse-time nNLSE via a matrix version of binary Darboux transformation (DT). They derived the formula of multi-soliton and high order solitons in a determinant form. It is seen that the single-soliton could exponentially blow up or decay. The asymptotic analysis has been established on two-soliton solutions for the nonlocal complex coupled dispersionless equation [21].

In fact, the soliton solutions of the integrable equation can be constructed by many methods, such as the inverse scattering method [22–26], the Hirota bilinear method [27–29], the DT [30–32] and so on. As we know that, for a physical system it is more interesting to find the bounded non-singular multi-soliton solutions. For a nonlocal integrable model, there will be problems such as singularity and boundedness in the process of searching for soliton solutions, and there are few studies in this area [33, 34]. In this work, we would like to explore a sufficient condition for the bounded multi-soliton solution for the nonlocal time-reversal NLS equation. What is more, the asymptotic analysis of bounded multi-soliton solutions to the reverse-time nNLSE can be established.

In this paper we use the DT method to construct the multi-soliton solutions for nNLSE (1) with the aid of loop group method [35]. By the construction of loop group method, the soliton solutions admit the compact determinant representation, which is beneficial to analyze the singularity of soliton solutions. Then we present a sufficient condition for a symmetric bounded multi-soliton solution. Furthermore, the asymptotic analysis for the multi-soliton solution is performed by the determinant technique, in which the modulus of multi-soliton solutions can be approximately decomposed into the sum of single-soliton solutions. The multi-soliton solutions exhibit the similar structure as the classical NLS equation, but they can not be decomposed directly since the phase term has the indefinite limitation. But we find the modulus of the multi-soliton solutions can be decomposed as $t \to \pm\infty$. Our proposed method provides a way to implement the singularity and asymptotic analysis for the time-reversal nonlocal integrable system, and it can be extended to other nonlocal NLS equation and vector nonlocal NLS equation [36–38].

The rest of the paper is organized as follows. In section 2, we will develop the N-fold DT for equation (1) by the loop group method. In section 3, the soliton solutions for equation (1) will be constructed through DT with zero seed. Besides, we will analyze the singularity and asymptoticity of the solutions that we have obtained. In the final section, we will give a few discussions and conclusions.

2. DT for the time-reversal nonlocal nonlinear Schödinger equation

As the nNLSE (1) can be regarded as a special reduction of the AKNS system, we firstly consider the AKNS system without reduction in order to obtain the DT for nNLSE (1) later. The AKNS system that we are going to explore is as follows:

$$\begin{aligned} &\boldsymbol{\Psi}_x = \boldsymbol{U}\boldsymbol{\Psi}, \qquad \boldsymbol{U}(\boldsymbol{Q};\lambda) \equiv \mathrm{i}(\lambda\boldsymbol{\sigma}_3 + \boldsymbol{Q}),\\ &\boldsymbol{\Psi}_t = \boldsymbol{V}\boldsymbol{\Psi}, \qquad \boldsymbol{V}(\boldsymbol{Q};\lambda) \equiv \mathrm{i}(\lambda^2\boldsymbol{\sigma}_3 + \lambda\boldsymbol{Q}) + \boldsymbol{V}_0,\\ &\boldsymbol{V}_0(\boldsymbol{Q}) = \frac{1}{2}\boldsymbol{\sigma}_3(\boldsymbol{Q}_x - \mathrm{i}\boldsymbol{Q}^2), \end{aligned} \tag{2}$$

where

$$\boldsymbol{Q} = \begin{bmatrix} 0 & \sigma p(x,-t)\\ p(x,t) & 0\end{bmatrix}, \qquad \boldsymbol{\sigma}_3 = \begin{bmatrix} 1 & 0\\ 0 & -1\end{bmatrix}.$$

Meanwhile, note that the reverse-time character implies the following symmetric relationship between $\boldsymbol{U}$ and $\boldsymbol{V}$ matrices [18, 39]:

$$\begin{aligned} \boldsymbol{U}^\top(x,-t;-\lambda) &= -\boldsymbol{C}\boldsymbol{U}(x,t;\lambda)\boldsymbol{C}^\top,\\ \boldsymbol{V}^\top(x,-t;-\lambda) &= \boldsymbol{C}\boldsymbol{V}(x,t;\lambda)\boldsymbol{C}^\top, \end{aligned} \tag{3}$$

where

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0\\ 0 & -\sigma\end{bmatrix}.$$

Based on the symmetric relation (3) and the existence and uniqueness of ordinary differential equation, we have

$$\boldsymbol{\Psi}^\top(x,-t;-\lambda)\boldsymbol{C}\boldsymbol{\Psi}(x,t;\lambda)\boldsymbol{C}^{-1} = \mathbb{I} \tag{4}$$

with the condition $\boldsymbol{\Psi}(0,0;\lambda) = \mathbb{I}$. Moreover, if the column vector solution $\psi_1(x,t;\lambda_1)$ is the solution of Lax pair (2) at $\lambda = \lambda_1$, then the row vector solution $\psi_1^\top(x,-t;\lambda_1)\boldsymbol{C}$ satisfies the following adjoint Lax pair:

$$-\boldsymbol{\Phi}_x = \boldsymbol{\Phi}\boldsymbol{U}, \quad -\boldsymbol{\Phi}_t = \boldsymbol{\Phi}\boldsymbol{V} \tag{5}$$

at $\lambda = -\lambda_1$.

Combining the loop group method [35] and the above symmetric relationship (3), it follows that the DT for system (2) can be represented as

$$\begin{aligned} \boldsymbol{T}_1(\lambda;x,t) &= \mathbb{I} - \frac{2\lambda_1}{\lambda+\lambda_1}\boldsymbol{P}_1(x,t),\\ \boldsymbol{P}_1(x,t) &= \frac{\psi_1(x,t;\lambda_1)\psi_1^\top(x,-t;\lambda_1)\boldsymbol{C}}{\psi_1^\top(x,-t;\lambda_1)\boldsymbol{C}\psi_1(x,t;\lambda_1)}, \end{aligned} \tag{6}$$

where the $\psi_1(x,t;\lambda_1)$ in DT is a special solution to system (2) at $\lambda = \lambda_1$. As $\psi_1(x,t;\lambda_1)$ can be written as $\begin{pmatrix}\psi_1^{(1)}(x,t;\lambda_1)\\ \psi_1^{(2)}(x,t;\lambda_1)\end{pmatrix}$, the corresponding Bäcklund transformation between old potential function and new one is given by

$$\begin{aligned} \boldsymbol{Q}^{[1]} &= \boldsymbol{Q} + 2\lambda_1[\boldsymbol{P}_1,\boldsymbol{\sigma}_3], \quad \text{i.e. } p^{[1]}(x,t)\\ &= p(x,t) + 2\lambda_1\frac{\psi_1^{(2)}(x,t;\lambda_1)\psi_1^{(1)}(x,-t;\lambda_1)}{\psi_1^\top(x,-t;\lambda_1)\boldsymbol{C}\psi_1(x,t;\lambda_1)}. \end{aligned} \tag{7}$$

In what follows, we verify the validity of above DT (6) and the corresponding Bäcklund transformation (7). Then, we can establish the following theorem:

Theorem 1. *The DT (6) converts Lax pair (2) into a new one*

$$\boldsymbol{\Psi}_x^{[1]} = \boldsymbol{U}(\boldsymbol{Q}^{[1]};\lambda)\boldsymbol{\Psi}^{[1]}, \qquad \boldsymbol{\Psi}_t^{[1]} = \boldsymbol{V}(\boldsymbol{Q}^{[1]};\lambda)\boldsymbol{\Psi}^{[1]},$$

where $\boldsymbol{\Psi}^{[1]}(x,t;\lambda) = \boldsymbol{T}_1(x,t;\lambda)\boldsymbol{\Psi}(x,t;\lambda)$, *and* $\boldsymbol{Q}^{[1]}$ *is given by equation (7).*

Proof 1. The proof of this theorem is equivalent to verify the following two equations:

$$\begin{aligned}\boldsymbol{T}_{1,x}\boldsymbol{T}_1^{-1} + \boldsymbol{T}_1\boldsymbol{U}(\boldsymbol{Q};\,\lambda)\boldsymbol{T}_1^{-1} = \boldsymbol{U}(\boldsymbol{Q}^{[1]};\,\lambda),\\ \boldsymbol{T}_{1,t}\boldsymbol{T}_1^{-1} + \boldsymbol{T}_1\boldsymbol{V}(\boldsymbol{Q};\,\lambda)\boldsymbol{T}_1^{-1} = \boldsymbol{V}(\boldsymbol{Q}^{[1]};\,\lambda),\end{aligned}$$

and the symmetric relationship (3). Defining

$$\boldsymbol{F}(x,\,t;\,\lambda) \equiv \boldsymbol{T}_{1,x}\boldsymbol{T}_1^{-1} + \boldsymbol{T}_1\boldsymbol{U}\boldsymbol{T}_1^{-1} - \boldsymbol{U}(\boldsymbol{Q}^{[1]};\,\lambda),$$

through direct calculation, we have

$$\begin{aligned}\boldsymbol{T}_{1,x}\boldsymbol{T}_1^{-1} &= -\frac{2\lambda_1}{\lambda+\lambda_1}\boldsymbol{P}_{1,x}\left(\mathbb{I} + \frac{2\lambda_1}{\lambda-\lambda_1}\boldsymbol{P}_1\right),\\ \boldsymbol{T}_1\boldsymbol{U}\boldsymbol{T}_1^{-1} &= \left(\mathbb{I} - \frac{2\lambda_1}{\lambda+\lambda_1}\boldsymbol{P}_1\right)\boldsymbol{U}(\boldsymbol{Q};\,\lambda)\left(\mathbb{I} + \frac{2\lambda_1}{\lambda-\lambda_1}\boldsymbol{P}_1\right).\end{aligned}$$

Based on above equations, we can obtain the residue for function $\boldsymbol{F}(x,\,t;\,\lambda)$ as

$$\begin{aligned}&\operatorname*{Res}_{\lambda=\lambda_1}\boldsymbol{F}(x,\,t;\,\lambda) = -2\lambda_1\boldsymbol{P}_{1,x}\boldsymbol{P}_1 + 2\lambda_1\boldsymbol{P}_1(\mathbb{I}-\boldsymbol{P}_1)\boldsymbol{U}(\boldsymbol{Q};\,\lambda)\\ &= -2\lambda_1\Bigg[\frac{\boldsymbol{\psi}_1(x,t;\lambda_1)}{\boldsymbol{\psi}_1^{\top}(x,-t;-\lambda_1)\boldsymbol{C}\boldsymbol{\psi}_1(x,t;\lambda_1)}\boldsymbol{\psi}_{1,x}^{\top}(x,\,-t;\,-\lambda_1)\boldsymbol{C}\boldsymbol{P}_1\\ &+\left(\frac{\boldsymbol{\psi}_1(x,t;\lambda_1)}{\boldsymbol{\psi}_1^{\top}(x,-t;-\lambda_1)\boldsymbol{C}\boldsymbol{\psi}_1(x,t;\lambda)}\right)_x\boldsymbol{\psi}_1^{\top}(x,\,-t;\,-\lambda_1)\\ &\boldsymbol{C}\boldsymbol{P}_1 - \boldsymbol{P}_1(\mathbb{I}-\boldsymbol{P}_1)\boldsymbol{U}(\boldsymbol{Q};\,\lambda)\Bigg]\\ &= 0.\end{aligned}$$

Similarly, we have $\operatorname*{Res}_{\lambda=-\lambda_1}\boldsymbol{F}(x,\,t;\,\lambda) = 0$. Thus the function $\boldsymbol{F}(x,\,t;\,\lambda)$ is an analytic function in the whole complex plane. Due to the Bäcklund transformation (7), the function $\boldsymbol{F}(x,\,t;\,\lambda)$ will vanish as $\lambda \to \infty$, which implies the function $\boldsymbol{F}(x,\,t;\,\lambda) = 0$ by the Liouville theorem.

Defining

$$\begin{aligned}\boldsymbol{G}(x,\,t;\,\lambda) &\equiv \boldsymbol{T}_{1,t}\boldsymbol{T}_1^{-1} + \boldsymbol{T}_1\boldsymbol{V}\boldsymbol{T}_1^{-1} - \widehat{\boldsymbol{V}^{[1]}},\\ \widehat{\boldsymbol{V}^{[1]}} &= \mathrm{i}(\lambda^2\boldsymbol{\sigma}_3 + \lambda\boldsymbol{Q}^{[1]} + \widehat{\boldsymbol{V}_0^{[1]}}),\end{aligned}$$

where $\widehat{\boldsymbol{V}_0^{[1]}} \equiv \boldsymbol{V}_0 + \mathrm{i}\boldsymbol{S}\boldsymbol{Q} - \mathrm{i}\boldsymbol{Q}^{[1]}\boldsymbol{S}$, $\boldsymbol{S} = \lambda_1 - 2\lambda_1\boldsymbol{P}_1$, and taking the similar procedure as above x-part, we can prove that

$$\boldsymbol{G}(x,\,t;\,\lambda) = 0.$$

Furthermore, with direct calculation, we can verify that

$$\widehat{\boldsymbol{V}_0^{[1]}} = \boldsymbol{V}_0(\boldsymbol{Q}^{[1]}).$$

Now we proceed to prove the symmetric properties of $\boldsymbol{U}^{[1]}$ and $\boldsymbol{V}^{[1]}$. Since $\boldsymbol{\Psi}^{\dagger}(x,\,-t;\,-\lambda)\boldsymbol{C}\boldsymbol{\Psi}(x,\,t;\,\lambda)\boldsymbol{C} = \mathbb{I}$, then

$$\boldsymbol{C}\boldsymbol{T}^{\dagger}(x,\,-t;\,-\lambda)\boldsymbol{C}\boldsymbol{T}(x,\,t;\,\lambda) = \mathbb{I}.$$

From $\boldsymbol{U}^{\dagger}(x,\,-t;\,-\lambda) = -\boldsymbol{C}\boldsymbol{U}(x,\,t;\,\lambda)\boldsymbol{C}^{\top}$ and $\boldsymbol{U}^{[1]}(x,\,t;\,\lambda) = \boldsymbol{T}_{1,x}\boldsymbol{T}_1^{-1} + \boldsymbol{T}_1\boldsymbol{U}\boldsymbol{T}_1^{-1}$, it follows that

$$\begin{aligned}&\boldsymbol{U}^{[1]\dagger}(x,\,-t;\,-\lambda) = -(\boldsymbol{T}_1(x,\,-t;\,\lambda)_x\boldsymbol{T}_1(x,\,-t;\,-\lambda))^{\dagger}\\ &+ (\boldsymbol{T}_1(x,\,-t;\,-\lambda)\boldsymbol{U}(x,\,-t;\,-\lambda)\boldsymbol{T}_1(x,\,-t;\,-\lambda))^{\dagger}\\ &= \boldsymbol{C}\boldsymbol{U}^{[1]}(x,\,t;\,\lambda)\boldsymbol{C}.\end{aligned}$$

and through $\boldsymbol{V}^{\dagger}(x,\,-t;\,-\lambda) = \boldsymbol{C}\boldsymbol{V}(x,\,t;\,\lambda)\boldsymbol{C}^{\top}$, we have

$$\boldsymbol{V}^{[1]\dagger}(x,\,-t;\,-\lambda) = \boldsymbol{C}\boldsymbol{V}^{[1]}(x,\,t;\,\lambda)\boldsymbol{C}^{\top}.$$

This completes the proof.

The above elementary DT can be iterated one by one to yield the N-fold DT. With the knowledge of linear algebra and complex variables functions, the iterated N-fold DT can be represented in a compact form. Then the multi-soliton solution can be constructed through the formula of corresponding Bäcklund transformation. In general, we can establish the following N-fold DT for nNLSE (1) as

Theorem 2. *If we have N different solutions to system (2): $\psi_1(x,\,t;\,\lambda_1)$, $\psi_2(x,\,t;\,\lambda_2)$, …, $\psi_N(x,\,t;\,\lambda_N)$ with $\lambda = \lambda_i$, $i = 1, 2, \ldots, N$, $\lambda_i \neq \lambda_j$ $(i \neq j)$, $\lambda_i \neq 0$, then the N-fold DT can be represented as*

$$\boldsymbol{T}(x,\,t;\,\lambda) = \mathbb{I} - \boldsymbol{Y}\boldsymbol{M}^{-1}\boldsymbol{D}^{-1}\boldsymbol{Z}\boldsymbol{C},$$

where

$$\begin{aligned}\boldsymbol{D} &= \mathrm{diag}(\lambda+\lambda_1,\,\lambda+\lambda_2,\,\ldots,\,\lambda+\lambda_N),\\ \boldsymbol{Y}(x,\,t) &= [|\boldsymbol{y}_1\rangle,\,|\boldsymbol{y}_2\rangle,\,\ldots,\,|\boldsymbol{y}_N\rangle],\\ \boldsymbol{Z} &= \begin{bmatrix}\langle\boldsymbol{y}_1|\\ \langle\boldsymbol{y}_2|\\ \vdots\\ \langle\boldsymbol{y}_N|\end{bmatrix},\qquad \boldsymbol{M} = \left(\frac{\langle\boldsymbol{y}_i|\boldsymbol{C}|\boldsymbol{y}_j\rangle}{\lambda_j+\lambda_i}\right)_{1\leqslant i,j\leqslant N},\end{aligned}$$

$\langle\boldsymbol{y}_j| \equiv \psi_j(x,\,-t)^{\top}$, $|\boldsymbol{y}_i\rangle \equiv \psi_i(x,\,t)$. *And the corresponding Bäcklund transformation between old and new potential functions is given by*

$$p^{[N]} = p - 2\boldsymbol{Y}_2\boldsymbol{M}^{-1}\boldsymbol{Z}_1. \tag{8}$$

Proof 2. The recursive DTs between matrix functions are given as

$$\begin{aligned}\boldsymbol{\Psi}^{[k]}(x,\,t;\,\lambda) = \boldsymbol{T}_k(x,\,t;\,\lambda)\boldsymbol{\Psi}^{[k-1]}(x,\,t;\,\lambda),\quad \boldsymbol{Q}^{[k]} = \boldsymbol{Q}^{[k-1]}\\ + 2\lambda_1[\boldsymbol{P}_k,\,\boldsymbol{\sigma}_3],\quad k = 1,\,\ldots,\,N,\end{aligned}$$

where

$$\begin{aligned}\boldsymbol{T}_1 &= \mathbb{I} - \frac{2\lambda_1}{\lambda+\lambda_1}\boldsymbol{P}_1,\\ \boldsymbol{P}_1 &= \frac{\psi_1(x,\,t;\,\lambda_1)\psi_1^{\top}(x,\,-t;\,\lambda_1)\boldsymbol{C}}{\psi_1^{\top}(x,\,-t;\,\lambda_1)\boldsymbol{C}\psi_1(x,\,t;\,\lambda_1)},\\ \boldsymbol{T}_2 &= \mathbb{I} - \frac{2\lambda_2}{\lambda+\lambda_2}\boldsymbol{P}_2,\\ \boldsymbol{P}_2 &= \frac{\psi_2^{[1]}(x,\,t;\,\lambda_2)(\psi_2^{[1]}(x,\,-t;\,\lambda_2))^{\top}\boldsymbol{C}}{(\psi_2^{[1]}(x,\,-t;\,\lambda_2))^{\top}\boldsymbol{C}\psi_2^{[1]}(x,\,t;\,\lambda_2)},\\ &\vdots\\ \boldsymbol{T}_i &= \mathbb{I} - \frac{2\lambda_i}{\lambda+\lambda_i}\boldsymbol{P}_i,\\ \boldsymbol{P}_i &= \frac{\psi_i^{[i-1]}(x,\,t;\,\lambda_i)(\psi_i^{[i-1]}(x,\,-t;\,\lambda_i))^{\top}\boldsymbol{C}}{(\psi_i^{[i-1]}(x,\,-t;\,\lambda_i))^{\top}\boldsymbol{C}\psi_i^{[i-1]}(x,\,t;\,\lambda_i)},\end{aligned}\tag{9}$$

and

$$\psi_i^{[i-1]}(x,t;\lambda_i)=(\boldsymbol{T}_{i-1}\boldsymbol{T}_{i-2}\ldots\boldsymbol{T}_1)|_{\lambda=\lambda_i}\,\psi_i(x,t;\lambda_i),\quad i=1,\ldots N.$$

Defining

$$\boldsymbol{T}(x,t;\lambda)=\boldsymbol{T}_N(x,t;\lambda)\boldsymbol{T}_{N-1}(x,t;\lambda)\ldots\boldsymbol{T}_1(x,t;\lambda) \tag{10}$$

which represents the N-fold DT, and analyzing the form of equation (10), it immediately comes out that $\boldsymbol{T}(x,t;\lambda)$ is a meromorphic function and ∞ is not the essential singularity, so it is a rational function with respect to λ. Then the N-fold DT $\boldsymbol{T}(x,t;\lambda)$ can be expressed in the form of

$$\boldsymbol{T}(x,t;\lambda)=\mathbb{I}-\sum_{i=1}^{N}\frac{\boldsymbol{A}_i(x,t)}{\lambda+\lambda_i}, \tag{11}$$

where $\boldsymbol{A}_i(x,t)$ is a matrix. By calculating residues on both sides of equation (11), an expression of $\boldsymbol{A}_i(x,t)$ is obtained:

$$\begin{aligned}\boldsymbol{A}_i(x,t)&=-\operatorname*{Res}_{\lambda=-\lambda_i}\boldsymbol{T}(x,t;\lambda)\\&=-(\boldsymbol{T}_N\ldots\boldsymbol{T}_{i+1})|_{\lambda=-\lambda_i}\,2\lambda_i\boldsymbol{P}_i(\boldsymbol{T}_{i-1}\ldots\boldsymbol{T}_1)|_{\lambda=-\lambda_i}.\end{aligned}$$

From equation (9) we know that $\mathrm{rank}(\boldsymbol{P}_i)=1$, thus $\mathrm{rank}(\boldsymbol{A}_i(x,t))=1$. According to the knowledge of linear algebra, $\boldsymbol{A}_i(x,t)$ could be rewritten as

$$\boldsymbol{A}_i(x,t)=|\boldsymbol{x}_i\rangle\langle\boldsymbol{y}_i|\boldsymbol{C},$$

where $|\boldsymbol{x}_i\rangle$ is a column vector, and $\langle\boldsymbol{y}_i|$ is a row vector. For simplicity, we denote that

$$\begin{aligned}\boldsymbol{R}&=\left[|\boldsymbol{x}_1\rangle,\ |\boldsymbol{x}_2\rangle,\ \ldots\ |\boldsymbol{x}_N\rangle\right],\\ \boldsymbol{D}&=\mathrm{diag}(\lambda+\lambda_1,\lambda+\lambda_2,\ldots,\lambda+\lambda_N),\\ \boldsymbol{Z}&=\begin{bmatrix}\langle\boldsymbol{y}_1|\\ \langle\boldsymbol{y}_2|\\ \vdots\\ \langle\boldsymbol{y}_N|\end{bmatrix}.\end{aligned}$$

Then $\boldsymbol{T}(x,t;\lambda)$ can be rewritten in matrix form as

$$\boldsymbol{T}(x,t;\lambda)=\mathbb{I}-\boldsymbol{R}\boldsymbol{D}^{-1}\boldsymbol{Z}. \tag{12}$$

Besides, after observation and inspection, we obtain that $\ker(\boldsymbol{T}(x,t;\lambda_i))=\psi_i(x,t;\lambda_i)$, i.e.

$$\left(\mathbb{I}-\sum_{j=1}^{N}\frac{|\boldsymbol{x}_j\rangle\langle\boldsymbol{y}_j|\boldsymbol{C}}{\lambda_i+\lambda_j}\right)\psi_i(x,t;\lambda_i)=0, \tag{13}$$

which implies that

$$\psi_i=\sum_{j=1}^{N}\frac{\langle\boldsymbol{y}_j|\boldsymbol{C}\psi_i}{\lambda_i+\lambda_j}|\boldsymbol{x}_j\rangle.$$

Thus

$$\begin{aligned}&\left[\psi_1,\ \psi_2,\ \ldots\ \psi_N\right]\\&=\boldsymbol{R}\begin{pmatrix}\frac{\langle\boldsymbol{y}_1|\boldsymbol{C}\psi_1}{\lambda_1+\lambda_1}&\frac{\langle\boldsymbol{y}_1|\boldsymbol{C}\psi_2}{\lambda_1+\lambda_2}&\cdots&\frac{\langle\boldsymbol{y}_1|\boldsymbol{C}\psi_N}{\lambda_1+\lambda_N}\\ \frac{\langle\boldsymbol{y}_2|\boldsymbol{C}\psi_1}{\lambda_2+\lambda_1}&\frac{\langle\boldsymbol{y}_2|\boldsymbol{C}\psi_2}{\lambda_2+\lambda_2}&\cdots&\frac{\langle\boldsymbol{y}_2|\boldsymbol{C}\psi_N}{\lambda_2+\lambda_N}\\ \vdots&\vdots&\ddots&\vdots\\ \frac{\langle\boldsymbol{y}_N|\boldsymbol{C}\psi_1}{\lambda_N+\lambda_1}&\frac{\langle\boldsymbol{y}_N|\boldsymbol{C}\psi_2}{\lambda_N+\lambda_2}&\cdots&\frac{\langle\boldsymbol{y}_N|\boldsymbol{C}\psi_N}{\lambda_N+\lambda_N}\end{pmatrix}.\end{aligned} \tag{14}$$

Furthermore, it is readily verified that

$$\boldsymbol{T}(x,t;\lambda)\boldsymbol{C}\boldsymbol{T}^{\top}(x,-t;-\lambda)\boldsymbol{C}=\mathbb{I},$$

i.e.

$$\left(\mathbb{I}-\sum_{i=1}^{N}\frac{|\boldsymbol{x}_i\rangle\langle\boldsymbol{y}_i|\boldsymbol{C}}{\lambda+\lambda_i}\right)\boldsymbol{C}\left(\mathbb{I}+\sum_{i=1}^{N}\frac{\boldsymbol{C}|\boldsymbol{y}_i\rangle\langle\boldsymbol{x}_i|}{\lambda-\lambda_i}\right)\boldsymbol{C}=\mathbb{I}. \tag{15}$$

Calculating residues on both sides of equation (15), then we obtain that

$$\left(\mathbb{I}-\sum_{j=1}^{n}\frac{|\boldsymbol{x}_j\rangle\langle\boldsymbol{y}_j|\boldsymbol{C}}{\lambda_i+\lambda_j}\right)|\boldsymbol{y}_i\rangle\langle\boldsymbol{x}_i|=0. \tag{16}$$

Because $\langle\boldsymbol{x}_i|\neq\boldsymbol{0}$, comparing equations (13) and (16), we find that

$$|\boldsymbol{y}_i\rangle=\psi_i(x,t;\lambda_i). \tag{17}$$

Denote

$$\boldsymbol{Y}=[|\boldsymbol{y}_1\rangle,|\boldsymbol{y}_2\rangle,\ldots,|\boldsymbol{y}_N\rangle],\qquad \boldsymbol{M}=\left(\frac{\langle\boldsymbol{y}_i|\boldsymbol{C}|\boldsymbol{y}_j\rangle}{\lambda_j+\lambda_i}\right)_{1\leqslant i,j\leqslant N},$$

then equation (14) can be rewritten as

$$\boldsymbol{Y}=\boldsymbol{R}\boldsymbol{M},$$

thus

$$\boldsymbol{R}=\boldsymbol{Y}\boldsymbol{M}^{-1}. \tag{18}$$

Finally, by substituting equations (18) into (12), we obtain that

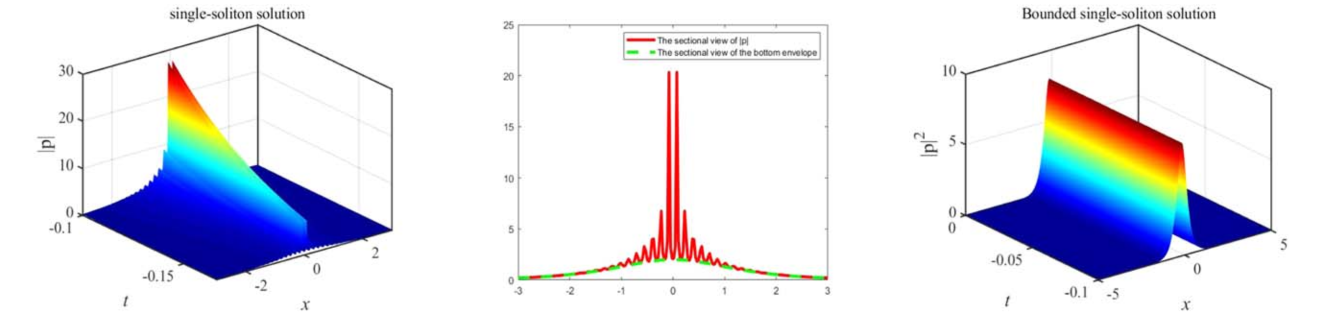
$$\boldsymbol{T}=\mathbb{I}-\boldsymbol{Y}\boldsymbol{M}^{-1}\boldsymbol{D}^{-1}\boldsymbol{Z}.$$

The Bäcklund transformation (8) can be obtained through the formula

$$\boldsymbol{T}_x+\boldsymbol{T}\boldsymbol{U}=\boldsymbol{U}^{[N]}\boldsymbol{T} \tag{19}$$

by expanding with respect to λ at the neighborhood of ∞.

So far we have constructed the N-fold DT, and in the next section it will be utilized to derive the N-soliton solution, based on which the singularity and asymptoticity analysis on these solutions will have been done.



(a) Unbounded non-singular single-soliton solution (b) Sectional view of a soliton solution and its lower envelope (c) Bounded non-singular single-soliton solution

Figure 1. (a) By choosing the parameters $a_1 = 10$, $b_1 = \frac{1}{2}$, $c = 1$, $\sigma = -1$, we obtain an unbounded non-singular single-soliton solution. (b) The corresponding sectional view of the single-soliton solution in (a) and its lower envelope. (c) By setting $a_1 = 0$, $b_1 = 1$, $c = 1 + \mathrm{i}$, $\sigma = -1$, we obtain a bounded non-singular single-soliton solution.

3. Bounded multi-solitons solutions and their asymptotic analysis

In this section, some exact solutions with zero seed solution will be derived. It is worth noting that the parameter σ in nNLSE represents different optical meanings, that is, σ takes ± 1 to correspond to focusing case and defocusing case respectively. As mentioned before, Ye and Zhang [20] have studied a reverse-time nNLSE with $\sigma = 1$, so we mainly discuss the situation when $\sigma = -1$ here, because this situation has never been studied before. When discussing the asymptoticity of the solution, we quote two lemmas proposed by Zhang *et al* [40] to illustrate the properties of the modulus of the soliton solution. What is more, we utilize a kind of method proposed by Ling *et al* [15] to calculate the limit form of M-matrix. Also, another method proposed by Faddeev and Takhtajan [41] has been used to determine the exponential decay term of the remainder.

3.1. The dynamics for the single-soliton solution

Taking $\lambda_1 = a_1 + \mathrm{i}b_1$, $a_1, b_1 \in \mathbb{R}$ and the seed solution $p(x, t) = 0$, then the vector solution of Lax pair (2) can be solved with the form:

$$\psi_1 = \begin{bmatrix} \mathrm{e}^{\eta_1(x,t)} \\ c_1 \mathrm{e}^{-\eta_1(x,t)} \end{bmatrix},$$

where $\eta_1(x, t) = \mathrm{i}\lambda_1(x + \lambda_1 t)$, and c_1 is a complex constant. Then, by the formula (7), we derive the following single-soliton solution:

$$p^{[1]}(x, t) = \frac{4(a_1 + \mathrm{i}b_1)c_1 \mathrm{e}^{[4a_1b_1 + 2\mathrm{i}(b_1^2 - a_1^2)]t}}{\mathrm{e}^{2(\mathrm{i}a_1 - b_1)x} - \sigma c_1^2 \mathrm{e}^{-2(\mathrm{i}a_1 - b_1)x}}. \tag{20}$$

Now we analyze the dynamics for the single-soliton.

3.1.1. Singularity. If $\ln(\sigma c_1^2)/(\mathrm{i}\lambda_1) \in \mathbb{R}$, the above solution will appear the singularity. For the other case, there is no singularity for the single-soliton solutions. For instance, by setting $c_1 = \mathrm{i}$, then we can obtain a non-singular single-soliton solution.

3.1.2. Boundedness. When $a_1 = 0$, $\sigma = -1$, we can obtain the bounded single-soliton solution with $\Re(c_1) \neq 0$:

$$p^{[1]}(x, t) = \frac{4\mathrm{i}b_1 c_1 \mathrm{e}^{2b_1(x + \mathrm{i}b_1 t)}}{\mathrm{e}^{4b_1 x} c_1^2 \ + \ 1}.$$

After calculation, we can deduce that the extreme point of the soliton solution is

$$x = -\frac{\ln(|c_1|)}{2\, b_1},$$

and its amplitude is

$$\frac{4b_1^2 |c_1|^2}{(\Re(c_1))^2}. \tag{21}$$

Similarly, we can consider the boundedness of solution for the case $\sigma = 1$. It is seen that the boundedness of solutions is valid for the stationary one.

3.1.3. Oscillation effect. It is seen that the oscillation effect will appear in the solution as a_1 large enough, which had never been pointed out in the previous studies. After calculation, we obtain the bottom envelope of the oscillation (see the figure 1(b)), the expression of which is as follows

$$b(x) = \frac{16\Re(c)^2(a^2 + b^2)}{\Re(c)^4 \mathrm{e}^{4bx} + \mathrm{e}^{-4bx} + 2\Re(c)^2}. \tag{22}$$

In principle, the expression of an envelope at the top can also be calculated, but the top envelope here is singular and has lost its practical significance, so it is not shown. In order to better represent the oscillation effect of soliton solution, we define its width.

Definition 1 The width of the soliton solution oscillation. The distance between the two abscissa corresponding to $1/2$ of the maximum of the lower envelope is defined as the width of the soliton solution oscillation.

By equation (22) we obtain the expression of the width

$$d = |x_+ - x_-|, \quad x_\pm = \frac{1}{4b}\ln \left| \frac{(\Re(c)^4 + \Re(c)^2 + 1) \pm (\Re(c)^2 + 1)\sqrt{\Re(c)^4 + 1}}{\Re(c)^4} \right|. \tag{23}$$

And it is readily proved that the period of oscillation is

$$T = \frac{\pi}{2a}, \tag{24}$$

so combining equation (22) with equation (24), we can deduce that the number of times that soliton solution oscillates is

$$N = \left[\frac{d}{T}\right], \tag{25}$$

where [] represents the ceil function. The single-soliton solution with oscillation effect and the sectional view of soliton solution and lower envelope are shown in figure 1.

Different from the single-soliton of classic NLSE, the amplitude of single-soliton solution of which is determined by the fixed spectral parameter λ_1. From equation (21) we know that for the nNLSE (1) studied in this paper, the amplitude of the soliton solution is jointly determined by the spectral parameter λ_1 and the solution parameter c_1. What is more, its amplitude can tend to ∞. Two examples of single-soliton solution are shown in figure 1.

Example 1. Choosing the parameters appropriately we have drawn graphs of unbounded single-soliton solutions and bounded single-soliton solutions. And from figure 1(a), we can see the single-soliton emerging the effect of periodic oscillation. Also, we provide the sectional view of the single-soliton solution and the lower envelope.

In fact, it is not easy-to find the singularity conditions of the soliton solution, as the conditions are already very complicated only in the case of a single-soliton. In physics, we always pay much attention on non-singular and bounded soliton solutions. We have discussed these two features in the case of single-soliton. Fortunately, we have not only found the non-singular conditions and bounded conditions of single-soliton solutions, we have also found those for the multi-soliton solution.

3.2. The non-singular symmetric multi-soliton solution

For the convenience of discussion, we will take the solution parameter c_i of each matrix function $\psi_i(x, t)$ as $e^{\frac{\pi}{2}i\frac{1+\sigma}{2}}$ below, then the matrix function of multi-soliton can be expressed as follows:

$$\psi_i(x, t) = \begin{pmatrix} e^{\eta_i(x,t)} \\ e^{\frac{\pi}{2}i\frac{1+\sigma}{2}}e^{-\eta_i(x,t)} \end{pmatrix}, \tag{26}$$

where

$$\eta_i(x, t) = i\lambda_i(x + \lambda_i t), \qquad i = 1 \dots n.$$

By the formula (8), we can construct the multi-soliton solution with the setting of $\psi_i(x, t)$ (26):

$$p^{[N]}(x, t) = \boldsymbol{Y}_2(x, t)\boldsymbol{M}^{-1}(x, t)\boldsymbol{Y}_1^\top(x, -t), \tag{27}$$

where

$$\begin{aligned} \boldsymbol{Y}_1(x, t) &= \left(e^{\eta_1(x,t)}, \ e^{\eta_2(x,t)}, \ \dots \ e^{\eta_n(x,t)}\right), \\ \boldsymbol{Y}_2(x, t) &= \left(e^{-\eta_1(x,t)}, \ e^{-\eta_2(x,t)}, \ \dots \ e^{-\eta_n(x,t)}\right), \\ \boldsymbol{M}(x, t) &= \left(\frac{e^{\eta_i(x,-t)+\eta_j(x,t)} + e^{-\eta_i(x,-t)-\eta_j(x,t)}}{2(\lambda_j + \lambda_i)}\right). \end{aligned}$$

With the aim of better illustrating the properties of the soliton solutions, we first invoke two lemmas about the relations among the solution, the reverse-time solution and the *M*-matrix.

Lemma 1. *[40]*

$$\begin{aligned} (p^{[N]}(x, t))^* &= -p^{[N]}(x, -t), \\ p^{[N]}(x, t)p^{[N]}(x, -t) &= -|p^{[N]}(x, t)|^2. \end{aligned}$$

Proof 3. From equation (27), we can deduce that

$$\begin{aligned} -p^{[N]}(x, -t) &= -\boldsymbol{Y}_2(x, -t)\boldsymbol{M}^{-1}(x, -t)\boldsymbol{Y}_1^\top(x, t) \\ &= (\boldsymbol{Y}_2(x, t)\boldsymbol{M}^{-1}(x, t)\boldsymbol{Y}_1^\top(x, -t))^* \\ &= (p^{[N]}(x, t))^*. \end{aligned}$$

Then, naturally we have that

$$\begin{aligned} p^{[N]}(x, t)p^{[N]}(x, -t) &= -p^{[N]}(x, t)(p^{[N]}(x, t))^* \\ &= -|p^{[N]}(x, t)|^2. \end{aligned}$$

Following the method in [40], the following lemma can be established:

Lemma 2. *[40]*

$$p^{[N]}(x, t)p^{[N]}(x, -t) = -\partial_x^2 \ln(|\boldsymbol{M}|)$$

Proof 4. From theorem 2 we can perform the *N*-fold Darboux matrix as follows:

$$\boldsymbol{T}(x, t; \lambda) = \mathbb{I} + \sum_{j=1}^{N} \frac{\boldsymbol{K}_j}{\lambda + \lambda_j}.$$

That Darboux matrix converts the system (2) into the invariant form

$$\begin{cases} \psi_x^{[N]} = \boldsymbol{U}^{[N]}\psi_{[N]}, & \boldsymbol{U}^{[N]}(x, t; \lambda) := i(\lambda\boldsymbol{\sigma}_3 + \boldsymbol{Q}^{[N]}), \\ \psi_t^{[N]} = \boldsymbol{V}^{[N]}\psi_{[N]}, & \boldsymbol{V}^{[N]}(x, t; \lambda) := \lambda\boldsymbol{U}^{[N]} \\ +\dfrac{1}{2}\boldsymbol{\sigma}_3(\boldsymbol{Q}_x^{[N]} - i(\boldsymbol{Q}^{[N]})^2) & \end{cases}$$

with

$$Q^{[N]} = \begin{bmatrix} 0 & \sigma p^{[N]}(x, -t) \\ p^{[N]}(x, t) & 0 \end{bmatrix}.$$

By $T_x + TU^{bg}(\lambda) = U^{[N]}T$ and matching the term $\mathcal{O}(\lambda^{-1})$, one yields

$$\left(\sum_{j=1}^{N} K_j\right)_x = \mathrm{i}(Q^{[N]}\sum_{j=1}^{N} K_j - \sum_{j=1}^{N} K_j Q^{bg}).$$

By $T_t + TV^{bg}(\lambda) = V^{[N]}T$ and matching the term $\mathcal{O}(1)$, one yields

$$(Q^{[N]})^2 = Q^{bg} + \mathrm{i}(Q^{bg} - Q^{[N]})_x + 2\sigma_3(Q^{[N]}\sum_{j=1}^{N} K_j - \sum_{j=1}^{N} K_j Q^{bg}).$$

Combining the above two equations, one deduces

$$(Q^{[N]})^2 = Q^{bg} + \mathrm{i}(Q^{bg} - Q^{[N]})_x - 2\mathrm{i}\sigma_3\left(\sum_{j=1}^{N} K_j\right)_x.$$

Then we derive

$$|p^{[N]}|^2 = -2\mathrm{i}\frac{\partial^2}{\partial x}\left(\sum_{j=1}^{N} K_j\right).$$

Note that

$$\sum_{j=1}^{N} K_j = -YM^{-1}Z, \qquad Y = Z^{\dagger}(x, -t)C.$$

Then one obtains

$$|p^{[N]}|^2 = \frac{\partial^2}{\partial x^2}\ln(|M|),$$

i.e.

$$p^{[N]}(x, t)p^{[N]}(x, -t) = -\frac{\partial^2}{\partial x^2}\ln(|M|).$$

Then we provide a sufficient condition for a non-singular $2n$-soliton solution.

Proposition 1. *For a $2n$-soliton solution, if the parameters satisfy*

$$\lambda_2 = -\lambda_1^*, \dots, \lambda_{2i} = -\lambda_{2i-1}^*, \qquad i = 1, \dots, n,$$

then the solutions constructed by theorem 2 are non-singular.

Note that throughout the paper we use * to denote complex conjugation.

Proof 5. Firstly, we verify that M is non-degenerate. By theorem 2, M can be written as $M = (m_{ij})_{1\leqslant i,j\leqslant 2n}$, where

$$m_{ij} = \frac{\psi_i^{\top}(x, -t)C\psi_j(x, t)}{2(\lambda_j + \lambda_i)} = \frac{\mathrm{e}^{\eta_i(x,-t)+\eta_j(x,t)} + \mathrm{e}^{-\eta_i(x,-t)-\eta_j(x,t)}}{2(\lambda_j + \lambda_i)}.$$

So the matrix M can also be expressed in matrix form as

$$M = M_1UM_2 + M_3UM_4,$$

where

$$\begin{aligned} M_1 &= \mathrm{diag}(\mathrm{e}^{\eta_1(x,-t)}, \dots, \mathrm{e}^{\eta_{2n}(x,-t)}),\\ M_2 &= \mathrm{diag}(\mathrm{e}^{\eta_1(x,t)}, \dots, \mathrm{e}^{\eta_{2n}(x,t)}),\\ M_3 &= \mathrm{diag}(\mathrm{e}^{-\eta_1(x,-t)}, \dots, \mathrm{e}^{-\eta_{2n}(x,-t)}),\\ M_4 &= \mathrm{diag}(\mathrm{e}^{-\eta_1(x,t)}, \dots, \mathrm{e}^{-\eta_{2n}(x,t)}),\\ U &= \left(\frac{1}{2(\lambda_j + \lambda_i)}\right)_{1\leqslant i,j\leqslant 2n}, \end{aligned}$$

among which obviously U is a Cauchy matrix. Moreover, we readily come up with the following parameter relations:

$$\begin{aligned} \eta_{2k-1}^*(x, t) &= \eta_{2k}(x, -t),\\ \eta_{2k}^*(x, t) &= \eta_{2k-1}(x, -t), \qquad k = 1, \dots, n. \end{aligned} \tag{28}$$

Besides, by equation (28) we can deduce that

$$\begin{aligned} M_1 &= \mathrm{diag}(\mathrm{e}^{\eta_2^*(x,t)}, \mathrm{e}^{\eta_1^*(x,t)}, \mathrm{e}^{\eta_4^*(x,t)}, \mathrm{e}^{\eta_3^*(x,t)}, \dots, \mathrm{e}^{\eta_{2n}^*(x,t)}, \mathrm{e}^{\eta_{2n-1}^*(x,t)}),\\ M_3 &= \mathrm{diag}(\mathrm{e}^{-\eta_2^*(x,t)}, \mathrm{e}^{-\eta_1^*(x,t)}, \mathrm{e}^{-\eta_4^*(x,t)}, \mathrm{e}^{-\eta_3^*(x,t)}, \dots, \mathrm{e}^{-\eta_{2n}^*(x,t)}, \mathrm{e}^{-\eta_{2n-1}^*(x,t)}). \end{aligned} \tag{29}$$

In order to make the proof clearer, we define a block matrix

$$N = \begin{pmatrix} N_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & N_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & N_n \end{pmatrix},$$

where

$$N_1 = N_2 = \dots = N_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Furthermore, equation (29) implies that

$$\begin{aligned} NM_1N^{-1} &= \mathrm{diag}(\mathrm{e}^{\eta_1^*(x,t)}, \mathrm{e}^{\eta_2^*(x,t)}, \mathrm{e}^{\eta_3^*(x,t)}, \mathrm{e}^{\eta_4^*(x,t)}, \dots, \mathrm{e}^{\eta_{2n-1}^*(x,t)}, \mathrm{e}^{\eta_{2n}^*(x,t)}),\\ NM_3N^{-1} &= \mathrm{diag}(\mathrm{e}^{-\eta_1^*(x,t)}, \mathrm{e}^{-\eta_2^*(x,t)}, \mathrm{e}^{-\eta_3^*(x,t)}, \mathrm{e}^{-\eta_4^*(x,t)}, \dots, \mathrm{e}^{-\eta_{2n-1}^*(x,t)}, \mathrm{e}^{-\eta_{2n}^*(x,t)}). \end{aligned} \tag{30}$$

For simplicity, we denote that

$$\tilde{M}_1 = NM_1N^{-1}, \quad \tilde{M}_3 = NM_3N^{-1}, \quad \hat{M} = NU,$$

and it is readily verified that $\hat{M}^{\dagger} = \hat{M}$, i.e. M is a Hermitian matrix. We ulteriorly derive that

$$\begin{aligned} NM &= N(M_1UM_2 + M_3UM_4)\\ &= \tilde{M}_1\hat{M}M_2 + \tilde{M}_3\hat{M}M_4. \end{aligned} \tag{31}$$

By setting that H equals to equation (31), H can be written as $H = (h_{ij})_{1\leqslant i,j\leqslant 2n}$, where

$$h_{ij} = \left(\frac{\mathrm{e}^{\eta_i^*(x,t)+\eta_j(x,t)} + \mathrm{e}^{-\eta_i^*(x,t)-\eta_j(x,t)}}{2(\lambda_i + \lambda_j)}\right).$$

And because for any ξ_i, ξ_j,

$$\sum_{i,j}^{2n,2n} h_{ij}\xi_i\xi_j^* = \left|\sum_{j=1}^{\infty}\xi_j\int_x^{+\infty} \mathrm{e}^{\eta_j(s,t)}ds\right|^2 + \left|\sum_{j=1}^{\infty}\xi_j\int_{-\infty}^{x} \mathrm{e}^{-\eta_j(s,t)}ds\right|^2 > 0$$

thus $\boldsymbol{H}$ is a positive definite matrix, which implies that $\boldsymbol{M}$ is a non-degenerate matrix.

By theorem 2, a solution can be constructed in the form of

$$p^{[N]}(x,t) = \boldsymbol{X}_2\boldsymbol{M}^{-1}\boldsymbol{X}_1 = \frac{\boldsymbol{X}_2^{\mathrm{adj}}\boldsymbol{M}\boldsymbol{X}_1}{\det(\boldsymbol{M})}, \tag{32}$$

where $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ are matrices obtained by the transformation of system matrix function. Because $\boldsymbol{M}$ is non-degenerate, i.e. $\det(\boldsymbol{M}) \neq 0$, under the circumstances $p^{[N]}(x,t)$ must be a non-singular soliton solution.

Remark 1. In the above proposition, we give a sufficient condition for the $2n$-soliton solution with pair parameters. Actually, for the case $\lambda_i \in \mathrm{i}\mathbb{R}$, we can also obtain the similar result in a similar way. In this case, what we obtain are also the bounded non-singular soliton solutions.

3.3. Asymptotic analysis for the multi-soliton solution

In what follows, we will illustrate the asymptoticity of multi-soliton solution under $\sigma = -1$. For the case $\sigma = 1$, it can be analyzed similarly. In section 3.3, we set $c_i = 1$ for convenience. We take that $\lambda_1 = a_1 + \mathrm{i}b_1$, $\lambda_2 = -a_1 + \mathrm{i}b_1$, because other forms of parameter selection result in that the derivation result in any direction equals to 0 in the limiting case. The corresponding matrix functions are as follows:

$$\psi_1(x,t) = \begin{pmatrix}\mathrm{e}^{\eta_1(x,t)}\\ \mathrm{e}^{-\eta_1(x,t)}\end{pmatrix}, \qquad \psi_2(x,t) = \begin{pmatrix}\mathrm{e}^{\eta_2(x,t)}\\ \mathrm{e}^{-\eta_2(x,t)}\end{pmatrix},$$

where $\eta_i(x,t) = \mathrm{i}\lambda_i(x+\lambda_i t)$, $i = 1, 2$. And then, throughout the theorem 2 we can derive the 2nd order $\boldsymbol{M}_2$ matrix:

$$\boldsymbol{M}_2 = \left(\frac{\psi_i^{\top}(x,-t;\lambda_i)\boldsymbol{C}\psi_j(x,t;\lambda_j)}{\lambda_j+\lambda_i}\right)_{1\leqslant i,j\leqslant 2}.$$

For simplicity of expression, we denote that

$$[\psi_1(x,t), \psi_2(x,t)] = \begin{bmatrix}\boldsymbol{X}_1(x,t)\\ \boldsymbol{X}_2(x,t)\end{bmatrix}.$$

Then the solution could be written as:

$$p^{[1]}(x,t) = \boldsymbol{X}_2(x,t;\lambda_1,\lambda_2)\boldsymbol{M}^{-1}\boldsymbol{X}_1^{\top}(x,-t;\lambda_1,\lambda_2). \tag{33}$$

Due to the symmetry of the solution with respect to the time variable t, here we only prove the case that $t \to +\infty$. If we fix the direction as $x + 2a_1 t = \theta_1$, thus

$$|\boldsymbol{M}_2| \to \tilde{\boldsymbol{M}}_{21},$$

where

$$\tilde{\boldsymbol{M}}_{21} = \begin{vmatrix}\frac{\mathrm{e}^{2\eta_1(x,t)}}{2(\lambda_1+\lambda_1)} & \frac{1}{2(-\lambda_1^*+\lambda_1)}\\ \frac{\mathrm{e}^{2(\eta_1(x,t)+\eta_1^*(x,t))}+1}{2(\lambda_1-\lambda_1^*)} & \frac{\mathrm{e}^{2\eta_1^*(x,t)}}{2(-\lambda_1^*-\lambda_1^*)}\end{vmatrix}.$$

Similarly, we can obtain that if the direction is fixed as $x - 2a_1 t = \theta_2$,

$$|\boldsymbol{M}| \to |\tilde{\boldsymbol{M}}_{22}| = \begin{vmatrix}\frac{1}{2(\lambda_1+\lambda_1)} & \frac{\mathrm{e}^{2(\eta_1(x,-t)+\eta_1^*(x,-t))}+1}{2(-\lambda_1^*+\lambda_1)}\\ \frac{1}{2(\lambda_1-\lambda_1^*)} & \frac{1}{2(-\lambda_1^*-\lambda_1^*)}\end{vmatrix}.$$

If we fix the direction to any other value, then we have

$$\partial_x^2 \ln(|\boldsymbol{M}|) \to 0.$$

Therefore, from lemmas 2 and 3, we can derive that

$$|p^{[2]}|^2 = 4b_1^2\sum_{i=1}^{2}\mathrm{sech}^2\left(\frac{\tilde{w}_i}{2}\right) + \mathcal{O}(\mathrm{e}^{-|c|t}),$$

where

$$c = 4a_1b_1,$$
$$\tilde{w}_1 = -4b_1\theta_1 + \ln\frac{a_1^2}{a_1^2+b_1^2},$$
$$\tilde{w}_2 = -4b_1\theta_2 + \ln\frac{a_1^2+b_1^2}{a_1^2}.$$

So far, the asymptotic form of two-soliton solution has been obtained.

Example 2. Choosing the parameters as $a_1 = 1$, $b_1 = 1$, $\sigma = -1$, we construct the asymptotic form of the two-soliton solution. Due to the symmetry of the solution with respect to the time variable t, here we only show the sectional view when $t \to +\infty$. The results are shown in figure 2.

Note that the asymptotic analysis merely works for the multi-soliton with different velocity. Next, on the premise of the sufficient condition for non-singular $2n$-soliton solution mentioned above, we give the asymptotic analysis of $2n$-soliton solutions.

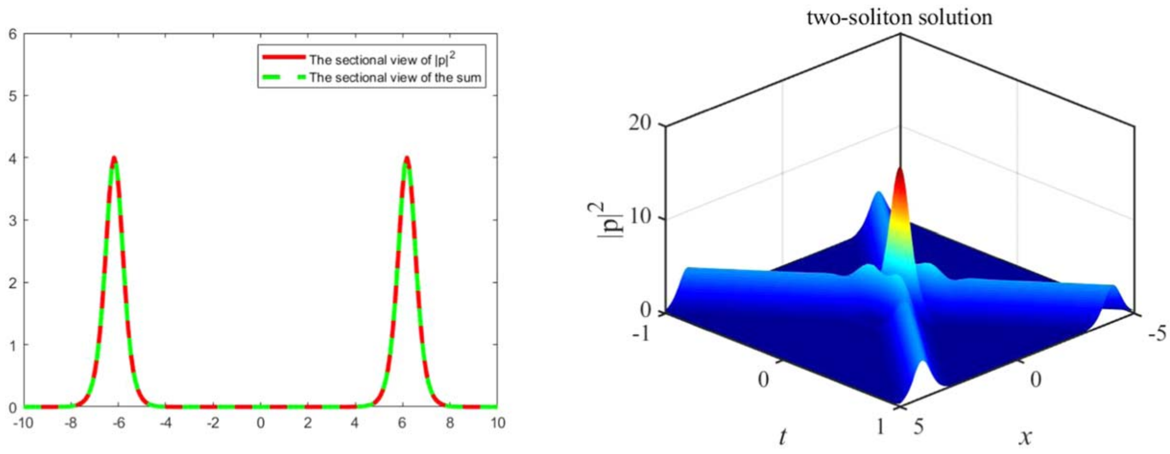
Theorem 3. *For a $2n$-soliton solution $p^{[2n]}$, if the selected parameters satisfy the following conditions*

$$\lambda_{2i} = -\lambda_{2i-1}^*, \qquad i = 1, \ldots, n$$

then

$$|p^{[2n]}|^2 = \sum_{i=1}^{n}4b_i^2\left[\mathrm{sech}^2\left(\frac{\tilde{w}_{2i-1}}{2}\right) + \mathrm{sech}^2\left(\frac{\tilde{w}_{2i}}{2}\right)\right] + \mathcal{O}(\mathrm{e}^{-|ct|}), \quad t \to \pm\infty$$

where $b_i = \Im(\lambda_i)$, and $\tilde{w}_{2i-1}$, $\tilde{w}_{2i}$, $|c|$ are given by equations (37), (39), (41) respectively. In other words, when



(a) $t=3$, the sum and $|p|^2$ (b) two-soliton solution

Figure 2. $a_1 = 1$, $b_1 = 1$ (a) The red solid lines represent the sectional view of $|p^{[2]}|^2$ when $t = 3$. The green dotted lines represent the sectional view of the sum of the two decomposed single-soliton solutions with $t = 3$. It is shown that the sum matches $|p^{[2]}|^2$ very well. (b) two-soliton solution: $a_1 = 2$, $b_1 = 1$, $\sigma = -1$.

$t \to \infty$, the square of the modulus of a 2n-soliton solution can be written as the sum of solutions of 2n single-soliton.

Proof 6. We regard two adjacent matrix functions as a pair, and the expression of any pair is given as follows

$$\psi_{2k-1}(x,t) = \begin{pmatrix} e^{\eta_{2k-1}(x,t)} \\ e^{-\eta_{2k-1}(x,t)} \end{pmatrix}, \quad \psi_{2k}(x,t) = \begin{pmatrix} e^{\eta_{2k}(x,t)} \\ e^{-\eta_{2k}(x,t)} \end{pmatrix}, \quad k = 1, \ldots, n. \tag{34}$$

From proposition 2 we know that if the parameters satisfy

$$\lambda_{2k} = -\lambda_{2k-1}^*,$$

then there is the following symmetric relationship between $\eta(x, t)$

$$\eta_{2k-1}^*(x,t) = \eta_{2k}(x,-t), \qquad \eta_{2k}^*(x,t) = \eta_{2k-1}(x,-t),$$

which yields that

$$\psi_{2k}(x,t) = \psi_{2k-1}^*(x,-t). \tag{35}$$

Choosing the $2n$ parameters as follows

$$\lambda_1, \ldots, \lambda_n, \lambda_1^*, \ldots, \lambda_n^*,$$

then the corresponding matrix functions could be listed out as follows

$$\psi_1(x,t), \ldots, \psi_n(x,t), \psi_1^*(x,-t), \ldots, \psi_n^*(x,-t).$$

For each parameter $\lambda_k = a_k + \mathrm{i}b_k (a_k, b_k \in \mathbb{R})$ we assume that $a_k < a_{k-1} < \ldots < a_2 < a_1 < 0 (k \neq 1)$, and $b_k < 0$. The two directions corresponding to any pair of matrix function, which is shown in equation (34), are

$$x + 2a_k t = \theta_{2k-1}, \qquad x - 2a_k t = \theta_{2k}.$$

From lemma 3 we know that the square of the modulus of the soliton solution is symmetric about t, so below we only prove the situation when $t \to +\infty$.

When $t \to +\infty$, fixed the direction as $x + 2a_i t = \theta_{2i-1}$, we can deduce that

$$|\boldsymbol{M}| \to |\tilde{\boldsymbol{M}}_{2i-1}|,$$

where

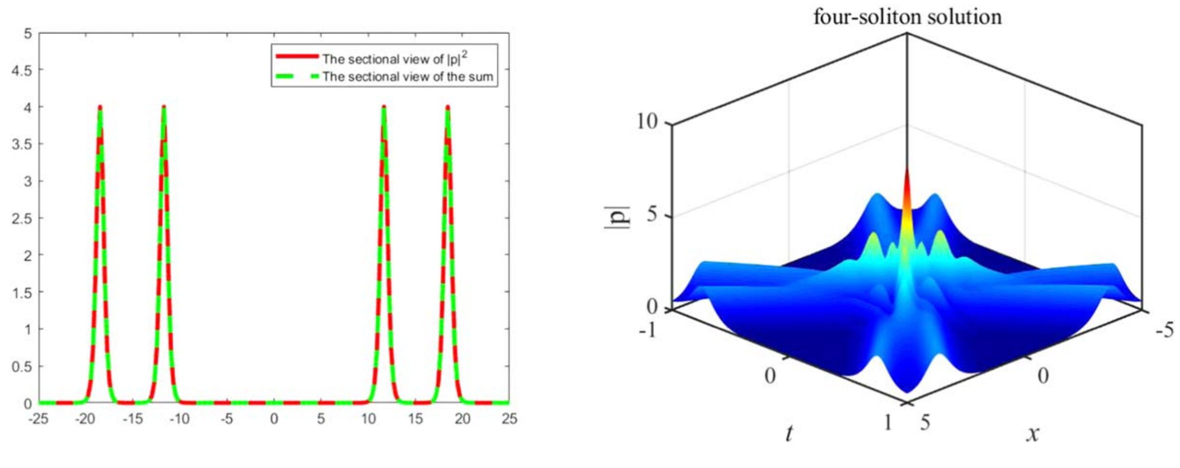
$$|\tilde{\boldsymbol{M}}_{2i-1}| = \frac{1}{2^{2n}} \begin{vmatrix} \frac{1}{\lambda_1+\lambda_1} & \cdots & \frac{1}{\lambda_{i-1}+\lambda_1} & \frac{e^{2\eta_i(x,t)}}{\lambda_i+\lambda_1} & 0 & \cdots & 0 & \frac{1}{-\lambda_1^*+\lambda_1} & \cdots & \frac{1}{-\lambda_n^*+\lambda_1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \frac{1}{\lambda_1-\lambda_{i-1}^*} & \cdots & \frac{1}{\lambda_{i-1}-\lambda_{i-1}^*} & \frac{e^{2\eta_i(x,t)}}{\lambda_i-\lambda_{i-1}^*} & 0 & \cdots & 0 & \frac{1}{-\lambda_1^*-\lambda_{i-1}^*} & \cdots & \frac{1}{-\lambda_n^*-\lambda_{i-1}^*} \\ \frac{e^{2\eta_i^*(x,t)}}{\lambda_1-\lambda_i^*} & \cdots & \frac{e^{2\eta_i^*(x,t)}}{\lambda_{i-1}-\lambda_i^*} & \frac{e^{2(\eta_i^*(x,t)+\eta_i(x,t))}}{\lambda_i-\lambda_i^*} & 0 & \cdots & 0 & \frac{e^{2\eta_i^*(x,t)}}{-\lambda_1^*-\lambda_i^*} & \cdots & \frac{e^{2\eta_i^*(x,t)}}{-\lambda_n^*-\lambda_i^*} \\ 0 & \cdots & 0 & \frac{1}{\lambda_i-\lambda_{i+1}^*} & \frac{1}{\lambda_{i+1}-\lambda_{i+1}^*} & \cdots & \frac{1}{\lambda_n-\lambda_{i+1}^*} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \frac{1}{\lambda_i-\lambda_n^*} & \frac{1}{\lambda_{i+1}-\lambda_n^*} & \cdots & \frac{1}{\lambda_n-\lambda_n^*} & 0 & \cdots & 0 \end{vmatrix}.$$

We can transform this determinant into the following easy-to-solve form [15]

$$
\begin{aligned}
|\tilde{\boldsymbol{M}}_{2i-1}| = &\begin{vmatrix}
\frac{1}{2(\lambda_{i+1}-\lambda_{i+1}^*)} & \cdots & \frac{1}{2(\lambda_n-\lambda_{i+1}^*)} & \frac{1}{2(\lambda_i-\lambda_{i+1}^*)} & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\frac{1}{2(\lambda_{i+1}-\lambda_n^*)} & \cdots & \frac{1}{2(\lambda_n-\lambda_n^*)} & \frac{1}{2(\lambda_i-\lambda_n^*)} & 0 & \cdots & 0 \\
\frac{1}{2(\lambda_{i+1}-\lambda_i^*)} & \cdots & \frac{1}{2(\lambda_n-\lambda_i^*)} & \frac{1}{2(\lambda_i-\lambda_i^*)} & \frac{e^{\eta_i^*(x,t)}}{2(\lambda_1-\lambda_i^*)} & \cdots & \frac{e^{\eta_i^*(x,t)}}{2(-\lambda_n^*-\lambda_i^*)} \\
0 & \cdots & 0 & 0 & \frac{1}{2(\lambda_1+\lambda_1)} & \cdots & \frac{1}{2(-\lambda_n^*+\lambda_1)} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & \frac{1}{2(\lambda_1-\lambda_{i-1}^*)} & \cdots & \frac{1}{2(-\lambda_n^*-\lambda_{i-1}^*)}
\end{vmatrix} \\
&+ \begin{vmatrix}
\frac{1}{2(\lambda_{i+1}-\lambda_{i+1}^*)} & \cdots & \frac{1}{2(\lambda_n-\lambda_{i+1}^*)} & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
\frac{1}{2(\lambda_{i+1}-\lambda_n^*)} & \cdots & \frac{1}{2(\lambda_n-\lambda_n^*)} & 0 & 0 & \cdots & 0 \\
\frac{1}{2(\lambda_{i+1}-\lambda_i^*)} & \cdots & \frac{1}{2(\lambda_n-\lambda_i^*)} & \frac{e^{2(\eta_i(x,t)+\eta_i^*(x,t))}}{2(\lambda_i-\lambda_i^*)} & \frac{e^{\eta_i^*(x,t)}}{2(\lambda_1-\lambda_i^*)} & \cdots & \frac{e^{\eta_i^*(x,t)}}{2(-\lambda_n^*-\lambda_i^*)} \\
0 & \cdots & 0 & \frac{e^{2\eta_i(x,t)}}{2(\lambda_i+\lambda_1)} & \frac{1}{2(\lambda_1+\lambda_1)} & \cdots & \frac{1}{2(-\lambda_n^*+\lambda_1)} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & \frac{e^{2\eta_i(x,t)}}{2(\lambda_i-\lambda_{i-1}^*)} & \frac{1}{2(\lambda_1-\lambda_{i-1}^*)} & \cdots & \frac{1}{2(-\lambda_n^*-\lambda_{i-1}^*)}
\end{vmatrix} \\
&= \frac{1}{2^{2n}}[C_1C_2 + C_3C_4 e^{2(\eta_i(x,t)+\eta_i^*(x,t))}] \\
&= \frac{1}{2^{2n}}\left[\frac{1}{\lambda_i-\lambda_i^*}C_2C_3(\gamma_{2i-1,1} + \gamma_{2i-1,2}e^{2(\eta_i(x,t)+\eta_i^*(x,t))})\right],
\end{aligned}
$$

where

$$
\begin{aligned}
C_1 &= \frac{\prod_{i\leqslant m<k\leqslant n}(\lambda_m-\lambda_k)(-\lambda_m^*+\lambda_k^*)}{\prod_{i\leqslant i,j\leqslant n}(\lambda_m-\lambda_k^*)}, \\
C_3 &= \frac{\prod_{i+1\leqslant m<k\leqslant n}(\lambda_m-\lambda_k)(-\lambda_m^*+\lambda_k^*)}{\prod_{i+1\leqslant m,k\leqslant n}(\lambda_m-\lambda_k^*)}, \\
C_2 &= \frac{\prod_{1\leqslant m<k\leqslant i-1}(\lambda_m-\lambda_k)(-\lambda_m^*+\lambda_k^*)\prod_{1\leqslant m<k\leqslant n}(-\lambda_m^*+\lambda_k^*)(\lambda_m-\lambda_k)\prod_{\substack{1\leqslant m\leqslant i-1\\ 1\leqslant k\leqslant n}}(\lambda_m+\lambda_k^*)(\lambda_m^*+\lambda_k)}{\prod_{\substack{1\leqslant m\leqslant i-1\\ 1\leqslant k\leqslant n}}(\lambda_m+\lambda_k)(-\lambda_m^*-\lambda_k^*)}, \\
C_4 &= \frac{\prod_{1\leqslant m<k\leqslant i}(\lambda_m-\lambda_k)(-\lambda_m^*+\lambda_k^*)\prod_{1\leqslant m<k\leqslant n}(-\lambda_m^*+\lambda_k^*)(\lambda_m-\lambda_k)\prod_{\substack{1\leqslant m\leqslant i\\ 1\leqslant k\leqslant n}}(\lambda_m+\lambda_k^*)(\lambda_m^*+\lambda_k)}{\prod_{\substack{1\leqslant m\leqslant i\\ 1\leqslant k\leqslant n}}(\lambda_m+\lambda_k)(-\lambda_m^*-\lambda_k^*)},
\end{aligned}
$$



(a) $t = 3$, the sum and $|p|^2$

(b) four-soliton solution

Figure 3. $a_1 = 1$, $b_1 = 1$, $a_2 = 3$, $b_2 = 1$ (a) The red solid lines represent the sectional view of $|p^{[4]}|^2$ when $t = 3$. The green dotted lines represent the sectional view of the sum of the four decomposed single-soliton solutions with $t = 3$. It is shown that the sum matches $|p^{[4]}|^2$ very well. (b) four-soliton solution: $a_1 = 2$, $b_1 = 1$, $a_2 = 3$, $b_2 = 1$, $\sigma = -1$.

and

$$\gamma_{2i-1,1} = \prod_{j=i+1}^{n} \left| \frac{\lambda_i - \lambda_j}{\lambda_j - \lambda_i^*} \right|^2, \qquad \gamma_{2i-1,2} = \prod_{i=1}^{i-1} \left| \frac{\lambda_i - \lambda_i}{\lambda_i - \lambda_i^*} \right|^2 \prod_{j=1}^{n} \left| \frac{\lambda_j + \lambda_i^*}{\lambda_j + \lambda_i} \right|^2.$$

We can calculate that

$$\partial_x^2 \ln(|\tilde{\boldsymbol{M}}_{2i-1}|) = 4b_i^2 \operatorname{sech}^2(\frac{\tilde{w}_{2i-1}}{2}), \tag{36}$$

where

$$\tilde{w}_{2i-1} = -4b_i\theta_{2i-1} + \ln(\frac{\gamma_{2i-1,2}}{\gamma_{2i-1,1}}). \tag{37}$$

Similarly, if the direction is fixed as $x - 2a_i t = \theta_{2i}$, then we can derive that

$$\partial_x^2 \ln(|\tilde{\boldsymbol{M}}_{2i}|) = 4b_i^2 \operatorname{sech}^2(\frac{\tilde{w}_{2i}}{2}), \tag{38}$$

where

$$\tilde{w}_{2i} = -4b_i\theta_{2i} + \ln(\frac{\gamma_{2i-1,1}}{\gamma_{2i-1,2}}). \tag{39}$$

It must be noted that if the speed direction is fixed to any value other than the direction between any pair of matrix function, then we have

$$\partial_x^2 \ln(|\boldsymbol{M}|) \to 0. \tag{40}$$

Combining the Lemma 3 and the results of equations (36), (38) and (40), we can finally infer that

$$|p^{[2n]}|^2 = \sum_{i=1}^{n} 4b_i^2 \left[\operatorname{sech}^2\left(\frac{\tilde{w}_{2i-1}}{2}\right) + \operatorname{sech}^2\left(\frac{\tilde{w}_{2i}}{2}\right) \right] + \mathcal{O}(\mathrm{e}^{-|ct|}),$$

where

$$|c| = \min_{j=1,\,\ldots,\,n} \{2|b_j|\} \min_{\substack{i \neq k \\ i,k=1,\,\ldots,\,2n}} \{|a_i - a_k|\}. \tag{41}$$

Example 3. Choosing the parameters as $a_1 = 2$, $b_1 = 1$, $a_2 = 3$, $b_2 = 1$, $\sigma = -1$, we construct the asymptotic form of a four-soliton solution to test the result of theorem 3. Due to the symmetry of the solution with respect to the time variable t, here we only show the sectional view when $t \to +\infty$. Then results are shown in figure 3, which verifies the asymptotic analysis by numeric graphs.

Remark 2. For a $(2n + 1)$-soliton solution, if the added parameter λ_{2n+1} satisfies $\Re(\lambda_{2n+1}) = 0$, then the result still holds.

For the $\sigma = 1$, by setting the solution parameter c_i of each matrix function $\psi_i(x, t)$ as i, we can not only ensure the non-singularity of the soliton solution, but also obtain the same $\mathbf{M}$ matrix as in the case of $\sigma = -1$. So it can be similarly verified that

$$|p^{[2n]}|^2 = \sum_{i=1}^{n} 4b_i^2 \left[\operatorname{sech}^2\left(\frac{\tilde{w}_{2i-1}}{2}\right) + \operatorname{sech}^2\left(\frac{\tilde{w}_{2i}}{2}\right) \right] + \mathcal{O}(\mathrm{e}^{-|ct|}),$$

where

$$|c| = \min_{j=1,\,\ldots,\,n} \{2|b_j|\} \min_{\substack{i \neq k \\ i,k=1,\,\ldots,\,2n}} \{|a_i - a_k|\}.$$

Remark 3. Compared to the classical NLSE, the asymptotic decomposition of the multi-soliton solutions of the nNLS equation under study is only applicable to the symmetric soliton solutions. In addition, the classical NLSE has not only the asymptotic expression of the square of the modulus of the solution, but also other forms of asymptotic expressions of the solution. What is more, there is no phase shift character in our asymptotic analysis, but in general it does exist in classical NLSE.

4. Discussions and conclusions

In this work, we obtain and analyze the bounded multi-soliton solution for the focusing and defocusing nNLSE (1) in a uniform frame by the method of DT. Throughout the studies in this work, we find that the feature of soliton for the nNLSE is different from the classic NLSE in the following aspects. The amplitude of the soliton solution to the nNLSE is jointly determined by the spectral parameter and the solution parameter, but for the solitons of NLSE the amplitude of soliton is uniquely determined by the spectral parameter. The exponentially blow up and decay solution can admit the oscillating effect. And some special parameter setting will result in the singularity for the solutions, which can not appear for the solitons of classic NLSE. The bounded multi-soliton solutions also have the elastic interaction. These interesting dynamics would enrich the dynamics for the field of nonlinear physics.

We construct the N-fold DT for the nNLSE (1) by the loop group method. Then we use the DT to the determinant representation of multi-soliton solutions by zero seed solution. Afterwards, the singular and asymptotic analysis on multi-soliton solutions are performed by the formula of determinant. Actually, we propose a way to analyze the singularity and asymptotic analysis for the nonlocal type of NLS equation. This method can be readily extended to the multi-component equation [26], two-place and four-place nonlocal integrable equation, multi-place nonlocal KP equation and so on.

As a matter of fact, there are lots of work to be performed. When constructing the solitonic solution, we only consider the zero seed solution. We can construct the solitonic solution by the plane wave solution or elliptic function seed solution. Meanwhile, the high order or the multi-pole solitons with large order are also deserved to study for these models. In addition, besides singularity and asymptotics, soliton solutions have many other noteworthy properties to be further explored. The above mentioned problems will be studied in the near future.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Grant No. 11771151), the Guangzhou Science and Technology Program of China (Grant No. 201904010362), the Fundamental Research Funds for the Central Universities of China (Grant No. 2019MS110).

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Statistics and Probability Letters 168 (2021) 108922

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Random covering sets in metric space with exponentially mixing property

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ARTICLE INFO

Article history:
Received 19 May 2020
Received in revised form 27 August 2020
Accepted 28 August 2020
Available online 6 September 2020

Keywords:
Random covering sets
Exponentially mixing
Dynamical system
Metric space
Huasdorff dimension

ABSTRACT

Let $\{B(\xi_n, r_n)\}_{n\geq 1}$ be a sequence of random balls whose centers $\{\xi_n\}_{n\geq 1}$ is a stationary process, and $\{r_n\}_{n\geq 1}$ is a sequence of positive numbers decreasing to 0. Our object is the random covering set $E = \limsup_{n\to\infty} B(\xi_n, r_n)$, that is, the points covered by $B(\xi_n, r_n)$ infinitely often. The sizes of E are investigated from the viewpoint of measure, dimension and topology.

1. Introduction

Let (X, d) be a complete metric space. Given a sequence of points $\{x_n\}_{n\geq 1}$ in X, let $\{r_n\}_{n\geq 1}$ be a sequence of positive numbers decreasing to 0. A general covering problem concerns the sets

$$\limsup_{n\to\infty} B(x_n, r_n) = \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} B(x_n, r_n),$$

where $B(x_n, r_n)$ denotes the ball centered at x_n of radius r_n. Many authors have investigated the size and structure of these limsup sets.

One of classical models is to let $\{x_n\}_{n\geq 1}$ be a sequence of random variables on $(\Omega, \mathcal{B}, \mathbb{P})$. The limsup set $E = \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} B(x_n, r_n)$ is usually called random covering set, since it consists of the points covered by random balls $\{B(x_n, r_n)\}$ infinitely many times. The study of random covering sets has a long and convoluted history (Durand, 2010; Ekström, 2019; Ekström et al., 2018; Fan, 2002; Li et al., 2013; Li and Suomala, 2014; Persson, 2015).

In 1956, Dvoretzky (1956) called the attention on the study of such random covering sets in the circle $\mathbb{T}$ with $\{x_n\}_{n\geq 1}$ being independent and uniformly distributed. He asked the question when $E = \mathbb{T}$ a.s. or not. There was a series of contributions. In 1971, Shepp (1972) gave a sufficient and necessary condition: $E = \mathbb{T}$ a.s. if and only if $\sum_{n=1}^{\infty}(1/n^2)\exp(r_1 + \cdots + r_n) = \infty$. Kahane (1985) proved that E is a.s. dense on $\mathbb{T}$ and moreover of second category. The applications of the Borel–Cantelli lemma and Fubini's theorem give that the Lebesgue measure of E is 0 or 1 a.s. according to the convergence or divergence of the series $\sum_{n=1}^{\infty} r_n$. Many authors have studied the Hausdorff dimension and other fractal properties of random covering set E. Fan and Wu (2004) considered the special case $r_n = a/n^{\alpha}$ with $a > 0$ and $\alpha > 1$. They proved that $\dim_{\mathrm{H}}(E) = 1/\alpha$ a.s., where $\dim_{\mathrm{H}}$ denotes Hausdorff dimension. Durand (2010) considered a

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https://doi.org/10.1016/j.spl.2020.108922
0167-7152/

general sequence $\{r_n\}_{n\geq1}$ and proved $\dim_{\mathrm{H}} E = \inf\{s > 0 : \sum_{n=1}^{\infty} r_n^s < \infty\}$ and $\dim_{\mathrm{P}} E = 1$ a.s., where $\dim_{\mathrm{P}}$ denotes packing dimension.

Many variations of the random covering problems have been addressed by many mathematicians. For example, Järvenpää et al. (2014) covered the torus by self-affine sets instead of balls. Feng et al. (2018) extended it to any open sets. The covering model in Ahlfors regular metric space has been studied in Järvenpää et al. (2017).

Instead of a sequence of random variables, many authors considered the case that $\{x_n\}_{n\geq1}$ is the orbit of a dynamical system (X, T). Define the dynamical covering set $E(x, r_n) = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B(T^{n-1}x, r_n)$. Fan et al. (2013) computed the Hausdorff dimension of $E(x, r_n)$ when X is the unit interval and $T : x \mapsto 2x \pmod 1$. In 2013, Liao and Seuret (2013) successfully enlarged the setting to finite expanding Markov map with Gibbs measure m. They proved that if $r_n = n^{-\alpha}$, then $\dim_{\mathrm{H}} E(x, r_n) = 1/\alpha$ a.e. provided that $1/\alpha$ is not larger than the dimension of the measure m. Persson and Rams (2017) considered more general piecewise expanding maps than Markov maps. In 2017, Wang et al. (2017) considered the dynamical covering problem on the triadic cantor set.

In this paper, we consider the covering set with $\{\xi_n\}_{n\geq1}$ which is a sequence of points in a compact metric space (X, d), chosen randomly. And independence of $\{\xi_n\}_{n\geq1}$ is not necessary. The purpose of this article is to study some properties of random covering set in general probabilistic setting, including measure, density, fractal dimensions and so on. Next we will state the main results and provide some discussions. Section 2 is devoted to the proof of the results. In Section 3, we will give an application to the dynamical covering problem.

Let $\{\xi_n\}_{n\geq1}$ be a stationary process on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and take values in a compact metric space (X, d). Let μ be the probability measure defined by

$$\mu(A) = \mathbb{P}(\xi_1 \in A) \tag{1.1}$$

for any Borel set $A \subset X$. Assume that X is the support of μ.

We say that $\{\xi_n\}_{n\geq1}$ is *exponentially mixing* if for any $n \geq 1$, there exist two constants $c > 0$ and $0 < \gamma < 1$ such that

$$\left|\mathbb{P}(\xi_1 \in A|D) - \mathbb{P}(\xi_1 \in A)\right| \leq c\gamma^n$$

holds for any ball $A \subset X$ and $D \in \mathcal{B}^{n+1}$, where $\mathcal{B}^{n+1}$ is the sub-σ-field generated by $\{\xi_{n+i}\}_{i\geq1}$.

Recall that a Borel measure μ is *Ahlfors s-regular* $(0 < s < \infty)$ if there exists a constant $0 < C < \infty$ such that

$$C^{-1}r^s \leq \mu\big(B(x, r)\big) \leq Cr^s \tag{1.2}$$

holds for all $x \in X$ and $0 < r \leq \operatorname{diam} X$, where $\operatorname{diam} X$ is the diameter of X.

A metric space X is said to be *Ahlfors s-regular* if there exists a Borel measure on X satisfying formula (1.2).

Let $\{r_n\}_{n\geq1}$ be a sequence of positive real numbers decreasing to zero. For every $n \geq 1$, denote $B_n := B(\xi_n, r_n)$. Define

$$E := \limsup_{n\to\infty} B_n = \{y \in X : y \in B_n \text{ for infinitely many } n \geq 1\}.$$

The set E is a *random covering set* and consists of the points which are covered by $\{B_n\}_{n\geq1}$ infinitely often (i.o. for short).

Theorem 1.1. *Let $\{\xi_n\}_{n\geq1}$ be exponentially mixing and the probability measure μ defined in (1.1) be Ahlfors s-regular. Then we have*

$$\mu\big(E\big) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} r_n^s < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} r_n^s = \infty \end{cases} \quad a.s.$$

A dimension function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and non-decreasing function such that $f(r) \to 0$ as $r \to 0$. If there exists a constant $\eta > 1$ such that for $r > 0$, $f(2r) \leq \eta f(r)$, then we say that function f is *doubling*.

Theorem 1.2. *Let $\{\xi_n\}_{n\geq1}$ be exponentially mixing and the probability measure μ defined in (1.1) be Ahlfors s-regular. Suppose that f is a doubling dimension function with $f(r)/r^s$ being nondecreasing as $r \to 0$. Then, with probability one, for any ball B of X,*

$$\mathcal{H}^f(E \cap B) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} f(r_n) < \infty \\ \mathcal{H}^f(B) & \text{if } \sum_{n=1}^{\infty} f(r_n) = \infty. \end{cases}$$

Furthermore,

$$\dim_{\mathrm{H}} E = \alpha \quad a.s.,$$

where $\alpha = \inf\{t \leq s : \sum_{n=1}^{\infty} r_n^t < \infty\}$.

Theorem 1.3. *Let $\{\xi_n\}_{n\geq1}$ be exponentially mixing and the probability measure μ be Ahlfors s-regular. Then random covering set E is dense in X almost surely.*

Corollary 1.4. *Assume that the conditions of Theorem 1.2 hold. We have $\dim_{\mathrm{B}} E = s$ almost surely, where $\dim_{\mathrm{B}}$ denotes box dimension.*

Recall that a set is called *residual* if the complement of the set is a first category set.

Theorem 1.5. *Let $\{\xi_n\}_{n\geq 1}$ be exponentially mixing and the probability measure μ be Ahlfors s-regular. Then random covering set E is a residual set almost surely. And E is also a set of second category almost surely. In particular* $\dim_P E = s$ *almost surely.*

Remark 1.1.

(i) Heinonen (2001) proved that, if X is a metric space admitting a Borel measure μ which is *Ahlfors s-regular* $(0 < s < \infty)$, then X has Hausdorff dimension precisely s.
(ii) From the proof of Corollary 1.4, we see that the box dimension of the space we considered is s.
(iii) By Cutler (1995, Theorem 3.16), we derive that the packing dimension of the space we considered is s.

2. Proofs of the main results

In this section we proved Theorems 1.1–1.5 and Corollary 1.4.

Lemma 2.1. *Suppose that $\{\xi_n\}_{n\geq 1}$ is an exponentially mixing stationary process. Let $\{h_n\}_{n\geq 1}$ be a decreasing sequence. For any point $y \in X$, if the series $\sum_{n=1}^{\infty} h_n^s$ diverges, we have $\mathbb{P}(\xi_n \in B(y, h_n)$ i.o.$) = 1$.*

Proof. Let $y \in X$ and denote $\widetilde{J}_n = \{\omega \in \Omega : \xi_n(\omega) \in B(y, h_n)\}$. Let $N \geq 1$ and $S_N = \sum_{n=1}^{N} \chi_{\widetilde{J}_n}$, where χ is the indicator function. Then

$$\mathbb{E}(S_N) = \sum_{n=1}^{N} \mathbb{P}(\widetilde{J}_n) = \sum_{n=1}^{N} \mu\big(B(y, h_n)\big).$$

Since μ is Ahlfors s-regular, by (1.2), we have

$$\mathbb{E}(S_N) \geq C^{-1} \sum_{n=1}^{N} h_n^s \to \infty, \quad \text{as } N \to \infty.$$

Thus $\sum_{n=1}^{\infty} \mathbb{P}(\widetilde{J}_n) = \lim_{N\to\infty} \mathbb{E}(S_N) = \infty$.

By the Paley–Zygmund inequality, for all $0 < \lambda < 1$, we have

$$\begin{aligned} \mathbb{P}\left(S_N \geq \lambda \mathbb{E}(S_N)\right) &\geq (1-\lambda)^2 \frac{\mathbb{E}^2(S_N)}{\mathbb{E}(S_N^2)} \\ &= (1-\lambda)^2 \frac{\left(\sum_{n=1}^{N} \mathbb{P}(\widetilde{J}_n)\right)^2}{\mathbb{E}(S_N^2)}. \end{aligned} \tag{2.1}$$

Now we estimate $\mathbb{E}(S_N^2)$,

$$\begin{aligned} \mathbb{E}(S_N^2) &= \mathbb{E}\left(\sum_{n=1}^{N} \chi_{\widetilde{J}_n}\right)^2 = \mathbb{E}\Big(\sum_{n=1}^{N} \chi_{\widetilde{J}_n} + \sum_{n=1}^{N} \sum_{\substack{m=1 \\ m\neq n}}^{N} \chi_{\widetilde{J}_n \cap \widetilde{J}_m}\Big) \\ &= \sum_{n=1}^{N} \mathbb{P}(\widetilde{J}_n) + 2 \sum_{n=1}^{N} \sum_{m=1}^{n-1} \mathbb{P}(\widetilde{J}_n \cap \widetilde{J}_m). \end{aligned} \tag{2.2}$$

Using the stationarity and the exponentially mixing property of the process $\{\xi_n\}$, we get

$$\begin{aligned} \mathbb{P}(\widetilde{J}_n \cap \widetilde{J}_m) &= \mathbb{P}\{\xi_n \in B(y, h_n), \xi_m \in B(y, h_m)\} = \mathbb{P}\{\xi_1 \in B(y, h_m), \xi_{n-m+1} \in B(y, h_n)\} \\ &\leq \mathbb{P}\big(\xi_1 \in B(y, h_m)\big)\mathbb{P}\big(\xi_{n-m+1} \in B(y, h_n)\big) + c\gamma^{n-m}\mathbb{P}\big(\xi_{n-m+1} \in B(y, h_n)\big) \\ &= \mathbb{P}(\widetilde{J}_n)\mathbb{P}(\widetilde{J}_m) + c\gamma^{n-m}\mathbb{P}(\widetilde{J}_n). \end{aligned}$$

Hence the equality (2.2) reads as follows

$$\begin{aligned}\mathbb{E}(S_N^2) &\le \sum_{n=1}^{N}\mathbb{P}(\widetilde{J}_n)+2\sum_{n=1}^{N}\sum_{m=1}^{n-1}\left(\mathbb{P}(\widetilde{J}_n)\mathbb{P}(\widetilde{J}_m)+c\gamma^{n-m}\mathbb{P}(\widetilde{J}_n)\right)\\ &\le \sum_{n=1}^{N}\mathbb{P}(\widetilde{J}_n)+\left(\sum_{n=1}^{N}\mathbb{P}(\widetilde{J}_n)\right)^2+\frac{2c\gamma}{1-\gamma}\sum_{n=1}^{N}\mathbb{P}(\widetilde{J}_n)\\ &\le c'\sum_{n=1}^{N}\mathbb{P}(\widetilde{J}_n)+\left(\sum_{n=1}^{N}\mathbb{P}(\widetilde{J}_n)\right)^2,\end{aligned} \tag{2.3}$$

where c' is a constant. Combining (2.1) and (2.3), we derive that

$$\mathbb{P}\left(S_N\ge\lambda\mathbb{E}(S_N)\right)\ge(1-\lambda)^2\frac{\left(\sum_{n=1}^{N}\mathbb{P}(\widetilde{J}_n)\right)^2}{c'\sum_{n=1}^{N}\mathbb{P}(\widetilde{J}_n)+\left(\sum_{n=1}^{N}\mathbb{P}(\widetilde{J}_n)\right)^2}\to 1, \tag{2.4}$$

as $N\to\infty$ and $\lambda\to 0$ due to the divergence of the series $\sum_{n=1}^{\infty}\mathbb{P}(\widetilde{J}_n)$. We notice that

$$\{\omega\in\widetilde{J}_n\ i.o.\}=\{\lim_{N\to\infty}S_N=\infty\}\supset\{S_N\ge\lambda\mathbb{E}(S_N)\}.$$

By (2.4) we have

$$\mathbb{P}(\widetilde{J}\ i.o.)\ge\lim_{N\to\infty}\mathbb{P}\big(S_N\ge\lambda\mathbb{E}(S_N)\big)=1,$$

which derives $\mathbb{P}(\widetilde{J}\ i.o.)=1$. $\square$

Proof of Theorem 1.1. First we show that $\mu\big(E\big)=0$ for any $\omega\in\Omega$ if $\sum_{n=1}^{\infty}r_n^s<\infty$.

Since μ is Ahlfors s-regular, by (1.2), we have $\sum_{n=1}^{\infty}\mu(B_n)\le C\sum_{n=1}^{\infty}r_n^s<\infty$. By the Borel–Cantelli lemma, $\mu(\limsup_{n\to\infty}B_n)=0$, that is $\mu\big(E\big)=0$.

Now we consider the divergence case. Let $y\in X$ and

$$F(y)=\{\omega\in\Omega\colon\xi_n(\omega)\in B(y,r_n)\ i.o.\}.$$

By Lemma 2.1, we have $\mathbb{P}\big(F(y)\big)=1$. Since $y\in E\Leftrightarrow\omega\in F(y)$, applying Fubini's theorem gives

$$\begin{aligned}\mathbb{P}\left(\mu(E)\right)&=\int\int 1_E(y)d\mu(y)d\mathbb{P}(\omega)=\int\int 1_{F(y)}(\omega)d\mathbb{P}(\omega)d\mu(y)\\ &=\int\mathbb{P}\big(F(y)\big)d\mu(y)=1.\end{aligned}$$

Hence, for $\mathbb{P}$-almost all ω, $\mu\big(E\big)=1$. $\square$

We recall a general case of the mass transference principle (Beresnevich and Velani, 2006, Theorem 3), suitable for the proof of our Theorem 1.2.

Let (Y,ρ) be a locally compact metric space. Let g be a doubling dimension function and suppose there exist constants $0<c_1<1<c_2<\infty$ and $r_0>0$ such that

$$c_1g\big(r(B)\big)\le\mathcal{H}^g(B)\le c_2g\big(r(B)\big),$$

for any ball $B=B(x,r)$ with $x\in Y$ and $r\le r_0$. Next, given a dimension function f and a ball $B=B(x,r)$ we define $B^f:=B\big(x,g^{-1}f(r)\big)$.

Theorem 2.2 (*Beresnevich–Velani*). *Let (Y,ρ) and g be as above and let $\{B_i\}_{i\in\mathbb{N}}$ be a sequence of balls in Y with $r(B_i)\to 0$ as $i\to\infty$. Let f be a dimension function such that $f(r)/g(r)$ is monotonic and suppose that for any ball B in Y*

$$\mathcal{H}^g(B\cap\limsup_{i\to\infty}B_i^f)=\mathcal{H}^g(B).$$

Then, for any ball B in Y

$$\mathcal{H}^f(B\cap\limsup_{i\to\infty}B_i^g)=\mathcal{H}^f(B).$$

Proof of Theorem 1.2. For any $\delta > 0$, there is an integer $n_0 \geq 1$ such that $0 < 2r_n < \delta$ for any $n \geq n_0$. Since $f(r)/r^s$ is increasing as $r \to 0$ and $E \subset \bigcup_{n=n_0}^{\infty} B_n$, we have

$$\mathcal{H}^f_\delta(E) \leq \sum_{n=n_0}^{\infty} f(2r_n) \leq 2^s \sum_{n=n_0}^{\infty} f(r_n).$$

If the series $\sum_{n=1}^{\infty} f(r_n)$ converges, then $\sum_{n=n_0}^{\infty} f(r_n) \to 0$ as $\delta \to 0$ since it deduces that $n_0 \to \infty$. Therefore $\mathcal{H}^f(E) = \lim_{\delta\to 0} \mathcal{H}^f_\delta(E) = 0$ for any $\omega \in \Omega$.

For the divergence part, denote $B^f_n = B\big(\xi_n, (f(r_n))^{1/s}\big)$. By Theorem 1.1, we obtain

$$\mu\big(\limsup_{n\to\infty} B^f_n\big) = 1 \text{ a.s.},$$

since

$$\sum_{n=1}^{\infty} \big((f(r_n))^{1/s}\big)^s = \sum_{n=1}^{\infty} f(r_n) = \infty.$$

By Theorem 2.2, we have

$$\mathcal{H}^f(E \cap B) = \mathcal{H}^f(B) \text{ a.s.} \quad \square$$

For proving Theorems 1.3, 1.5 and Corollary 1.4 we will use the following elementary lemma, whose proof can be found in Munkres (2000).

Lemma 2.3. *If the metric space (X, d) is compact, then X is bounded, separated and complete.*

Proof of Theorem 1.3. The compact metric space (X, d) is separated, hence there exists a countable dense subset denoted by A. Letting $\mathcal{A} = \{B(a, 1/k) | a \in A, k \geq 1\}$, now we prove that $\mathcal{A}$ is a countable base of (X, d).

For any open subset U in X and $x \in U$, we can find a $\epsilon_x > 0$ such that $B(x, \epsilon_x) \subset U$. There exists $k_x \geq 1$ with $2/k_x < \epsilon_x$. Since A is dense in X, we can choose $a_x \in A$ satisfying $d(x, a_x) < 1/k_x$. Hence $x \in B(a_x, 1/k_x) \subset B(x, \epsilon_x) \subset U$. It yields that $U = \bigcup_{x\in U}\{x\} \subset \bigcup_{x\in U} B(a_x, 1/k_x) \subset U$, that is $U = \bigcup_{x\in U} B(a_x, 1/k_x)$, where $B(a_x, 1/k_x) \in \mathcal{A}$. Therefore $\mathcal{A}$ is a countable base of (X, d), since $\{B(a_x, 1/k_x) | x \in U\}$ is at most countable.

For any $B = B(a, r) \in \mathcal{A}$, we will show $\mathbb{P}(E \cap B \neq \varnothing) = 1$.

Denote $\widetilde{A}_n = \{\omega \in \Omega : B_n(\omega) \cap B \neq \varnothing\}$. We notice that $\widetilde{A}_n = \{\xi_n \in B(a, r + r_n)\}$. Then we have

$$\sum_{n=1}^{N} (r + r_n)^s \geq \sum_{n=1}^{N} r^s \to \infty, \quad \text{as } N \to \infty.$$

Thus the series $\sum_{n=1}^{\infty} (r + r_n)^s$ diverges. Hence $\mathbb{P}(E \cap B \neq \varnothing) = \mathbb{P}(\widetilde{A}_n \ i.o.) = 1$ due to Lemma 2.1.

For convenience, we denote $\mathcal{A} = \{B^i\}_{i\geq 1}$. For any $B^i \in \mathcal{A}$, let $A_i = \{\omega | B^i \cap E = \varnothing\}$, then $\mathbb{P}(A_i) = 0$. Thus

$$\mathbb{P}\{\omega | E \cap B^i = \varnothing \text{ for some } i \geq 1\} = \mathbb{P}\big(\bigcup_{i\geq 1} A_i\big) \leq \sum_{i\geq 1} \mathbb{P}(A_i) = 0.$$

So there exists a null probability event outside which for all balls B^i in $\mathcal{A}$, we have $E \cap B^i \neq \varnothing$. For any $x \in X$ and open set $U \ni x$, there exist $\{B^{i_n}\}_{n\geq 1} \subset \mathcal{A}$ with $U = \bigcup_{n\geq 1} B^{i_n}$. Hence

$$\mathbb{P}\{E \cap U \neq \varnothing\} \geq \mathbb{P}\{\omega | E \cap B^i \neq \varnothing \text{ for any } i \geq 1\} = 1.$$

Therefore E is dense in X a.s. $\square$

Proof of Corollary 1.4. Since X is bounded, let $\{y_k\}_{k=1}^n$ be a maximal r-separated set of X ($r > 0$). Then the balls $B(k) = B(y_k, r/3)$ are disjoint, that is $B(k) \cap B(j) = \varnothing$ for any $k \neq j$. Thus, by (1.2)

$$C^{-1} 3^{-s} n r^s \leq \mu\big(B(1)\big) + \cdots + \mu\big(B(n)\big) \leq 1.$$

On the other hand, the balls $\{3B(k)\}_{k=1}^n$ cover X where $3B(k) = B(y_k, r)$, so

$$1 \leq \mu\big(3B(1)\big) + \cdots + \mu\big(3B(n)\big) \leq C n r^s.$$

Combining these estimates, it follows that $\dim_{\mathrm{B}} X = s$. From Theorem 1.2, we have $\dim_{\mathrm{B}} E = \dim_{\mathrm{B}} X = s$ *a.s.* $\square$

Proof of Theorem 1.5. First we show that E is a residual set a.s.

Now we check if $E^C = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} B_n^C$ is a first category set almost surely. Denote $E_N^C = \bigcap_{n=N}^{\infty} B_n^C$. We suppose that there exists $N_1 \geq 1$ such that $E_{N_1}^C$ is not sparse. That means there is a ball $B(y, r)$ with $r > 0$ satisfying

$$\overline{B}(y, r) \subset \overline{E_{N_1}^C} = E_{N_1}^C = X \setminus \bigcup_{n=N_1}^{\infty} B_n.$$

Hence $B(y, r) \cap \bigcup_{n=N_1}^{\infty} B_n = \varnothing$. However $\bigcup_{n=N_1}^{\infty} B_n$ is dense in X a.s., since $E \subset \bigcup_{n=N_1}^{\infty} B_n$ is dense in X a.s., which implies $B(y, r) \cap \bigcup_{n=N_1}^{\infty} B_n \neq \varnothing$. Contradiction. Hence E^C is a set of first category almost surely. It derives that E is a residual set a.s.

From Lemma 2.3, we know the space X is complete. Then by Baire's category theorem, X is a set of second category. We have shown that E^C is a first category set a.s. Thus E is second category a.s. It follows that $\dim_P E = \dim_P X = s$ a.s. □

3. Dynamical covering problem

Our results can be applied to the dynamical covering problem as follows.

We say that a metric measure preserving system (m.m.p.s. for short) $(X, \mathcal{B}, \mu, T, d)$ is *exponentially mixing* if there exist two constants $c > 0$ and $0 < \gamma < 1$ such that

$$|\mu(E|T^{-n}F) - \mu(E)| \leq c\gamma^n \qquad (n \geq 1)$$

holds for any ball E and any measurable set $F \in \mathcal{B}$ with $\mu(F) > 0$. Here $\mu(A|B)$ denotes $\frac{\mu(A\cap B)}{\mu(B)}$. Sometimes we say μ is exponentially mixing.

We assume that (X, d) is compact, endowed with a Borel probability measure μ which is Ahlfors s-regular ($0 < s < \infty$). Define the *dynamical covering set* as

$$E(x, l_n) = \left\{y \in X : d(T^n x, y) < l_n \text{ for infinitely many } n \geq 0\right\},$$

where $x \in X$ and $\{l_n\}_{n\geq 0}$ is a sequence of positive real numbers which is decreasing to zero. The dynamical covering set $E(x, l_n) = \bigcap_{k=0}^{\infty} \bigcup_{n=k}^{\infty} B(T^n x, l_n)$. We study the size of $E(x, l_n)$ from the viewpoint of measure, dimension and topology.

Theorem 3.1. *Let $(X, \mathcal{B}, \mu, T, d)$ be an exponentially mixing m.m.p.s. and the measure μ be Ahlfors s-regular. For μ-almost all $x \in X$, we have*

$$\mu\big(E(x, l_n)\big) = \begin{cases} 0 & \text{if } \sum_{n=0}^{\infty} l_n^s < \infty \\ 1 & \text{if } \sum_{n=0}^{\infty} l_n^s = \infty. \end{cases}$$

Theorem 3.2. *Let $(X, \mathcal{B}, \mu, T, d)$ be an exponentially mixing m.m.p.s. and the measure μ be Ahlfors s-regular. Assume that f is a doubling dimension function with $f(r)/r^s$ being nondecreasing as $r \to 0$. Then for any ball B of X,*

$$\mathcal{H}^f(E(x, l_n) \cap B) = \begin{cases} 0 & \text{if } \sum_{n=0}^{\infty} f(l_n) < \infty \\ \mathcal{H}^f(B) & \text{if } \sum_{n=0}^{\infty} f(l_n) = \infty \end{cases}$$

holds for μ-almost all $x \in X$. Furthermore

$$\dim_H E(x, l_n) = \alpha \text{ a.e.},$$

where $\alpha = \inf\{t \leq s : \sum_{n=0}^{\infty} l_n^t < \infty\}$.

Theorem 3.3. *Let $(X, \mathcal{B}, \mu, T, d)$ be an exponentially mixing m.m.p.s. and μ be Ahlfors s-regular. Then the dynamical covering set $E(x, l_n)$ is dense in X a.e..*

Corollary 3.4. *Let $(X, \mathcal{B}, \mu, T, d)$ be an exponentially mixing m.m.p.s. and μ be Ahlfors s-regular. For μ-almost all $x \in X$, we have $\dim_B E(x, l_n) = s$.*

Theorem 3.5. *Let $(X, \mathcal{B}, \mu, T, d)$ be an exponentially mixing m.m.p.s. and μ be Ahlfors s-regular. Therefore $E(x, l_n)$ is a residual set and moreover of second category for μ-almost all $x \in X$. In particular, $\dim_P E(x, l_n) = s$ a.e.*

Remark 3.1. Now we give some systems which are exponentially mixing.

(i) For the doubling map $Tx = 2x \pmod 1$ on the interval $[0, 1)$, Gibbs measure μ associated to Hölder potentials is exponentially mixing.
(ii) For the β-shift $T_\beta x = \beta x \pmod 1$ on the interval $[0, 1)$, the Parry measure is exponentially mixing.
(iii) For the Gauss map $Sx = \{\frac{1}{x}\} \pmod 1$ on the interval $[0, 1)$, the Gauss measure is exponentially mixing.

Acknowledgments

This work is supported by NSFC 11671151 and Guangdong Natural Science Foundation 2018B0303110005.

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报告编号：2025-2364

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中华人民共和国教育部科技查新站（SH01）

论文作者： 胡张楠

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论文发表年限： 2021-2025 年

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来源出版物: JOURNAL OF THE LONDON MATHEMATICAL SOCIETY-SECOND SERIES 卷: 112 期: 3 文献号: e70287 DOI: 10.1112/jlms.70287 Published Date: 2025 SEP
Web of Science 核心合集中的 "被引频次": 0
被引频次合计: 0
入藏号: WOS:001579470500015
文献类型: Article
地址: [Hu, Zhang-nan] China Univ Petr, Coll Sci, Beijing, Peoples R China.
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电子邮件地址: tomasp@gmx.com
ISSN: 0024-6107
eISSN: 1469-7750
2024 年期刊的影响因子：1.2
2025 年期刊的中科院分区（升级版）：大类 数学 2

第 2 条，共 8 条
标题: Hausdorff dimension of recurrence sets for matrix transformations of tori
作者: Hu, ZN (Hu, Zhangnan); Li, B (Li, Bing)
来源出版物: ACTA MATHEMATICA SCIENTIA 卷: 45 期: 4 页: 1659-1673 DOI: 10.1007/s10473-025-0422-1 Published Date: 2025 JUL
Web of Science 核心合集中的 "被引频次": 0
被引频次合计: 0
入藏号: WOS:001465507500002
文献类型: Article
地址: [Hu, Zhangnan] China Univ Petr, Coll Sci, Beijing 102249, Peoples R China.
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通讯作者地址: Li, B (通讯作者)，South China Univ Technol, Sch Math, Guangzhou 510641, Peoples R China.
电子邮件地址: hnlgdxhzn@163.com; scbingli@scut.edu.cn
ISSN: 0252-9602
eISSN: 1572-9087
2024 年期刊的影响因子：1.1
2025 年期刊的中科院分区（升级版）：大类 数学 3

第 3 条，共 8 条
标题: Hausdorff dimension of recurrence sets

作者: Hu, ZN (Hu, Zhang-nan); Persson, T (Persson, Tomas)
来源出版物: NONLINEARITY 卷: 37 期: 5 文献号: 055010 DOI: 10.1088/1361-6544/ad3597
Published Date: 2024 MAY 1
Web of Science 核心合集中的 "被引频次": 4
被引频次合计: 4
入藏号: WOS:001195286600001
文献类型: Article
地址: [Hu, Zhang-nan] China Univ Petr, Coll Sci, Beijing 102249, Peoples R China.
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电子邮件地址: hnlgdxhzn@163.com; tomasp@maths.lth.se
ISSN: 0951-7715
eISSN: 1361-6544
2024 年期刊的影响因子：1.6
2025 年期刊的中科院分区（升级版）：大类 数学 2

第 4 条，共 8 条
标题: Large Intersection Property for Limsup Sets in Metric Space
作者: Hu, ZN (Hu, Zhang-nan); Li, B (Li, Bing); Yang, LQ (Yang, Linqi)
来源出版物: POTENTIAL ANALYSIS 卷: 60 期: 1 页: 341-363 DOI: 10.1007/s11118-022-10052-7 Early Access Date: NOV 2022 Published Date: 2024 JAN
Web of Science 核心合集中的 "被引频次": 2
被引频次合计: 2
入藏号: WOS:000884154800001
文献类型: Article
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电子邮件地址: hnlgdxhzn@163.com; scbingli@scut.edu.cn; 2981502134@qq.com
ISSN: 0926-2601
eISSN: 1572-929X
2024 年期刊的影响因子：0.8
2025 年期刊的中科院分区（升级版）：大类 数学 3

第 5 条，共 8 条
标题: ON THE HITTING PROBABILITIES OF LIMSUP RANDOM FRACTALS
作者: Hu, ZN (Hu, Zhang-Nan); Cheng, WC (Cheng, Wen-Chiao); Li, B (Li, Bing)
来源出版物: FRACTALS-COMPLEX GEOMETRY PATTERNS AND SCALING IN NATURE AND SOCIETY 卷: 30 期: 03 文献号: 2250055 DOI: 10.1142/S0218348X22500554 Published Date: 2022 MAY
Web of Science 核心合集中的 "被引频次": 1
被引频次合计: 1
入藏号: WOS:000796311900006
文献类型: Article
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电子邮件地址: hnlgdxhzn@163.com; zwq2@faculty.pccu.edu.tw; scbingli@scut.edu.cn
ISSN: 0218-348X
eISSN: 1793-6543
2022 年期刊的影响因子：4.7
2022 年期刊的中科院分区（升级版）：大类 数学 2

第 6 条，共 8 条
标题: On the intersection of dynamical covering sets with fractals
作者: Hu, ZN (Hu, Zhang-Nan); Li, B (Li, Bing); Xiao, YM (Xiao, Yimin)
来源出版物: MATHEMATISCHE ZEITSCHRIFT 卷: 301 期: 1 页: 485-513 DOI: 10.1007/s00209-021-02924-2 Early Access Date: JAN 2022 Published Date: 2022 MAY
Web of Science 核心合集中的 "被引频次": 4
被引频次合计: 4
入藏号: WOS:000744411700001
文献类型: Article
地址: [Hu, Zhang-Nan; Li, Bing] South China Univ Technol, Sch Math, Guangzhou 510641, Peoples R China.
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电子邮件地址: hnlgdxhzn@163.com; scbingli@scut.edu.cn; xiaoy@msu.edu
ISSN: 0025-5874
eISSN: 1432-1823
2022 年期刊的影响因子：0.8
2022 年期刊的中科院分区（升级版）：大类 数学 2

第 7 条，共 8 条
标题: Bounded multi-soliton solutions and their asymptotic analysis for the reversal-time nonlocal nonlinear Schrodinger equation
作者: Tang, WJ (Tang, Wei-Jing); Hu, ZN (Hu, Zhang-nan); Ling, LM (Ling, Liming)
来源出版物: COMMUNICATIONS IN THEORETICAL PHYSICS 卷: 73 期: 10 文献号: 105001 DOI: 10.1088/1572-9494/ac08fb Published Date: 2021 OCT 1
Web of Science 核心合集中的 "被引频次": 5
被引频次合计: 5
入藏号: WOS:000685181000001
文献类型: Article
地址: [Tang, Wei-Jing; Hu, Zhang-nan; Ling, Liming] South China Univ Technol, Dept Math, Guangzhou 510641, Guangdong, Peoples R China.
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电子邮件地址: hnlgdxhzn@163.com
ISSN: 0253-6102

eISSN: 1572-9494
2021 年期刊的影响因子：2.877
2021 年期刊的中科院分区（升级版）：大类　物理与天体物理　3

第 8 条，共 8 条
标题: Random covering sets in metric space with exponentially mixing property
作者: Hu, ZN (Hu, Zhang-nan); Li, B (Li, Bing)
来源出版物: STATISTICS & PROBABILITY LETTERS　卷: 168　文献号: 108922　DOI: 10.1016/j.spl.2020.108922　Published Date: 2021 JAN
Web of Science 核心合集中的 "被引频次": 4
被引频次合计: 4
入藏号: WOS:000579526800004
文献类型: Article
地址: [Hu, Zhang-nan; Li, Bing] South China Univ Technol, Sch Math, Guangzhou 510641, Guangdong, Peoples R China.
通讯作者地址: Li, B (通讯作者)，South China Univ Technol, Sch Math, Guangzhou 510641, Guangdong, Peoples R China.
电子邮件地址: hnlgdxhzn@163.com; scbingli@scut.edu.cn
ISSN: 0167-7152
eISSN: 1879-2103
2021 年期刊的影响因子：0.718
2021 年期刊的中科院分区（升级版）：大类　数学　4

（End）

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