中国石油大学（北京）

2025 年申报硕士研究生指导教师评审材料目录

申报人姓名	陈家麒	申报学科专业	物理学

序号	送审材料名称
1	国自然青年基金（C 类）批准通知（执行年限 2026-2028 年）
2	国自然理论物理专款博士后基金批准通知（执行年限 2023 全年）
3	论文索引证明 1（论文 1 - 国产期刊 中科院一区 top）
4	论文索引证明 2（论文 2-8）
5	论文 1
6	论文 2
7	论文 3
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12	论文 8

国家自然科学基金资助项目批准通知

（包干制项目）

陈家麒　　先生/女士：

根据《国家自然科学基金条例》、相关项目管理办法规定和专家评审意见，国家自然科学基金委员会（以下简称自然科学基金委）决定资助您申请的项目。项目批准号：12505094，项目名称：量子场论计算方法新发展及其在弯曲时空的应用，资助经费：30.00万元，项目起止年月：2026年01月至 2028年 12月，有关项目的评审意见及修改意见附后。

请您尽快登录科学基金网络信息系统（https://grants.nsfc.gov.cn），**认真阅读《国家自然科学基金资助项目计划书填报说明》并按要求填写《国家自然科学基金资助项目计划书》（以下简称计划书）**。对于有修改意见的项目，请您按修改意见及时调整计划书相关内容；如您对修改意见有异议，须在电子版计划书报送截止日期前向相关科学处提出。

请您将电子版计划书通过科学基金网络信息系统（https://grants.nsfc.gov.cn）提交，由依托单位审核后提交至自然科学基金委。自然科学基金委审核未通过者，将退回的电子版计划书修改后再行提交；审核通过者，打印纸质版计划书（一式两份，双面打印）并在项目负责人承诺栏签字，由依托单位在承诺栏加盖依托单位公章，且将申请书纸质签字盖章页订在其中一份计划书之后，一并报送至自然科学基金委项目材料接收工作组。纸质版计划书应当保证与审核通过的电子版计划书内容一致。**自然科学基金委将对申请书纸质签字盖章页进行审核，对存在问题的，允许依托单位进行一次修改或补齐。**

向自然科学基金委提交电子版计划书、报送纸质版计划书并补交申请书纸质签字盖章页截止时间节点如下：

1. **2025年9月5日16点：**提交电子版计划书的截止时间；

2. **2025年9月12日16点：**提交修改后电子版计划书的截止时间；

3. **2025年9月23日：**报送纸质版计划书（一式两份，其中一份包含申请书纸质签字盖章页）的截止时间。

4. **2025年10月9日：**报送修改后的申请书纸质签字盖章页的截止时间。

请按照以上规定及时提交电子版计划书，并报送纸质版计划书和申请书纸质签字盖章页，逾期不报计划书或申请书纸质签字盖章页且未说明理由的，视为自动放弃接受资助；未按要求修改或逾期提交申请书纸质签字盖章页者，将视情况给予暂缓拨付经费等处理。

附件：项目评审意见及修改意见表

国家自然科学基金委员会

2025年8月27日

附件：项目评审意见及修改意见表

项目批准号	12505094	项目负责人	陈家麒	申请代码1	A2601
项目名称	量子场论计算方法新发展及其在弯曲时空的应用				
资助类别	青年科学基金项目（C类）[原青年科学基金项目]		亚类说明		
附注说明					
依托单位	中国石油大学（北京）				
直接费用	30.00 万元		起止年月	2026年01月 至 2028年12月	

通讯评审意见:

<1>具体评价意见:

一、请评述申请项目研究思想的创新性。请详细阐述判断理由。

该项目讨论微扰量子场论中类费曼积分的计算方法及其应用。通过分部积分约化与微分方程方法发展微扰场论，特别是德西特时空中关联函数的计算。

二、请评述申请项目所提出科学问题的价值以及对相关前沿领域的贡献。

申请人特别强调了所发展出的方法在宇宙论中的应用，比如暴胀宇宙学。项目中对费曼积分的约化的想法是非常有启发性的。

三、请评述申请人的创新潜力与研究方案的可行性；如有可能，请对完善研究方案提出建议。

申请人有充分的前期工作基础，并对研究内容提出了详细可行的实施方案，对结果的预测可靠，并对应用有充分考虑。建议优先资助。

四、其他建议

<2>具体评价意见:

一、请评述申请项目研究思想的创新性。请详细阐述判断理由。

本项目致力于发展量子场论中的散射振幅与关联函数的微扰计算方法，并将结果应用于平直以及dS空间中的模型。创新点包括：将IBP约化和微分方程方法进一步应用于dS空间的关联函数树图和圈图的计算，解析计算超出多对数函数的微分方程，继续提升闵氏空间场论的IBP约化效率等。

二、请评述申请项目所提出科学问题的价值以及对相关前沿领域的贡献。

平直时空微扰散射振幅的高圈高点计算在高能物理的理论与唯象学中至关重要，而dS空间的低阶关联函数的结果对宇宙学以及与粒子物理交叉的方向也十分重要。本项目围绕这些方向开展研究十分有意义。

三、请评述申请人的创新潜力与研究方案的可行性；如有可能，请对完善研究方案提出建议。

申请人在高能物理唯象学以及散射振幅基本方法方面都有很好的研究基础，在与本项目直接相关的课题上也有前期工作基础。本项目研究方案可行。

四、其他建议

无

<3>具体评价意见:

一、请评述申请项目研究思想的创新性。请详细阐述判断理由。

de Sitter时空的关联函数对理解早期宇宙起源具有重要意义，该项目试图将已有的平直时空中的散射振幅技巧推广到de Sitter space,是近几年来出现的一个有意义的新方向，具有时效性。

二、请评述申请项目所提出科学问题的价值以及对相关前沿领域的贡献。
目前学界对于de sitter 时空的关联函数的结构了解到并不清楚，该项目试图将 IBP 约化与微分方程方法推广到包含有质量情形的任意dS 时空微扰场论情形，试图讨论其数学结构，能为理解宇宙学关联函数提供帮助。

三、请评述申请人的创新潜力与研究方案的可行性；如有可能，请对完善研究方案提出建议。
申请人前期有大量的关于散射振幅的研究，且已有将其应用到de sitter 时空的准备工作，研究方案预估可行。

四、其他建议

修改意见:

数理科学部

2025年8月27日

国家自然科学基金资助项目批准通知

（预算制项目）

陈家麒　　先生/女士：

根据《国家自然科学基金条例》、相关项目管理办法规定和专家评审意见，国家自然科学基金委员会（以下简称自然科学基金委）决定资助您申请的项目。项目批准号：12247120，项目名称：费曼积分约化关系的数学结构与算法，直接费用：18.00万元，项目起止年月：2023年01月至 2023年 12月，有关项目的评审意见及修改意见附后。

请您尽快登录科学基金网络信息系统（https://grants.nsfc.gov.cn），**认真阅读《国家自然科学基金资助项目计划书填报说明》并按要求填写《国家自然科学基金资助项目计划书》（以下简称计划书）**。对于有修改意见的项目，请您按修改意见及时调整计划书相关内容；如您对修改意见有异议，须在电子版计划书报送截止日期前向相关科学处提出。

请您将电子版计划书通过科学基金网络信息系统（https://grants.nsfc.gov.cn）提交，由依托单位审核后提交至自然科学基金委。自然科学基金委审核未通过者，将退回的电子版计划书修改后再行提交；审核通过者，打印纸质版计划书（一式两份，双面打印）并在项目负责人承诺栏签字，由依托单位科研、财务管理等部门审核、签章并在承诺栏加盖依托单位公章，且将申请书纸质签字盖章页订在其中一份计划书之后，一并报送至自然科学基金委项目材料接收工作组。纸质版计划书应当保证与审核通过的电子版计划书内容一致。**自然科学基金委将对申请书纸质签字盖章页进行审核，对存在问题的，允许依托单位进行一次修改或补齐。**

向自然科学基金委提交电子版计划书、报送纸质版计划书并补交申请书纸质签字盖章页截止时间节点如下：

1. **2022年12月26日16点：**提交电子版计划书的截止时间；

2. **2023年01月02日16点：**提交修改后电子版计划书的截止时间；

3. **2023年01月09日：**报送纸质版计划书（一式两份，其中一份包含申请书纸质签字盖章页）的截止时间。

请按照以上规定及时提交电子版计划书，并报送纸质版计划书和申请书纸质签字盖章页，逾期不报计划书或申请书纸质签字盖章页且未说明理由的，视为自动放弃接受资助；未按要求修改或逾期提交申请书纸质签字盖章页者，将视情况给予暂缓拨付经费等处理。

附件：项目评审意见及修改意见表

国家自然科学基金委员会

2022年12月06日

附件：项目评审意见及修改意见表

项目批准号	12247120	项目负责人	陈家麒	申请代码1	A25
项目名称	费曼积分约化关系的数学结构与算法				
资助类别	专项项目		亚类说明	研究项目	
附注说明	理论物理专款研究项目				
依托单位	北京计算科学研究中心				
直接费用	18.00 万元		起止年月	2023年01月 至 2023年12月	

通讯评审意见:

<1>该项目研究费曼积分约化关系的数学结构及算法。费曼积分的计算是量子场论中的重要方法。费曼积分的约化虽然已经有一些广为应用的自动计算程序包，但随着粒子物理实验精度的提高对理论预言的精度提高提出了更高的要求，现有程序包还不尽人意。该项目将充分利用约化关系线性方程矩阵的稀疏性、重复性， 尽可能避开其冗余性，提高约化算法的效率。实现新方法与传统的自动化软件包进行效率对比。同时探究约化关系与费曼积分数学性质的联系并对新方法进行唯象应用来检验优越性。该项目在打开费曼积分约化研究的黑盒子，搜寻前人未知新结果方面有一定的创新性。项目申请人在该方向已经有一定的研究基础和成果积累，有望顺利完成项目并取得一定的新成果。

<2>本项目拟研究费曼积分约化关系中的迭代结构，利用约化关系线性方程矩阵的稀疏性、重复性，避开其冗余性，提高约化算法的效率。另外研究多圈的约化，探索降低分子上的不可约标量积，辅助标量积以及分母上的传播子的幂次迭代约化关系式，并最终获得完整的任意费曼积分的迭代约化。项目研究具有理论意义，申请人对相关研究积累了一些经验，获得了不少研究成果，合作导师是著名的理论物理学家罗民兴院士，相信他的项目能顺利完成。建议优先资助。

<3>量子场论的微扰计算中的费曼积分约化关系中的迭代结构研究在粒子物理中具有重要意义，费曼积分的约化在微扰场论的计算与分析费曼积分的数学结构中都有着极为重要的作用。该项目运用计算代数几何、相交理论、辅助质量展开等方法探究约化关系，具有较好的科学价值。
项目立项依据撰写存在一定的缺陷，如选题相关工作阐述不够充分，参考文献偏少，在文献综述后未提出目前研究存在急需解决的问题。缺乏必要的可行性分析。
从整体情况分析来看，申请人的创新潜力尚可，近期研究成果较好。与本人收到的参与评审的其他项目进行综合比较，得出的结果是建议资助。

修改意见:

数理科学部

2022年12月06日

报告编号：2025-2234

论文收录引用

检索证明报告

中华人民共和国教育部科技查新站（SH01）

论文作者： 陈家麒

委托单位： 中国石油大学（北京）

论文发表年限： 2024 年

检索数据库：

SCI-EXPANDED	2001- present	网络版

检索结果：（作者提供文章）

1） SCI-E 收录：有 1 篇被收录

（详细结果见附件）

特此证明！

（盖章）

检索报告人： 王艳敏 **2025 年 10 月 27 日**

附件：

一、 SCI-E 收录情况

第 1 条，共 1 条

标题: Intersection theory rules symbology

作者: Chen, JQ (Chen, Jiaqi); Feng, B (Feng, Bo); Yang, LL (Yang, Lilin)

来源出版物: SCIENCE CHINA-PHYSICS MECHANICS & ASTRONOMY 卷: 67 期: 2 文献号: 221011 DOI: 10.1007/s11433-023-2239-8 Published Date: 2024 FEB

Web of Science 核心合集中的 "被引频次": 19

被引频次合计: 23

入藏号: WOS:001143022500001

文献类型: Article

地址: [Chen, Jiaqi; Feng, Bo] Beijing Computat Sci Res Ctr, Beijing 100084, Peoples R China.

[Feng, Bo; Yang, Lilin] Zhejiang Univ, Sch Phys, Zhejiang Inst Modern Phys, Hangzhou 310027, Peoples R China.

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Feng, B (通讯作者)，Peng Huanwu Ctr Fundamental Theory, Hefei 230026, Peoples R China.

电子邮件地址: jiaqichen@csrc.ac.cn; fengbo@csrc.ac.cn; yanglilin@zju.edu.cn

ISSN: 1674-7348

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2024 年期刊的影响因子：7.5

2025 年期刊中科院分区（升级版）：大类 物理与天体物理 1 TOP

（End）

经委托人确认签字：

报告编号：2025-2307

论文收录引用

检索证明报告

中华人民共和国教育部科技查新站（SH01）

论文作者： **陈家麒**

委托单位： **中国石油大学（北京）**

论文发表年限： **2022-2025 年**

检索数据库：

SCI-EXPANDED	2001- present	网络版

检索结果：（作者提供文章）

1） SCI-E 收录：有 7 篇被收录

（详细结果见附件）

特此证明！

（盖章）

检索报告人： 王艳敏

2025 年 10 月 31 日

附件：

一、 SCI-E 收录情况

第 1 条，共 7 条
标题: The canonical differential equations of the one-loop-like integrals
作者: Chen, JQ (Chen, Jiaqi); Feng, B (Feng, Bo); Zhang, L (Zhang, Liang)
来源出版物: JOURNAL OF HIGH ENERGY PHYSICS 期: 6 文献号: 245 DOI: 10.1007/JHEP06(2025)245 Published Date: 2025 JUN 26
Web of Science 核心合集中的 "被引频次": 0
被引频次合计: 0
入藏号: WOS:001518088800006
文献类型: Article
地址: [Chen, Jiaqi] China Univ Petr, Beijing Key Lab Opt Detect Technol Oil & Gas, Beijing 102249, Peoples R China.
[Chen, Jiaqi] China Univ Petr, Coll Sci, Basic Res Ctr Energy Interdisciplinary, Beijing 102249, Peoples R China.
[Chen, Jiaqi; Feng, Bo; Zhang, Liang] Beijing Computat Sci Res Ctr, Beijing 100084, Peoples R China.
[Feng, Bo] Peng Huanwu Ctr Fundamental Theory, Hefei 230026, Anhui, Peoples R China.
通讯作者地址: Zhang, L (通讯作者)，Beijing Computat Sci Res Ctr, Beijing 100084, Peoples R China.
电子邮件地址: jiaqichen@cup.edu.cn; fengbo@csrc.ac.cn; liangzhang@csrc.ac.cn
eISSN: 1029-8479
2024 年期刊的影响因子：5.5
2025 年期刊中科院分区（升级版）：大类 物理与天体物理 2 TOP

第 2 条，共 7 条
标题: Multivariate hypergeometric solutions of cosmological (dS) correlators by d log-form differential equations
作者: Chen, JQ (Chen, Jiaqi); Feng, B (Feng, Bo); Tao, YX (Tao, Yi-Xiao)
来源出版物: JOURNAL OF HIGH ENERGY PHYSICS 期: 3 文献号: 75 DOI: 10.1007/JHEP03(2025)075 Published Date: 2025 MAR 11
Web of Science 核心合集中的 "被引频次": 6
被引频次合计: 6
入藏号: WOS:001443699700003
文献类型: Article
地址: [Chen, Jiaqi] China Univ Petr, Beijing Key Lab Opt Detect Technol Oil & Gas, Beijing 102249, Peoples R China.
[Chen, Jiaqi] China Univ Petr, Coll Sci, Basic Res Ctr Energy Interdisciplinary, Beijing 102249, Peoples R China.
[Chen, Jiaqi; Feng, Bo] Beijing Computat Sci Res Ctr, Beijing 100084, Peoples R China.
[Feng, Bo] Peng Huanwu Ctr Fundamental Theory, Hefei 230026, Anhui, Peoples R China.
[Tao, Yi-Xiao] Tsinghua Univ, Dept Math Sci, Beijing 100084, Peoples R China.
通讯作者地址: Tao, YX (通讯作者)，Tsinghua Univ, Dept Math Sci, Beijing 100084, Peoples R China.
电子邮件地址: jiaqichen@csrc.ac.cn; fengbo@csrc.ac.cn; taoyx21@mails.tsinghua.edu.cn
ISSN: 1029-8479
2024 年期刊的影响因子：5.5

2025 年期刊中科院分区（升级版）：大类　物理与天体物理　2　TOP

第 3 条，共 7 条
标题: Notes on selection rules of canonical differential equations and relative cohomology
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Intersection theory rules symbology

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We propose a novel method to determine the structure of symbols for any family of polylogarithmic Feynman integrals. Using the d log-bases and simple formulas for the leading order and next-to-leading contributions to the intersection numbers, we give a streamlined procedure to compute the entries in the coefficient matrices of canonical differential equations, including the symbol letters and the rational coefficients. We also provide a selection rule to decide whether a given matrix element must be zero. The symbol letters are deeply related to the poles of the integrands and also have interesting connections to the geometry of Newton polytopes. Our method can be applied to many cutting-edge multi-loop calculations. The simplicity of our results also hints at the possible underlying structure in perturbative quantum field theories.

scattering amplitude, Feynman integral, symbology

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1 Introduction

Perturbative quantum field theories (pQFTs) play a pivotal role in high-precision phenomenology of high energy physics. In many perturbative calculations, one encounters a class of analytic functions called multiple polylogarithms (MPLs) [1, 2]. They can be mapped to symbols [3, 4], which are sequences of $\mathrm{d}\log W_i$, where the W_i's are algebraic functions of kinematic variables known as symbol letters. For a given scattering process, the complete set of symbol letters is called the "alphabet". The knowledge of the alphabet can be used to bootstrap multi-loop integrals and amplitudes [5-29]. This has stimulated extensive research on the construction of symbol alphabets [14, 27, 30-40]. In particular, the symbol letters of one-loop integrals have been fully understood [41-46]. However, beyond one-loop, there are no general results available. On the other hand, from experiences in multi-loop calculations, the expressions of the symbol letters usually turn out to be much simpler than those in the intermediate steps of the calculations. Hence, in addition to the phenomenological motivations, it is also theoretically interesting to investigate the source of such simplicity and to ask whether it implies the existence of simpler rules for symbology.

The symbols in a polylogarithmic integral family are deeply related to the method of canonical differential equations (CDEs) [47-51]. This method has become the most streamlined approach to obtain analytic expressions of Feynman integrals. One chooses a canonical basis of master integrals with uniform transcendentality (UT) [51], and de-

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rives their differential equations with the help of integration-by-parts (IBP) reduction [52]. These differential equations are ϵ-factorized (where $d = 4 - 2\epsilon$ in dimensional regularization) and are dubbed "canonical". The entries of the coefficient matrix, if can be written as total derivatives, directly give the symbol letters. The symbols of the solutions to the CDEs can then be iteratively obtained order-by-order in ϵ. However, converting the coefficient matrix elements to total derivatives can be rather challenging in multivariate situations. Moreover, the procedure of performing the IBP reduction and deriving the CDEs offers little insight into the origin of the symbol letters.

The method of intersection theory [53-56] provides an alternative way to reduce the Feynman integrals to master integrals. It is also useful in the construction of UT bases with d log-form integrands [51, 57-63] in the Baikov representation [64]. In both the computation of intersection numbers and the construction of UT bases, information of poles in the integrands plays a crucial role. In this paper, we show that the information of poles also determines the symbol letters to a certain extent. We employ the method of computing intersection numbers from higher-order partial differential equations [56], and apply it to the differential equations of the UT bases of refs. [58, 59]. We find that with the universal formulas of the leading order (LO) and next-to-leading order (NLO) contributions to intersection numbers, the symbol letters can be generated by localizing the d log-integrands to the multivariate poles. We provide a streamlined procedure to derive all symbol letters in an integral family, that involves the factorization of degenerate poles, followed by simple algebraic operations. This can be applied to many cutting-edge multi-loop calculations in pQFTs.

2 Symbols from intersection theory

The Feynman integrals in the Baikov representation are hypergeometric functions of the form:

$$I[u,\varphi] \equiv \int u\,\varphi\,, \tag{1}$$

where

$$u = \prod_i [P_i(z)]^{\beta_i}\,, \quad \varphi \equiv \hat{\varphi}(z) \bigwedge_j \mathrm{d}z_j = \frac{Q(z)}{\prod_i P_i^{a_i}} \bigwedge_j \mathrm{d}z_j\,. \tag{2}$$

The sequence of Baikov variables is denoted by $\boldsymbol{z} = (z_1, \ldots, z_n)$. The polynomials $P_i(\boldsymbol{z})$ include the Baikov variables themselves, and the Gram determinants $\mathrm{G}(\boldsymbol{q}) \equiv \det(q_i \cdot q_j)$ of loop and external momenta. The exponents take the general form $\beta_i = n_i + m_i\epsilon + l_i\delta_i$, where n_i, m_i, l_i are rational numbers, ϵ is the dimensional regulator, and δ_i is an optional extra regulator. One usually needs to introduce δ_i into the computation if $m_i = 0$, n_i is integer and P_i appears in the denominator, e.g., when P_i is an inverse propagator. The numerator $Q(z)$ is an arbitrary polynomial of z. All integrals with the same u form an integral family, within which one can define IBP-equivalence classes of cocycles [53-56]:

$$\langle\varphi_L| \equiv \varphi_L \sim \varphi_L + \sum_i \nabla_i \xi_i\,, \qquad \nabla_i = \mathrm{d}z_i \wedge (\partial_{z_i} + \hat{\omega}_i), \tag{3}$$

where $\omega \equiv \sum_i \hat{\omega}_i \mathrm{d}z_i$ with $\hat{\omega}_i \equiv \partial_{z_i} \log(u)$. The dual space consists of equivalence classes $|\varphi\rangle$ of integrals $I[u^{-1}, \varphi]$. The intersection number between $\langle\varphi_L|$ and $|\varphi_R\rangle$ is given by

$$\langle\varphi_L|\varphi_R\rangle = \sum_{\boldsymbol{p}} \mathrm{Res}_{z=\boldsymbol{p}}\,(\psi_L \hat{\varphi}_R)\,, \tag{4}$$

where ψ_L is a function satisfying $\nabla_n \cdots \nabla_1 \psi_L = \varphi_L$. The summation goes over all n-variable poles $\boldsymbol{p}$ determined by the zeros of the polynomial factors P_i in u [65]. One complication is that some of these poles can be non-factorized, such that the residue can not be computed variable-by-variable in terms of z_i. A non-factorized pole can also be degenerate, roughly meaning that more than n factors vanish at the pole. A simple example is $u = z_1^{\beta_1} z_2^{\beta_2} (z_1 + z_2)^{\beta_3}$, for which the pole $\boldsymbol{p} = (0, 0)$ is non-factorized and degenerate.

To compute the multivariate residues in the presence of non-factorized poles, one can carry out a factorization procedure [56]. The idea is similar in spirit to the method of sector decomposition [66-72]. This involves a change of variables (labelled by (α)) from z to $\boldsymbol{x}^{(\alpha)}$, such that the pole at $z = \boldsymbol{p}$ corresponds to $\boldsymbol{x}^{(\alpha)} = \boldsymbol{\rho}^{(\alpha)}$, and in the vicinity of the pole

$$u(\boldsymbol{x}^{(\alpha)})\Big|_{\boldsymbol{x}^{(\alpha)} \to \boldsymbol{\rho}^{(\alpha)}} = \bar{u}_\alpha(\boldsymbol{\rho}^{(\alpha)}) \prod_i \left[x_i^{(\alpha)} - \rho_i^{(\alpha)}\right]^{\gamma_i^{(\alpha)}}, \tag{5}$$

where $\bar{u}_\alpha(\boldsymbol{\rho}^{(\alpha)})$ is non-vanishing. This expression defines the u-powers $\gamma_i^{(\alpha)}$ for the variable change (α). Note that for each degenerate pole $\boldsymbol{p}$, one usually needs to sum over several different factorization to correctly reproduce the multivariate residue. To unify the notation, we also give a label (α) for already factorized poles. Hence the summation in eq. (4) is replaced by a summation over α with the residue at $\boldsymbol{x}^{(\alpha)} = \boldsymbol{\rho}^{(\alpha)}$. For the simple example $u = z_1^{\beta_1} z_2^{\beta_2} (z_1 + z_2)^{\beta_3}$, one possible variable change is $z_1 = x_1, z_2 = x_1(x_2 - 1)$. This leads to $u = x_1^{\beta_1+\beta_2+\beta_3} x_2^{\beta_3} (x_2 - 1)^{\beta_2}$. We will discuss more about the factorization of degenerate poles later.

With above discussions, we now study the CDE satisfied by a d log basis $\{\langle\varphi_I|\}$ constructed using the method of refs. [58, 59]. For later convenience, we choose to keep only the dimensional regulator ϵ and the regulators δ_i for propagators in β_i of eq. (2), and absorb all other powers into a_i in φ. There are two types of building blocks (which will be called the "rational-type" and the "sqrt-type" in the following):

$$\begin{aligned} &\mathrm{d}\log(z-c) = \frac{\mathrm{d}z}{z-c}, \\ &\mathrm{d}\log(\tau[z,c;c_\pm]) = \frac{\sqrt{(c-c_+)(c-c_-)}\mathrm{d}z}{(z-c)\sqrt{(z-c_+)(z-c_-)}}, \\ &\tau[z,c;c_\pm] \equiv \frac{\sqrt{c-c_+}\sqrt{z-c_-}+\sqrt{c-c_-}\sqrt{z-c_+}}{\sqrt{c-c_+}\sqrt{z-c_-}-\sqrt{c-c_-}\sqrt{z-c_+}}, \end{aligned} \tag{6}$$

where c and $c_\pm$ are independent of z. The CDE is

$$\langle\dot{\varphi}_I| \equiv \langle\hat{\mathrm{d}}\varphi_I + \varphi_I\hat{\mathrm{d}}\log u| = (\hat{\mathrm{d}}\Omega)_{IJ}\langle\varphi_J|\,, \tag{7}$$

where $\hat{\mathrm{d}}$ denotes the total derivative with respect to external parameters, such as masses and scalar products (to distinguish, d is used for integration variables z). The matrix $\hat{\mathrm{d}}\Omega$ contains all information about the symbol letters, and can be computed by intersection numbers:

$$(\hat{\mathrm{d}}\Omega)_{IK} = \langle\dot{\varphi}_I|\varphi_J\rangle(\eta^{-1})_{JK}\,, \tag{8}$$

where η^{-1} is the inverse of the matrix η with elements $\eta_{IJ} = \langle\varphi_I|\varphi_J\rangle$. Apparently, $(\hat{\mathrm{d}}\Omega)_{IK}$ can be nonzero only if there exist at least one J such that the two factors in the above formula are both nonzero. Note that the $\hat{\mathrm{d}}\Omega$ matrix is independent of the choice of the ket-basis. Here, we choose the ket-basis with the same representatives as the bra-basis. This choice is convenient for computing intersection numbers, and also helps to reveal the selection rule to be discussed later.

We now consider the contributions from the factorized pole $\boldsymbol{x}^{(\alpha)} = \boldsymbol{\rho}^{(\alpha)}$ to the intersection number $\langle\varphi_L|\varphi_R\rangle$. Around the pole, an n-form φ can be Laurent-expanded and organized by the powers $\boldsymbol{b} = (b_1,\ldots,b_n)$. Such a term can be written as:

$$\varphi^{(\boldsymbol{b})} = C^{(\boldsymbol{b})}\bigwedge_i\left[x_i^{(\alpha)}-\rho_i^{(\alpha)}\right]^{b_i}\mathrm{d}x_i^{(\alpha)}\,. \tag{9}$$

In the computation of the intersection number, we know that the contributing terms must have $b_{L,i}+b_{R,i}\le -2$ for all i. A key point is that a d log-form φ_I or φ_J exhibits only multivariate simple poles, i.e., $b_i \ge -1$. The action of $\hat{\mathrm{d}}$ may generate terms with one $b_i=-2$. Hence, we only need to consider two kinds of contributions. The LO contribution [55] has all $b_{L,i}+b_{R,i}=-2$, and can be written as:

$$\frac{C_L^{(\boldsymbol{b}_L)}C_R^{(\boldsymbol{b}_R)}}{\tilde{\gamma}_1^{(\alpha)}\cdots\tilde{\gamma}_n^{(\alpha)}}\,, \tag{10}$$

where $\tilde{\gamma}_i^{(\alpha)} = \gamma_i^{(\alpha)} - b_{R,i} - 1$. The NLO contribution has one $b_{L,j}+b_{R,j}=-3$ and the other $b_{L,i}+b_{R,i}=-2$, and can be written as:

$$-\frac{C_L^{(\boldsymbol{b}_L)}C_R^{(\boldsymbol{b}_R)}}{\tilde{\gamma}_1^{(\alpha)}\cdots\tilde{\gamma}_n^{(\alpha)}}\,\frac{\partial_{\rho_j^{(\alpha)}}\log\left(\bar{u}_\alpha(\boldsymbol{\rho}^{(\alpha)})\right)}{\tilde{\gamma}_j^{(\alpha)}-1}\,. \tag{11}$$

Here and in the following, we treat each component of $\boldsymbol{\rho}^{(\alpha)}$ as an independent external variable. They will be set to the actual expressions in the symbol letters. The detailed derivation of the above results are given in Appendix A1.

From the above discussion, one finds that $\langle\dot{\varphi}_I|\varphi_J\rangle$ can receive nonzero contributions from the pole $\boldsymbol{x}^{(\alpha)} = \boldsymbol{\rho}^{(\alpha)}$ only if $\boldsymbol{b}_I$ and $\boldsymbol{b}_J$ satisfy either of the following two conditions. The first condition is that one component $b_{I,k}+b_{J,k}=-1$, while all other $b_{I,i}=b_{J,i}=-1$. We say that φ_I and φ_J share the $(n-1)$-variable simple pole ($(n-1)$-SP) for $\boldsymbol{x}_{\hat{k}}^{(\alpha)}$, where the subscript $\hat{k}$ means that the kth variable is removed from the sequence. The derivative $\partial_{\rho_k^{(\alpha)}}$ in $\hat{\mathrm{d}}$ generates LO contributions to $\langle\dot{\varphi}_I|\varphi_J\rangle$. Since $C_I^{(\boldsymbol{b}_I)}$ and $C_J^{(\boldsymbol{b}_J)}$ may also depend on $\rho_k^{(\alpha)}$, we need to perform an integration to get a total derivative. The contribution can then be written as:

$$-\frac{\gamma_k^{(\alpha)}}{\boldsymbol{\gamma}^{(\alpha)}}\,\hat{\mathrm{d}}\int C_I^{(\boldsymbol{b}_I)}C_J^{(\boldsymbol{b}_J)}\,\hat{\mathrm{d}}\rho_k^{(\alpha)}\,, \tag{12}$$

where $\boldsymbol{\gamma}^{(\alpha)} \equiv \gamma_1^{(\alpha)}\cdots\gamma_n^{(\alpha)}$. Note that here $\gamma_i = \tilde{\gamma}_i$ for $i \neq k$. In many cases, the product $C_I^{(\boldsymbol{b}_I)}C_J^{(\boldsymbol{b}_J)}$ is proportional to $(\rho_k^{(\alpha)}-c)^{-1}$, and the integral simply gives rise to the letter $\log(\rho_k^{(\alpha)}-c)$. The more complicated situations will be discussed in the next section, where we will show that the letters can always be obtained via purely algebraic operations without performing any integration.

The second condition for a nonzero contribution to $\langle\dot{\varphi}_I|\varphi_J\rangle$ is $\boldsymbol{b}_I = \boldsymbol{b}_J = \boldsymbol{-1}$, i.e., φ_I and φ_J share the n-variable simple pole (n-SP) for $\boldsymbol{x}^{(\alpha)}$. The derivative $\hat{\mathrm{d}}$ now generates NLO contributions. Using eq. (11), the sum of the NLO contributions are

$$\frac{C_I^{(\boldsymbol{-1})}C_J^{(\boldsymbol{-1})}}{\boldsymbol{\gamma}^{(\alpha)}}\,\hat{\mathrm{d}}\log\left(\bar{u}_\alpha(\boldsymbol{\rho}^{(\alpha)})\right). \tag{13}$$

Note that $\bar{u}_\alpha(\boldsymbol{\rho}^{(\alpha)})$ still contains powers β_i. After taking the $\hat{\mathrm{d}}\log$, they become coefficients in front, and the remaining arguments of the logarithms are the symbol letters. We also note that if u has any z-independent constant factors such as $P_0^{\beta_0}$, it is automatically included in $\bar{u}_\alpha$.

We now turn to the matrix element $\eta_{IJ} = \langle\varphi_I|\varphi_J\rangle$, which receives LO contributions (10) if and only if φ_I and φ_J share an n-SP for at least one factorization (α) (hence, η_{II} is always nonzero). To understand when does $(\eta^{-1})_{IJ} \neq 0$, we introduce the concept of n-SP chains. If φ_I and φ_J share an n-SP, we say that they are n-SP related (denoted as $\varphi_I \sim \varphi_J$). If $\varphi_I \sim \varphi_K$ and $\varphi_I \sim \varphi_J$, the three n-forms belong to an n-SP chain. This concept straightforwardly generates to more than three n-forms. One can see that if φ_I and φ_J do not belong to an n-SP chain, then $(\eta^{-1})_{IJ} = 0$ [1)].

1) $(\eta^{-1})_{IJ}$ is proportional to the IJ-minor of η. If φ_I and φ_J do not belong to an n-SP chain, all terms in the minor vanish.

Combining the condition for nonzero $(\eta^{-1})_{IJ}$ and that for nonzero $\langle\dot{\varphi}_I|\varphi_J\rangle$, we arrive at the selection rule for nonzero entries in $\hat{\mathrm{d}}\Omega$: $(\hat{\mathrm{d}}\Omega)_{IJ}$ can be nonzero only if there exists at least one φ_K belonging to an n-SP chain with φ_J, and sharing at least one n-SP or $(n-1)$-SP with φ_I. This selection rule, together with the expressions (12) and (13) of the symbol letters, serves as the most important results of this paper.

Before closing this section, we show from our results that the differential equation of the d log-basis is indeed canonical. Let us assign a transcendental weight of -1 to β_i (contains ϵ and δ_i) in eq. (2). Then, all $\gamma_i^{(\alpha)}$ in eq. (5) have weight-(-1). Since η_{IJ} has the form of eq. (10) (with $\tilde{\gamma}_i^{(\alpha)} = \gamma_i^{(\alpha)}$), the η^{-1} has weight $-n$. Eqs. (12) and (13) have the form of a weight-$(n-1)$ coefficient times a weight-1 $\hat{\mathrm{d}}\log$. Using eq. (8), one can see that $(\hat{\mathrm{d}}\Omega)_{IJ}$ is a weight-(-1) coefficient times a $\hat{\mathrm{d}}\log$. Hence, we have proved that $\hat{\mathrm{d}}\Omega$ is proportional to ϵ when the regulators δ_i are taken to zero.

3 Structure of symbol letters

We now consider the computations of $\langle\dot{\varphi}_I|\varphi_J\rangle$ leading to eqs. (12) and (13) from a different perspective. The expansion of φ_I and φ_J in the form of eq. (9) helps to take the n-variable residue at once. However, we can always choose to take the $(n-1)$-variable residue of $\boldsymbol{x}_{\hat{k}}^{(\alpha)}$ first using eq. (10), and leave the dependence on $x_k^{(\alpha)}$ un-expanded. The leftover 1-form of $x_k^{(\alpha)}$ is a univariate d log-form. This operation applies to both $(n-1)$-SP contributions (where k is fixed) and the n-SP contributions (where one can freely choose any k). Hence, the problem with the single variable $z \equiv x_k^{(\alpha)}$ lies in all contributions to the symbol letters. In this section, we work out this univariate problem generically, and reveal the surprisingly simple structure of symbol letters in the meantime. Details of the derivation are given in Appendix A2.

To warm up, we first consider the case where $c_1 \equiv \rho_k^{(\alpha)}$ is the pole in a rational-type d log. In general, there can be further factors involving z after taking the $(n-1)$-variable residues. Without loss of generality, we take

$$u = P_0^{\beta_0}(z-c_1)^{\beta_1}(z-c_2)^{\beta_2}(z-c_3)^{\beta_3}\,, \tag{14}$$

and more factors can be easily added. The poles c_α do not necessarily correspond to the poles in the original multivariate problem. Nevertheless, we can use the formulas from the previous section to solve this univariate problem. The poles and the corresponding u-powers are

$$c_\alpha \in \{c_1, c_2, c_3, \infty\}\,, \quad \gamma^{(\alpha)} \in \left\{\beta_1, \beta_2, \beta_3, -\sum_{i=1}^{3}\beta_i\right\}, \tag{15}$$

with $\alpha = 1, 2, 3, 4$. The space has dimension 2, and the d log basis can be constructed as:

$$\varphi_I \in \left\{\frac{\mathrm{d}z}{z-c_1}, \frac{\mathrm{d}z}{z-c_2}\right\}. \tag{16}$$

Each φ_I involves two poles, c_I and $c_4 = \infty$. The relevant intersection numbers can be immediately obtained from eqs. (10), (12) and (13):

$$\begin{aligned}\langle\dot{\varphi}_I|\varphi_I\rangle &= \sum_{\alpha\neq I}\frac{\gamma^{(\alpha)}}{\gamma^{(I)}}\,\hat{\mathrm{d}}\log(c_I - c_\alpha) + \eta_{II}\beta_0\,\hat{\mathrm{d}}\log P_0,\\ \langle\dot{\varphi}_I|\varphi_J\rangle &= -\hat{\mathrm{d}}\log(c_I - c_J) + \eta_{IJ}\beta_0\,\hat{\mathrm{d}}\log P_0,\end{aligned} \tag{17}$$

and

$$\eta = \begin{pmatrix} \dfrac{1}{\gamma^{(1)}} + \dfrac{1}{\gamma^{(4)}} & \dfrac{1}{\gamma^{(4)}} \\ \dfrac{1}{\gamma^{(4)}} & \dfrac{1}{\gamma^{(2)}} + \dfrac{1}{\gamma^{(4)}} \end{pmatrix}. \tag{18}$$

It is interesting to see that, after taking the $(n-1)$-variable residues, each symbol letter is either the difference between two univariate poles, or the constant factor P_0 in u.

We now turn to the case where $\rho_k^{(\alpha)}$ appears in a sqrt-type d log. We take (here we drop the constant factor P_0 for simplicity)

$$u = (z-c_1)^{\beta_1}(z-c_2)^{\beta_2}(z-c_+)^{\beta_3}(z-c_-)^{\beta_4}\,. \tag{19}$$

The poles and their corresponding u-powers are

$$\begin{aligned}&c_\alpha \in \{c_1, c_2, \infty, c_+, c_-\},\\ &\gamma^{(\alpha)} \in \left\{\beta_1, \beta_2, -\sum_i \beta_i, \beta_3, \beta_4\right\}.\end{aligned} \tag{20}$$

The d log basis $\{\varphi_I\}$ can be constructed as:

$$\mathrm{d}\log\tau[z, c_1; c_\pm],\ \mathrm{d}\log\tau[z, c_2; c_\pm],\ \mathrm{d}\log\tau[z, \infty; c_\pm]\,. \tag{21}$$

Each φ_I has only one pole at c_I. However, for intersection numbers, the poles at $c_\pm$ can also contribute. We have

$$\begin{aligned}\langle\dot{\varphi}_I|\varphi_I\rangle &= \frac{1}{\gamma^{(I)}}\,\hat{\mathrm{d}}\log(\bar{u}_I(c_I)) - \hat{\mathrm{d}}\log(c_+ - c_-)\\ &\quad + \hat{\mathrm{d}}\log(c_I - c_+) + \hat{\mathrm{d}}\log(c_I - c_-),\\ \langle\dot{\varphi}_I|\varphi_J\rangle &= \langle\dot{\varphi}_J|\varphi_I\rangle = -\hat{\mathrm{d}}\log\tau[c_I, c_J; c_\pm].\end{aligned} \tag{22}$$

Again, it is interesting to note that the symbol letters (including those in $\bar{u}_I$) in eq. (22) takes the form of the difference between two univariate poles, except the last one. However, for the univariate problem, it is always possible to perform a rationalization to get rid of the square-roots in the context of polylogarithmic Feynman integrals. The last letter in eq. (22) then becomes one of those in eq. (17). In this sense, we arrive at a surprisingly simple structure of symbol letters: all symbol letters (except the constant factors in u) are the difference between two univariate poles after taking the $(n-1)$-variable residues. Combining eqs. (17) and (22) with the $(n-1)$-variable residue already obtained, we have completed the derivation of symbol letters.

4 Factorization of degenerate poles and Newton polytopes

As is evident, the first and the most important step of our method is the factorization of degenerate poles. While this can be done algorithmically following sector decomposition, it is instructive to use an example to get some feeling about the procedure. Let us consider the kite topology defined by

$$\begin{aligned} &z_1 = l_1^2 - m^2, \quad z_2 = (l_2 - p)^2 - m^2, \quad z_3 = (l_1 - l_2)^2, \\ &z_4 = l_2^2, \quad z_5 = (l_1 - p)^2, \quad p^2 = s. \end{aligned} \tag{23}$$

We impose cut on z_1, z_2, z_3, and hence the u function is given by $u = z_4^{\delta_1} z_5^{\delta_2} [\mathcal{G}(z_4, z_5)]^{-\epsilon}$, with

$$\begin{aligned} \mathcal{G} &\equiv 4\mathrm{G}(l_1, l_2, p)\Big|_{z_1=z_2=z_3=0} \\ &= -2m^6 + m^4(s + z_4 + z_5) \\ &\quad + m^2(2z_4z_5 - sz_4 - sz_5) + z_4z_5(s - z_4 - z_5). \end{aligned} \tag{24}$$

From u, we can determine the set of poles for (z_4, z_5):

$$\boldsymbol{p} \in \Big\{(0,0), (m^2, m^2), (\infty, 0), (0, \infty), (\infty, \infty)\Big\}. \tag{25}$$

Here we focus on the three-fold degenerate pole $(\infty, 0)$. The complete results for this family are given in Appendix A3. For convenience, we first introduce the variable change $z_4 = 1/t_4$, and rewrite $u = t_4^{2\epsilon-\delta_1} z_5^{\delta_2} \mathcal{G}_{\infty 0}^{-\epsilon}$. Here

$$\mathcal{G}_{\infty 0} \equiv t_4^2 \mathcal{G}(1/t_4, z_5) \equiv t_4[r_+(t_4) - z_5][z_5 - r_-(t_4)], \tag{26}$$

where the last equal sign defines the two roots $r_\pm(t_4)$ of $\mathcal{G}_{\infty 0}$ with respect to z_5. Noting that

$$z_5 - r_-(t_4) = z_5 - m^2(m^2 - s)t_4 + O(t_4^2), \tag{27}$$

we find 3 factors in u vanishing when $(t_4, z_5) = (0, 0)$: t_4, z_5 and $z_5 - r_-(t_4)$. There are 3 different factorizations, corresponding to 3 ways to organize the 3 factors into 2 groups:

$$\begin{aligned} &\boldsymbol{x}^{(4)} : (\{t_4\}, \{z_5, z_5 - r_-(t_4)\}), \\ &\boldsymbol{x}^{(5)} : (\{t_4, z_5 - r_-(t_4)\}, \{z_5\}), \\ &\boldsymbol{x}^{(6)} : (\{z_5 - r_-(t_4)\}, \{t_4, z_5\}). \end{aligned} \tag{28}$$

As an example, for $\boldsymbol{x}^{(5)}$ we have the variable change

$$t_4 = x_1^{(5)}, \quad z_5 = x_1^{(5)} x_2^{(5)}, \tag{29}$$

which leads to (see eq. (5))

$$\bar{u}_5(\boldsymbol{\rho}^{(5)}) = [m^2(m^2 - s)]^{-\epsilon}, \ \gamma_i^{(5)} \in \{\epsilon - \delta_1 + \delta_2, \delta_2\}, \tag{30}$$

where $\boldsymbol{\rho}^{(5)} = (0, 0)$.

The integral family has four master integrals and exhibits a symmetry under $z_4 \leftrightarrow z_5$ and $\delta_1 \leftrightarrow \delta_2$. The $\mathrm{d}\log$ basis can be constructed as:

$$\begin{aligned} &\varphi_1 = \frac{\mathrm{d}z_4 \mathrm{d}z_5}{z_4 z_5}, \quad \varphi_2 = \frac{\sqrt{s(s - 4m^2)}}{\mathcal{G}} \mathrm{d}z_4 \mathrm{d}z_5, \\ &\varphi_3 = \frac{z_4 - m^2}{\mathcal{G}} \mathrm{d}z_4 \mathrm{d}z_5, \quad \varphi_4 = \frac{z_5 - m^2}{\mathcal{G}} \mathrm{d}z_4 \mathrm{d}z_5. \end{aligned} \tag{31}$$

With the variable change to $\boldsymbol{x}^{(5)}$ and the expansion around $\boldsymbol{\rho}^{(5)}$, the leading terms of φ_1 and φ_3 are

$$\begin{aligned} &\varphi_1^{(-1,-1)} = \frac{\mathrm{d}x_1^{(5)} \mathrm{d}x_2^{(5)}}{x_1^{(5)} x_2^{(5)}}, \\ &\varphi_3^{(-1,0)} = \frac{\mathrm{d}x_1^{(5)} \mathrm{d}x_2^{(5)}}{x_1^{(5)} \left[\rho_2^{(5)} + m^2(m^2 - s)\right]}, \quad \rho_2^{(5)} = 0. \end{aligned} \tag{32}$$

Apparently, they share the $(n-1)$-SP for $\boldsymbol{x}_2^{(5)}$. We can then immediately obtain the letter in $\langle \dot{\varphi}_1 | \varphi_3 \rangle$ from eq. (12), or from eq. (17) as the difference between two univariate poles:

$$m^2(m^2 - s). \tag{33}$$

We now make an interesting observation: the letter in eq. (33) is just the ratio between the coefficient of t_4 and that of z_5 in eq. (27) (as well as in $\mathcal{G}_{\infty 0}$). These two terms are the leading ones in the limit $t_4 \to 0$ and $z_5 \to 0$. Newton polytopes provide a geometric view to study limits of multivariate polynomials. A Newton polytope is the convex hull of the exponent-vectors of a polynomial. This geometric view has been used to study singularities of Feynman integrands. See, e.g., ref. [73] for ultraviolet and infrared divergences, ref. [74] for the method of regions [75], and ref. [76] for sector decomposition [66-72]. These motivate us to understand the symbol letters from Newton polytopes. The Newton polytope of $\mathcal{G}_{\infty 0}$ is shown in Figure 1. It has five facets. Since the components of the outer normal vector of facet ③ are all negative, this facet is degenerate and the corresponding polynomial is exactly eq. (27). The letter (33) is essentially the ratio of the two coefficients at the vertices of the degenerate facet. The other $(n-1)$-SP contributions of the form $c_I - c_J$ to $\langle \dot{\varphi}_I | \varphi_J \rangle$ follow a similar pattern.

Similar observations can also be made for the n-SP contributions. There are two possibilities here. The first case is when there is a degenerate facet, and then the coefficient at one of its vertices gives the letter. For example, the contribution from $\boldsymbol{\rho}^{(4)}$ to $\langle \dot{\varphi}_1 | \varphi_1 \rangle$ is given by the vertex $(0, 1)$, and the letter is $\hat{\mathrm{d}} \log(-1) = 0$; while the contribution from $\boldsymbol{\rho}^{(5)}$ is

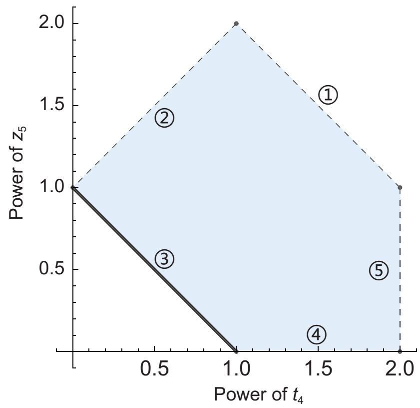

Figure 1 (Color online) The Newton polytope of $\mathcal{G}_{\infty 0}$. Horizontal and vertical axes are the power of t_4 and z_5.

given by the vertex $(1, 0)$, and the letter is the same as eq. (33). The second possibility is when there is no degenerate facet. In this case, the origin $(0, 0)$ must be a vertex of the polytope, and its coefficient gives a letter. For example, the contributions from $\boldsymbol{p} = (0, 0)$ to $\langle \dot{\varphi}_1 | \varphi_1 \rangle$ are related to vertex $(0, 0)$ of the polytope corresponding to $\mathcal{G}(z_4, z_5)$. Hence, the letter is given by the constant term of eq. (24):

$$\mathcal{G}(0, 0) = m^4(s - 2m^2)\,. \tag{34}$$

We have checked that the other contributions do not give rise to new letters, and eqs. (33) and (34) are already the full set of letters in this simple example.

The example discussed above is simple with only two integration variables and only involving rational letters depending on two kinematic variables. We emphasize that our method can be applied to problems with more variables and with irrational letters as well. In particular, we have tested our method in multivariate one-loop examples with irrational letters, and find agreement with existing results. Applications in more complicated multi-loop examples are in progress and will be presented in a forthcoming article.

5 Summary and outlooks

In this paper, we propose a novel method to determine the structure of symbols for any family of polylogarithmic Feynman integrals using intersection theory. The procedure is purely algebraic, involving factorization of degenerate poles and computation of residues at simple poles. The computation of intersection numbers also gives the rational coefficients in the CDEs, and hence completely determines the latter. In particular, we have found a selection rule for nonzero entries in the CDEs.

Our results also reveal some interesting structures underlying the symbol letters. We find that all symbol letters are either the constant factors in the u-function, or the differences between univariate poles after taking the residues for the other variables. We also take a first glance at the possible relationship between the symbol letters and the Newton polytopes associated with the polynomial factors in the u-function. We hope that these algebraic and geometric structures can be used to further simplify the calculation of symbol letters, and provide insights about the mathematical structure of QFT.

In recent years, there have been enormous efforts to extend the concept of pure functions to Feynman integrals beyond the polylogarithmic cases (see, e.g., refs. [77,78]). It is interesting to see whether our method can be generalized to those cases as well. Moreover, since differential equations can be regarded as iterative reduction relations [79], our result also serves as a development towards simplifying the reduction procedure, and shows the connection between the analytic and algebraic structures of Feynman integrals.

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Appendix

A1 Intersection numbers from factorized poles

In this Appendix, we review the calculation of intersection numbers that leads to the leading-order (LO) and next-to-leading order (NLO) contributions (10) and (11). For the moment we will suppress the superscript (α) labeling the factorization transformations, and assume $\boldsymbol{x} = \boldsymbol{\rho}$ is already a factorized pole. Around this pole, the u function can be written as:

$$u(\boldsymbol{x}) = \bar{u}(\boldsymbol{x}) \prod_i [x_i - \rho_i]^{\gamma_i} , \tag{a1}$$

where $\bar{u}(\boldsymbol{x})$ can be Taylor-expanded as:

$$\bar{u}(\boldsymbol{x}) = \bar{u}(\boldsymbol{\rho}) + \sum_i (x_i - \rho_i)\left[\frac{\partial}{\partial x_i}\bar{u}(\boldsymbol{x})\right]_{\boldsymbol{x}=\boldsymbol{\rho}} + \frac{1}{2}\sum_{i,j}(x_i-\rho_i)(x_j-\rho_j)\left[\frac{\partial^2}{\partial x_i \partial x_j}\bar{u}(\boldsymbol{x})\right]_{\boldsymbol{x}=\boldsymbol{\rho}} + \cdots . \tag{a2}$$

The n-form φ_L can be similarly decomposed as:

$$\varphi_L = \sum_{\boldsymbol{b}_L}\varphi_L^{(\boldsymbol{b}_L)} \equiv \sum_{\boldsymbol{b}_L} C_L^{(\boldsymbol{b}_L)} \bigwedge_i (x_i-\rho_i)^{b_{L,i}}\,\mathrm{d}x_i\,, \tag{a3}$$

where $\boldsymbol{b}_L = (b_{L,1},\ldots,b_{L,n})$ denotes a vector of powers. The covariant derivative ∇_i with respect to x_i is defined as:

$$\nabla_i = \mathrm{d}x_i \wedge (\partial_{x_i} + \omega_i)\,, \tag{a4}$$

where

$$\omega_i \equiv \partial_{x_i}\log(u) = \frac{\gamma_i}{x_i-\rho_i} + \partial_{x_i}\log(\bar{u})\,. \tag{a5}$$

We now need to look for a function ψ_L which satisfies $\nabla_n\cdots\nabla_1\psi_L = \varphi_L$ around the pole. The above equation is linear in ψ_L and φ_L. Hence we can decompose the solution as:

$$\psi_L = \sum_{\boldsymbol{b}_L}\psi_L^{(\boldsymbol{b}_L)}\,,\quad \nabla_n\cdots\nabla_1\psi_L^{(\boldsymbol{b}_L)} = \varphi_L^{(\boldsymbol{b}_L)}\,. \tag{a6}$$

We can write the Ansatz for $\psi_L^{(\boldsymbol{b}_L)}$ as:

$$\psi_L^{(\boldsymbol{b}_L)} = C_L^{(\boldsymbol{b}_L)}\left[A^{(0)} + \sum_j A_j^{(1)}(x_j-\rho_j) + \frac{1}{2}\sum_{j,k}A_{j,k}^{(2)}(x_j-\rho_j)(x_k-\rho_k) + \cdots\right]\prod_i (x_i-\rho_i)^{b_{L,i}+1}. \tag{a7}$$

Plugging the above into eq. (a6), the covariant derivatives give rise to

$$\left(\prod_i \nabla_i\right)\psi_L^{(\boldsymbol{b}_L)} = C_L^{(\boldsymbol{b}_L)}\left[\prod_i (x_i-\rho_i)^{b_{L,i}}\right] \times\left[A^{(0)}\prod_i(\gamma_i+b_{L,i}+1) + \sum_j (x_j-\rho_j)\left[A_j^{(1)}(\gamma_j+b_{L,j}+2) + A^{(0)}\partial_{\rho_j}\bar{u}(\boldsymbol{\rho})\right]\prod_{i\neq j}(\gamma_i+b_{L,i}+1) + \cdots\right]. \tag{a8}$$

Hence, we find that the coefficients are given by

$$A^{(0)} = \frac{1}{\prod_i(\gamma_i+b_{L,i}+1)}\,,\quad A_j^{(1)} = -\frac{A^{(0)}\partial_{\rho_j}\log(\bar{u}(\boldsymbol{\rho}))}{\gamma_j+b_{L,j}+2}\,. \tag{a9}$$

It is now straightforward to compute the intersection numbers. Supposing that φ_R is given by

$$\varphi_R = \sum_{\boldsymbol{b}_R}\varphi_R^{(\boldsymbol{b}_R)} \equiv \sum_{\boldsymbol{b}_R} C_R^{(\boldsymbol{b}_R)} \bigwedge_i (x_i-\rho_i)^{b_{R,i}}\,\mathrm{d}x_i\,, \tag{a10}$$

the contribution from the factorized pole $\boldsymbol{x} = \boldsymbol{\rho}$ to the intersection number between $\varphi_L^{(\boldsymbol{b}_L)}$ and $\varphi_R^{(\boldsymbol{b}_R)}$ is given by

$$\mathrm{Res}_{\boldsymbol{x}=\boldsymbol{\rho}}\left(\psi_L^{(\boldsymbol{b}_L)}\varphi_R^{(\boldsymbol{b}_R)}\right) = \mathrm{Res}_{\boldsymbol{x}=\boldsymbol{\rho}}\, C_L^{(\boldsymbol{b}_L)}C_R^{(\boldsymbol{b}_R)}\prod_i (x_i-\rho_i)^{b_{L,i}+b_{R,i}+1} \times\left[A^{(0)} + \sum_j A_j^{(1)}(x_j-\rho_j) + \cdots\right]. \tag{a11}$$

When $\boldsymbol{b}_L + \boldsymbol{b}_R = \mathbf{-2}$, the $A^{(0)}$ term gives rise to the so-called LO contribution (eq. (10)):

$$\mathrm{Res}_{\boldsymbol{x}=\boldsymbol{\rho}}\left(\psi_L^{(\boldsymbol{b}_L)}\varphi_R^{(\boldsymbol{b}_R)}\right) = \frac{C_L^{(\boldsymbol{b}_L)}C_R^{(\boldsymbol{b}_R)}}{\prod_i\tilde{\gamma}_i}\,, \tag{a12}$$

where $\tilde{\gamma}_i = \gamma_i - b_{R,i} - 1$. When all $b_{L,i}+b_{R,i} = -2$ except one $b_{L,j}+b_{R,j} = -3$, the $A_j^{(1)}$ term gives rise to the so-called NLO contribution (eq. (11)):

$$\mathrm{Res}_{\boldsymbol{x}=\boldsymbol{\rho}}\left(\psi_L^{(\boldsymbol{b}_L)}\varphi_R^{(\boldsymbol{b}_R)}\right) = -\frac{C_L^{(\boldsymbol{b}_L)}C_R^{(\boldsymbol{b}_R)}\,\partial_{\rho_j}\log(\bar{u}(\boldsymbol{\rho}))}{(\gamma_j+b_{L,j}+1)\prod_i\tilde{\gamma}_i}\,. \tag{a13}$$

At this point, it is worth noting that the contributions in eqs. (a12) and (a13) are invariant under a simultaneous rescaling of u, $\varphi_L^{(\boldsymbol{b}_L)}$ and $\varphi_R^{(\boldsymbol{b}_R)}$. In terms of the powers γ_i, $b_{L,i}$ and $b_{R,i}$, this rescaling amounts to the shifts:

$$\gamma_i \to \gamma_i + \xi_i\,,\quad b_{L,i}\to b_{L,i}-\xi_i\,,\quad b_{R,i}\to b_{R,i}+\xi_i\,. \tag{a14}$$

The shifts do not change the values of $b_{L,i}+b_{R,i}$, $\gamma_i+b_{L,i}$ and $\gamma_i - b_{R,i}$, and hence the expressions for the LO and NLO contributions are manifestly invariant. In the case $\boldsymbol{b}_L + \boldsymbol{b}_R = \mathbf{-2}$, we can employ this freedom to make $\boldsymbol{b}_L \to \mathbf{-1}$ and $\boldsymbol{b}_R \to \mathbf{-1}$, i.e., both $\varphi_L^{(\boldsymbol{b}_L)}$ and $\varphi_R^{(\boldsymbol{b}_R)}$ have only simple poles. The intersection numbers in this situation are well-understood in ref. [55], and agree with eq. (a12).

As a special case of the above general formulas, we consider the intersection numbers $\langle\dot{\varphi}_I|\varphi_J\rangle$, where both φ_I and φ_J are d log-forms. We again expand φ_I as:

$$\varphi_I = \sum_{\boldsymbol{b}_I}\varphi_I^{(\boldsymbol{b}_I)} \equiv \sum_{\boldsymbol{b}_I} C_I^{(\boldsymbol{b}_I)} \bigwedge_i (x_i-\rho_i)^{b_{I,i}}\,\mathrm{d}x_i\,, \tag{a15}$$

and similarly for φ_J. For each $\boldsymbol{b}_I$ and ρ_j, there is a term in $\dot{\varphi}_I$ given by

$$-(\gamma_j + b_{I,j})\,\hat{\mathrm{d}}\rho_j\, C_I^{(\boldsymbol{b}_I)}\bigwedge_i (x_i-\rho_i)^{b_{I,i}-\delta_{ij}}\,\mathrm{d}x_i\,. \tag{a16}$$

Hence, setting $b_{L,i} = b_{I,i} - \delta_{ij}$ and $b_{R,i} = b_{J,i}$, we can readily use eqs. (a12) and (a13) to compute the residues. If

$\boldsymbol{b}_L = \boldsymbol{b}_R = \boldsymbol{-1}$ (which means $b_{I,j} = 0$, i.e., $(n-1)$-SP), the term gives rise to a LO contribution

$$-\frac{\gamma_j}{\prod_i \gamma_i} C_I^{(\boldsymbol{b}_I)} C_J^{(\boldsymbol{b}_J)}\, \hat{\mathrm{d}}\rho_j\,. \tag{a17}$$

On the other hand, if $\boldsymbol{b}_I = \boldsymbol{b}_J = \boldsymbol{-1}$ (i.e., n-SP), the term leads to a NLO contribution

$$\frac{C_I^{(\boldsymbol{-1})} C_J^{(\boldsymbol{-1})}}{\prod_i \gamma_i}\, \partial_{\rho_j} \log(\bar{u}(\boldsymbol{\rho}))\, \hat{\mathrm{d}}\rho_j\,. \tag{a18}$$

A2 Reduction to univariate problems

In the previous section, we've seen that in the computation of $\langle \dot{\varphi}_I | \varphi_J \rangle$ for d log-forms φ_I and φ_J, the contributing terms $\varphi_I^{(\boldsymbol{b}_I)}$ and $\varphi_J^{(\boldsymbol{b}_J)}$ share at least $(n-1)$-variable simple poles. Without loss of generality, we denote these $(n-1)$ variables as $\boldsymbol{x}_{\hat{1}} = (x_2, \ldots, x_n)$, and denote the remaining variable as $z \equiv x_1$. In the computation of intersection numbers, one may take the $(n-1)$-variable residues at $\boldsymbol{x}_{\hat{1}} = \boldsymbol{\rho}_{\hat{1}}$ first, and deal with the single variable z in the last step.

To see how that works, we assume that both φ_L and φ_R have simple poles at $\boldsymbol{x}_{\hat{1}} = \boldsymbol{\rho}_{\hat{1}}$. They can then be written as:

$$\begin{aligned} \varphi_L &= f_L(z, \boldsymbol{x}_{\hat{1}})\, \mathrm{d}z \bigwedge_{i=2}^{n} (x_i - \rho_i)^{-1}\, \mathrm{d}x_i, \\ \varphi_R &= f_R(z, \boldsymbol{x}_{\hat{1}})\, \mathrm{d}z \bigwedge_{i=2}^{n} (x_i - \rho_i)^{-1}\, \mathrm{d}x_i, \end{aligned} \tag{a19}$$

where f_L and f_R are regular at $\boldsymbol{x}_{\hat{1}} = \boldsymbol{\rho}_{\hat{1}}$. The $u(\boldsymbol{x})$ function can also be written as:

$$u(\boldsymbol{x}) = \bar{u}(z, \boldsymbol{x}_{\hat{1}}) \prod_{i=2}^{n} [x_i - \rho_i]^{\gamma_i}\,. \tag{a20}$$

To compute $\langle \varphi_L | \varphi_R \rangle$, we need to find a ψ_L satisfying $\nabla_n \cdots \nabla_1 \psi_L = \varphi_L$ in the vicinity of the pole. Due to the simple pole structure, it is straightforward to perform the inversion of ∇_i for $i = 2, \ldots, n$. This leads to

$$\nabla_1 \psi_L = \frac{f_L(z, \boldsymbol{\rho}_{\hat{1}})\, \mathrm{d}z}{\gamma_2 \cdots \gamma_n} + O((\boldsymbol{x}_{\hat{1}} - \boldsymbol{\rho}_{\hat{1}})^0)\,, \tag{a21}$$

where the higher-power terms do not contribute since φ_R has simple poles. Hence, the computation of the n-variable intersection number is equivalent to a univariate problem with

$$u(z) \equiv \bar{u}(z, \boldsymbol{\rho}_{\hat{1}})\,, \quad \varphi_L \equiv \frac{f_L(z, \boldsymbol{\rho}_{\hat{1}})\, \mathrm{d}z}{\gamma_2 \cdots \gamma_n}\,, \quad \varphi_R \equiv f_R(z, \boldsymbol{\rho}_{\hat{1}})\, \mathrm{d}z\,. \tag{a22}$$

Now, we may collect all contributions to $\langle \dot{\varphi}_I | \varphi_J \rangle$ from the $(n-1)$-variable simple pole at $\boldsymbol{x}_{\hat{1}} = \boldsymbol{\rho}_{\hat{1}}$ and an additional pole (not necessarily simple) for the variable $z = x_1$. This allows us to study the symbol letters using only univariate d log-constructions and intersection numbers.

We first look at the case of rational-type d log-forms. The u-function can be factorized into

$$u(z) = P_0^{\beta_0} \prod_{\alpha=1}^{\nu+1} (z - c_\alpha)^{\beta_\alpha}\,. \tag{a23}$$

There are $\nu + 2$ different poles for z:

$$\rho^{(\alpha)} \in \{c_1, \ldots, c_{\nu+1}, \infty\}\,, \quad \gamma^{(\alpha)} \in \left\{\beta_1, \ldots, \beta_{\nu+1}, -\sum_{\alpha=1}^{\nu+1} \beta_\alpha\right\}. \tag{a24}$$

For this u-function, there are ν independent integrands. They can be chosen as $\varphi_I = \mathrm{d}z/(z - c_I)$ for $I = 1, \ldots, \nu$. We need to consider two kinds of intersection numbers: $\langle \dot{\varphi}_I | \varphi_I \rangle$ and $\langle \dot{\varphi}_I | \varphi_J \rangle$ with $I \neq J$. For the first kind, we take $\langle \dot{\varphi}_1 | \varphi_1 \rangle$ as an example. For that we need to consider $\partial_{\rho^{(1)}} \varphi_1$, $\partial_{\rho^{(\alpha)}} \varphi_1$ for $\alpha \neq 1$, and the symbol letters contained in P_0. Here with an abuse of the notation, $\partial_\rho \varphi$ actually denotes $\partial_\rho (u\, \varphi)/u$. Using the formulas for LO and NLO contributions to intersection numbers, we have

$$\begin{aligned} \langle \partial_{c_1} \varphi_1 \,|\, \varphi_1 \rangle &= \left\langle \frac{(1 - \beta_1)\, \mathrm{d}z}{(z - c_1)^2} \,\middle|\, \frac{\mathrm{d}z}{z - c_1} \right\rangle = \frac{1}{\beta_1}\, \partial_{c_1} \log(\bar{u}_1(c_1))\,, \\ \langle \partial_{c_\alpha} \varphi_1 \,|\, \varphi_1 \rangle &= \left\langle \frac{\beta_\alpha\, \mathrm{d}z}{(z - c_1)(c_\alpha - z)} \,\middle|\, \frac{\mathrm{d}z}{z - c_1} \right\rangle \\ &= \frac{\beta_\alpha}{\beta_1}\, \partial_{c_\alpha} \log(c_1 - c_\alpha) = \frac{1}{\beta_1}\, \partial_{c_\alpha} \log(\bar{u}_1(c_1))\,, \end{aligned} \tag{a25}$$

where

$$\bar{u}_1(z) = P_0^{\beta_0} \prod_{\alpha=2}^{\nu+1} (z - c_\alpha)^{\beta_\alpha}\,. \tag{a26}$$

From the above results, one may easily reconstruct $\langle \dot{\varphi}_1 | \varphi_1 \rangle$ in the form of $\hat{\mathrm{d}}$ logs, which coincides with eq. (13) and the first line of eq. (17). For $\langle \dot{\varphi}_I | \varphi_J \rangle$, we only need to consider the contributions from $\partial_{c_I} \varphi_I$ and $\partial_{c_J} \varphi_I$, as well as from P_0. Using the formula for LO intersection numbers, we have

$$\begin{aligned} \langle \partial_{c_I} \varphi_I \,|\, \varphi_J \rangle &= (1 - \beta_I) \left\langle \frac{\mathrm{d}z}{(z - c_I)^2} \,\middle|\, \frac{\mathrm{d}z}{z - c_J} \right\rangle \\ &= -\partial_{c_I} \log(c_I - c_J)\,, \\ \langle \partial_{c_J} \varphi_I \,|\, \varphi_J \rangle &= -\beta_J \left\langle \frac{\mathrm{d}z}{(z - c_I)(z - c_J)} \,\middle|\, \frac{\mathrm{d}z}{z - c_J} \right\rangle \\ &= -\partial_{c_J} \log(c_I - c_J)\,. \end{aligned} \tag{a27}$$

These agree with the results in eq. (12) and the second line of eq. (17)

We now move to sqrt-type d log-forms. The u-function is given by

$$u(z) = P_0^{\beta_0} (z - c_+)^{\beta_+} (z - c_-)^{\beta_-} \prod_{\alpha=1}^{\nu-1} (z - c_\alpha)^{\beta_\alpha}\,. \tag{a28}$$

There are again $\nu+2$ different poles for z:

$$\rho^{(\alpha)} \in \{c_1, \dots, c_{\nu-1}, \infty, c_+, c_-\},$$
$$\gamma^{(\alpha)} \in \left\{\beta_1, \dots, \beta_{\nu-1}, -\sum_{\alpha=1}^{\nu-1}\beta_\alpha - \beta_+ - \beta_-, \beta_+, \beta_-\right\}. \tag{a29}$$

The two poles $c_\pm$ are singled out to remind us that there is always a factor of $\sqrt{(z-c_+)(z-c_-)}$ in the integrands according to the second equation in eq. (6), which we reproduce here:

$$\begin{aligned}\mathrm{d}\log(\tau[z,c;c_\pm]) &\equiv \mathrm{d}\log\frac{\sqrt{c-c_+}\sqrt{z-c_-}+\sqrt{c-c_-}\sqrt{z-c_+}}{\sqrt{c-c_+}\sqrt{z-c_-}-\sqrt{c-c_-}\sqrt{z-c_+}}\\ &= \frac{\sqrt{(c-c_+)(c-c_-)}\,\mathrm{d}z}{(z-c)\sqrt{(z-c_+)(z-c_-)}}.\end{aligned} \tag{a30}$$

At this point, we note that the square root of a linear function is related to that of a quadratic function via a variable change. For example, setting $z = 1/t + c_+$, we have

$$\frac{\mathrm{d}z}{\sqrt{(z-c_+)(z-c_-)}} = \frac{\mathrm{d}t}{t\sqrt{1+t(c_+-c_-)}}. \tag{a31}$$

Hence, we do not have to consider the linear function case separately.

For each $I = 1, \dots, \nu-1$, there is an independent integrand $\varphi_I = \mathrm{d}\log(\tau_I) \equiv \mathrm{d}\log\tau[z, c_I; c_\pm]$. The νth independent integrand is associated with the pole $\rho^{(\nu)} = \infty$, and is given by

$$\begin{aligned}\varphi_\nu &= \mathrm{d}\log(\tau_\nu) \equiv \mathrm{d}\log\tau[z,\infty;c_\pm]\\ &= \mathrm{d}\log\frac{\sqrt{z-c_-}+\sqrt{z-c_+}}{\sqrt{z-c_-}-\sqrt{z-c_+}} = \frac{\mathrm{d}z}{\sqrt{(z-c_+)(z-c_-)}}.\end{aligned} \tag{a32}$$

The intersection numbers $\langle\dot\varphi_I|\varphi_J\rangle$ can now be computed as usual. Taking $\langle\dot\varphi_1|\varphi_1\rangle$ as an example. We need to consider the derivatives with respect to c_1, $c_\pm$ and c_α for $\alpha = 2, \dots, \nu-1$. We have

$$\begin{aligned}\partial_{c_1}\varphi_1 &= \frac{1-\beta_1}{(z-c_1)^2} + \frac{\beta_1}{2}\left[\frac{1}{c_1-c_+}+\frac{1}{c_1-c_-}\right]\frac{1}{z-c_1}\\ &\quad + O\big((z-c_1)^0\big),\\ \varphi_1 &= \frac{1}{z-c_1} - \frac{1}{2}\left[\frac{1}{c_1-c_+}+\frac{1}{c_1-c_-}\right] + O\big((z-c_1)^1\big),\\ \partial_{c_\pm}\varphi_1 &= \frac{1/2-\beta_\pm}{z-c_\pm}\varphi_1,\\ \partial_{c_\alpha}\varphi_1 &= -\frac{\beta_\alpha}{z-c_\alpha}\varphi_1.\end{aligned} \tag{a33}$$

There are two terms in $\partial_{c_1}\varphi_1$, leading to both LO and NLO contributions from the pole c_1 to the intersection number:

$$\begin{aligned}\langle\partial_{c_1}\varphi_1\,|\,\varphi_1\rangle &= \frac{1}{\beta_1}\partial_{c_1}\log\bar u_1(c_1)\\ &\quad + \partial_{c_1}\log(c_1-c_+) + \partial_{c_1}\log(c_1-c_-).\end{aligned} \tag{a34}$$

The intersection number $\langle\partial_{c_\pm}\varphi_1|\varphi_1\rangle$ receive LO contributions from the pole c_1 as well as $c_\pm$, which are given by

$$\begin{aligned}\langle\partial_{c_\pm}\varphi_1\,|\,\varphi_1\rangle &= \frac{\beta_\pm-1/2}{\beta_1}\partial_{c_\pm}\log(c_1-c_\pm)\\ &\quad - \partial_{c_\pm}\log(c_\pm-c_\mp) + \partial_{c_\pm}\log(c_1-c_\pm).\end{aligned} \tag{a35}$$

Finally, the intersection number $\langle\partial_{c_\alpha}\varphi_1|\varphi_1\rangle$ for $\alpha = 2, \dots, \nu-1$ receive LO contributions only from the c_1 pole:

$$\langle\partial_{c_\alpha}\varphi_1|\varphi_1\rangle = \frac{\beta_\alpha}{\beta_1}\partial_{c_\alpha}\log(c_1-c_\alpha) = \frac{1}{\beta_1}\partial_{c_\alpha}\bar u_1(c_1). \tag{a36}$$

Combining the above results, we can reproduce the first equation in eq. (22). Similarly, ∂_{c_I}, ∂_{c_j}, and $\partial_{c_\pm}$ give the same contribution as shown in the second equation in eq. (22).

Alternatively, one may perform a variable change to rationalize the square root, and compute the intersection numbers in the same way as the rational case. The relevant variable change is simply

$$z = \frac{c_+(\tau_\nu+1)^2 - c_-(\tau_\nu-1)^2}{4\tau_\nu}, \tag{a37}$$

where the variable τ_ν is defined in eq. (a32). The poles for the new variable τ_ν can be written in terms of a set of new constants:

$$t_I \equiv \tau[\infty, c_I; c_\pm] = \frac{\sqrt{c_I-c_+}+\sqrt{c_I-c_-}}{\sqrt{c_I-c_+}-\sqrt{c_I-c_-}}. \tag{a38}$$

The d log integrands can then be rewritten as $d\log(\tau_\nu)$ and

$$\mathrm{d}\log(\tau_I) = \mathrm{d}\log(\tau_\nu - t_I) - \mathrm{d}\log\left(\tau_\nu - \frac{1}{t_I}\right). \tag{a39}$$

As promised, all integrands are of the rational-type, and the symbol letters can be read off using the existing results.

A3 Details of the kite topology

In this Appendix, we show the details of the kite topology discussed in sect. 4. The relevant polynomials are given by (with $z_1 = z_2 = z_3 = 0$)

$$\begin{aligned}\mathcal{G}(z_4,z_5) &\equiv 4\mathrm{G}(l_1,l_2,p) = -2m^6 + m^4(s+z_4+z_5)\\ &\quad + m^2(2z_4z_5 - sz_4 - sz_5) + z_4z_5(s-z_4-z_5),\\ \mathcal{G}_1(z_5) &\equiv -4\mathrm{G}(l_1,p) = (z_5-s)^2 + m^4 - 2m^2(z_5+s),\end{aligned} \tag{a40}$$

and the u-function is

$$u(z_4,z_5) = z_4^{\delta_1}z_5^{\delta_2}\,[\mathcal{G}(z_4,z_5)]^{-\epsilon}. \tag{a41}$$

To reveal the singularities at ∞, we employ the variable changes $z_4 = 1/t_4$ and $z_5 = 1/t_5$. The resulting polynomials are

$$\begin{aligned}\mathcal{G}_{\infty\infty} &\equiv t_4^2 t_5^2\, \mathcal{G}(1/t_4, 1/t_5) = (-2m^6 + m^4 s)t_4^2 t_5^2 \\ &\quad + (m^4 - m^2 s)t_4 t_5^2 + (m^4 - m^2 s)t_4^2 t_5 + 2m^2 t_4 t_5 \\ &\quad + s t_4 t_5 - t_4 - t_5, \\ \mathcal{G}_{\infty 0} &\equiv t_4^2\, \mathcal{G}(1/t_4, z_5) = -2m^6 t_4^2 + m^4 s t_4^2 + m^4 t_4 \\ &\quad + m^4 t_4^2 z_5 - m^2 s t_4 - m^2 s t_4^2 z_5 + 2m^2 t_4 z_5 + s t_4 z_5 \\ &\quad - t_4 z_5^2 - z_5, \\ \mathcal{G}_{0\infty} &\equiv t_5^2\, \mathcal{G}(z_4, 1/t_5) = \mathcal{G}_{\infty 0}(t_4 \to t_5, z_5 \to z_4).\end{aligned} \tag{a42}$$

The four master integrals can be expressed as d log-forms

$$\begin{aligned}\varphi_1 &= \mathrm{d}\log(z_4) \wedge \mathrm{d}\log(z_5), \\ \varphi_2 &= \mathrm{d}\log(\tau[z_4, m^2; r_{1;\pm}]) \wedge \mathrm{d}\log\left(\frac{z_5 - r_{5+}}{z_5 - r_{5-}}\right), \\ \varphi_3 &= -\mathrm{d}\log(\tau[z_4, \infty; r_{1;\pm}]) \wedge \mathrm{d}\log\left(\frac{z_5 - r_{5+}}{z_5 - r_{5-}}\right), \\ \varphi_4 &= -\mathrm{d}\log(\tau[z_5, \infty; r_{1;\pm}]) \wedge \mathrm{d}\log\left(\frac{z_4 - r_{4+}}{z_4 - r_{4-}}\right),\end{aligned} \tag{a43}$$

where the various roots of quadratic polynomials are given by

$$\begin{aligned}&r_{1;\pm} \equiv r_\pm[\mathcal{G}_1; z_5], \quad r_{4\pm}(z_5) \equiv r_\pm[\mathcal{G}; z_4], \\ &r_{5\pm}(z_4) \equiv r_\pm[\mathcal{G}; z_5], \quad r_{5+}(\infty) = \infty, \\ &r_{5-}(\infty) = 0, \quad r_{5\pm}(m^2) = m^2.\end{aligned} \tag{a44}$$

Note that φ_3 and φ_4 are related by an exchange symmetry under $z_4 \leftrightarrow z_5$, that we will employ later.

The poles and the relevant variables after factorization are given as:

$$\begin{aligned}&(0,0) : \boldsymbol{x}^{(1)}, \quad (m^2, m^2) : \boldsymbol{x}^{(2,3)}, \quad (\infty, 0) : \boldsymbol{x}^{(4,5,6)}, \\ &(0, \infty) : \boldsymbol{x}^{(7,8,9)}, \quad (\infty, \infty) : \boldsymbol{x}^{(10,11,12)}.\end{aligned} \tag{a45}$$

The factorization transformations are related to the following grouping of the 3 factors in the u-function:

$$\begin{aligned}&\boldsymbol{x}^{(2)} : (\{z_4 - m^2, z_5 - r_{5+}\}, \{z_5 - r_{5-}\}), \\ &\boldsymbol{x}^{(3)} : (\{z_4 - m^2, z_5 - r_{5-}\}, \{z_5 - r_{5+}\}), \\ &\boldsymbol{x}^{(4)} : (\{t_4\}, \{z_5, z_5 - r_-[\mathcal{G}_{\infty 0}; z_5]\}), \\ &\boldsymbol{x}^{(5)} : (\{t_4, z_5 - r_-[\mathcal{G}_{\infty 0}; z_5]\}, \{z_5\}), \\ &\boldsymbol{x}^{(6)} : (\{z_5 - r_-[\mathcal{G}_{\infty 0}; z_5]\}, \{t_4, z_5\}), \\ &\boldsymbol{x}^{(7)} : (\{t_5\}, \{z_4, z_4 - r_-[\mathcal{G}_{0\infty}; z_4]\}), \\ &\boldsymbol{x}^{(8)} : (\{t_5, z_4 - r_-[\mathcal{G}_{0\infty}; z_4]\}, \{z_4\}), \\ &\boldsymbol{x}^{(9)} : (\{z_4 - r_-[\mathcal{G}_{0\infty}; z_4]\}, \{z_4, t_5\}), \\ &\boldsymbol{x}^{(10)} : (\{t_4\}, \{t_5, t_4 - r_+[\mathcal{G}_{\infty\infty}; t_4]\}), \\ &\boldsymbol{x}^{(11)} : (\{t_4, t_4 - r_+[\mathcal{G}_{\infty\infty}; t_4]\}, \{t_5\}), \\ &\boldsymbol{x}^{(12)} : (\{t_4 - r_+[\mathcal{G}_{\infty\infty}; t_4]\}, \{t_4, t_5\}).\end{aligned} \tag{a46}$$

The explicit transformations for the pole (m^2, m^2) are

$$\begin{aligned}&z_4 \to x_1^{(2)} x_2^{(2)} + r_+[\mathcal{G}(x_1^{(2)}, x_2^{(2)}); x_1^{(2)}], \quad z_5 \to x_2^{(2)}, \\ &z_5 \to x_1^{(3)} x_2^{(3)} + r_+[\mathcal{G}(x_1^{(3)}, x_2^{(3)}); x_2^{(3)}], \quad z_4 \to x_2^{(3)}.\end{aligned} \tag{a47}$$

For the pole $(\infty, 0)$:

$$\begin{aligned}&t_4 \to x_1^{(4)} x_2^{(4)}, \quad z_5 \to x_2^{(4)}, \\ &t_4 \to x_1^{(5)}, \quad z_5 \to x_1^{(5)} x_2^{(5)}, \\ &t_4 \to x_1^{(6)} x_2^{(6)} + r_+[\mathcal{G}_{\infty 0}(x_1^{(6)}, x_2^{(6)}); x_1^{(6)}], \quad z_5 \to x_2^{(6)}.\end{aligned} \tag{a48}$$

The factorization transformations of the pole $(0, \infty)$ is similar as the above due to the $z_4 \leftrightarrow z_5$ symmetry, and we do not show them explicitly. Finally, for the pole (∞, ∞), we have

$$t_4 \to x_1^{(10)} x_2^{(10)}, \quad t_5 \to x_2^{(10)}, \tag{a49}$$

$$t_4 \to x_1^{(11)}, \quad t_5 \to x_1^{(11)} x_2^{(11)}, \tag{a50}$$

$$t_4 \to x_1^{(12)} x_2^{(12)} + r_+[\mathcal{G}_{\infty\infty}(x_1^{(12)}, x_2^{(12)}); x_1^{(12)}], \quad t_5 \to x_2^{(12)}. \tag{a51}$$

Note that in all the above transformations, we have shifted the pole to $\boldsymbol{\rho}^{(\alpha)} = (0, 0)$. Namely, the u-function can be written as:

$$u(\boldsymbol{x}^{(\alpha)}) = \bar{u}_\alpha(\boldsymbol{x}^{(\alpha)}) \left(x_1^{(\alpha)}\right)^{\gamma_1^{(\alpha)}} \left(x_2^{(\alpha)}\right)^{\gamma_2^{(\alpha)}}. \tag{a52}$$

The corresponding u-powers are given by (recall that $\boldsymbol{\gamma}^{(\alpha)} = \gamma_1^{(\alpha)} \gamma_2^{(\alpha)}$)

$$\begin{aligned}&\boldsymbol{\gamma}^{(1)} = \delta_1 \delta_2, \quad \boldsymbol{\gamma}^{(2,3)} = (-2\epsilon)(-\epsilon), \\ &\boldsymbol{\gamma}^{(7,8,9)} = \boldsymbol{\gamma}^{(4,5,6)}\Big|_{\delta_1 \leftrightarrow \delta_2} \quad \boldsymbol{\gamma}^{(4)} = (\epsilon - \delta_1 + \delta_2)(2\epsilon - \delta_1), \\ &\boldsymbol{\gamma}^{(10)} = (3\epsilon - \delta_1 - \delta_2)(2\epsilon - \delta_1), \\ &\boldsymbol{\gamma}^{(5)} = (\epsilon - \delta_1 + \delta_2)\delta_2, \quad \boldsymbol{\gamma}^{(11)} = (3\epsilon - \delta_1 - \delta_2)(2\epsilon - \delta_2), \\ &\boldsymbol{\gamma}^{(6)} = (\epsilon - \delta_1 + \delta_2)(-\epsilon), \quad \boldsymbol{\gamma}^{(12)} = (3\epsilon - \delta_1 - \delta_2)(-\epsilon).\end{aligned} \tag{a53}$$

The residues $C_I^{(-1)}$ of φ_I at each $\boldsymbol{\rho}^{(\alpha)}$ are given by

$$\begin{aligned}&\varphi_1 : \{1, 0, 0, -1, 1, 0, -1, 1, 0, 1, -1, 0\}, \\ &\varphi_2 : \{0, -1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \\ &\varphi_3 : \{0, 0, 0, 1, 0, -1, 0, 0, 0, -1, 0, 1\}, \\ &\varphi_4 : \{0, 0, 0, 0, 0, 0, 1, 0, -1, 0, 1, -1\}.\end{aligned} \tag{a54}$$

The zero entries mean that the corresponding integrands are not singular at those poles.

The elements of the η-matrix, $\eta_{IJ} = \langle\varphi_I|\varphi_J\rangle$ can be easily obtained using eq. (10). The inverse matrix is given by

$$\eta^{-1} = \begin{pmatrix} \frac{\delta_1\delta_2(-\delta_1-\delta_2+\epsilon)}{\epsilon} & 0 & -2\delta_1\delta_2 & -2\delta_1\delta_2 \\ 0 & \epsilon^2 & 0 & 0 \\ -2\delta_1\delta_2 & 0 & -2\epsilon(\delta_2+\epsilon) & -\epsilon(\delta_1+\delta_2+\epsilon) \\ -2\delta_1\delta_2 & 0 & -\epsilon(\delta_1+\delta_2+\epsilon) & -2\epsilon(\delta_1+\epsilon) \end{pmatrix}. \tag{a55}$$

For the symbol letters contained in $(\hat{\mathrm{d}}\Omega)_{13} = \langle\dot{\varphi}_1|\varphi_J\rangle(\eta^{-1})_{J3}$, we need to compute $\langle\dot{\varphi}_1|\varphi_1\rangle$, $\langle\dot{\varphi}_1|\varphi_3\rangle$ and $\langle\dot{\varphi}_1|\varphi_4\rangle$. Due to the exchange symmetry, $\langle\dot{\varphi}_1|\varphi_4\rangle$ can be obtained from $\langle\dot{\varphi}_1|\varphi_3\rangle$ by $\delta_1 \leftrightarrow \delta_2$.

According to eq. (a54), the term $\langle\dot{\varphi}_1|\varphi_1\rangle$ receives n-SP contributions from the poles $\rho^{(1,5,8)}$. Note that $\rho^{(4,7,10,11)}$ gives $\mathrm{d}\log C = 0$ and does not contribute to the symbol letters. In the contributing poles, $\rho^{(8)}$ are apparently related to $\rho^{(5)}$ by the exchange symmetry. Therefore, the genuinely independent contributions to $\langle\dot{\varphi}_1|\varphi_1\rangle$ are that from $\rho^{(1)}$:

$$-\frac{\epsilon}{\delta_1\delta_2}\left(2\log(m^2)+\log(s-2m^2)\right), \tag{a56}$$

and that from, e.g., $\boldsymbol{\rho}^{(5)}$:

$$-\frac{\epsilon}{(\epsilon-\delta_1+\delta_2)\delta_2}\left(\log(m^2)+\log(s-m^2)\right). \tag{a57}$$

The term $\langle\dot{\varphi}_1|\varphi_3\rangle$ receives $(n-1)$-SP contributions from $\rho^{(4,5,7,8)}$. Again, the only independent non-zero contribution comes from, e.g., $\rho^{(5)}$, and can be written as:

$$\frac{1}{\epsilon-\delta_1+\delta_2}\left(\log(m^2)+\log(s-m^2)\right). \tag{a58}$$

Combining $\langle\dot{\varphi}_1|\varphi_J\rangle$ and $(\eta^{-1})_{J3}$, we are ready to obtain the result for $(\hat{\mathrm{d}}\Omega)_{13}$. For simplicity we take $\delta_1 = \delta_2 = \delta$. In this case $\langle\varphi_3| = \langle\varphi_4|$ and we have only 3 master integrals. The result reads

$$(\hat{\mathrm{d}}\Omega)_{13} = 4\epsilon\left(2\log(s-2m^2)-3\log(s-m^2)+\log(m^2)\right). \tag{a59}$$

It is interesting to note that the result does not depend on δ. We have checked this result by comparing it to the differential equations obtained from the traditional IBP, and find agreement. All other elements in $(\hat{\mathrm{d}}\Omega)$ can be easily read out since the integrands are 0-form or 1-form after maximal cut.

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The canonical differential equations of the one-loop-like integrals

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Abstract: Recently, a new approach for high loop integrals has been proposed in [1], where the whole parameter integration has been divided into two parts: a one-loop-like integration and the remaining parameter integration. In this paper, we systematically study the one-loop-like integrals. We establish the IBP relations for the integral family and show how to complete the reduction. We find the canonical master integrals and write down the corresponding canonical differential equations.

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Contents

1 Introduction

The increasing precision of high-energy physics experiments necessitates correspondingly accurate theoretical predictions. Perturbative quantum field theory provides the primary framework for achieving these high-precision predictions in particle physics. A fundamental, yet challenging, aspect of perturbative calculations lies in evaluating Feynman integrals arising from scattering amplitudes. To address this, numerous systematic methods for computing Feynman integrals have been developed.

Among these methods, Integration-By-Parts (IBP) identities [2–4] and the differential equation method [5–8] derived from them have been proven particularly effective. IBP relations establish linear dependencies among Feynman integrals, enabling the reduction of a large number of integrals to a linear combination of a finite set, known as master integrals. These master integrals, in turn, satisfy a system of first-order linear differential equations, forming the basis of the differential equation method.

For analytic computations, the canonical differential equation (CDE) approach [9, 10], building upon IBP and the differential equation method, represents the most powerful technique currently available. By judiciously selecting master integrals, the differential equations can be transformed into a dlog-form proportional to ϵ. When this transformation is fully rationalizable, it facilitates an iterative solution for the analytic expressions of the master integrals at each order in ϵ, typically expressed in terms of multiple polylogarithm (MPL) functions. In cases where full rationalization is unattainable, alternative approaches for obtaining analytic results have been recently developed [11, 12]. Consequently, obtaining the canonical form of the differential equations often signifies, or at least significantly facilitates, the derivation of analytic solutions for the master integrals. However, extending this method beyond MPL functions — for instance, to cases involving elliptic integrals — remains an active area of research.

For numerical computations, the generalized power series expansion method [13, 14], based on IBP and the differential equation approach, offers a systematic and efficient framework for numerically solving differential equations, with convenient packages such as DiffEXP [14] and SeaSyde [15, 16]. This method, when combined with dlog-form differential equations, has also recently demonstrated success in deriving analytic solutions for tree-level cosmological correlators in curved spacetime [17]. Solving differential equations also need the boundary conditions as import. For the determination of boundary conditions, one can choose Monte Carlo-based sector decomposition [18, 19], or numerical differential equations-based methods including the AMFlow method [20] and its associated package [21], as well as an alternative approach developed in [22] and implemented in the AmpRed package [23]. Furthermore, alternative methods exist for the computation of Feynman integrals, as demonstrated in one-loop calculations [43, 44] and extended to multi-loop scenarios [24, 25].

At first glance, numerical methods based on differential equations appear to offer a systematic approach for computing Feynman integrals. However, for cutting-edge physical processes demanding extremely high precision, these computations are becoming increasingly complex and resource-intensive. In many cases, the IBP reduction process for higher loops becomes the primary bottleneck due to its excessive computational demands, rendering it impractical for these applications. Therefore, the development of novel and efficient methods for computing Feynman integrals remains an urgent and crucial challenge.

Recently, a new approach proposed in [1, 26] has demonstrated significant potential for accelerating the computation of arbitrary two-loop and higher loop diagrams. In [1], leveraging either Feynman parameterization or the Lee-Pomeransky (LP) representation, the authors introduced a novel parameterized representation of Feynman integrals, which we refer to as the Huang-Huang-Ma (HHM) representation. Within this framework, integrals at any loop order can be reformulated into a structure comprising a one-loop-like integral kernel (it is called fixed-branch integrals in [1, 26] and the precise nature of this analogy will be discussed subsequently) and a series of additional integrals.

It has long been recognized that one-loop integrals exhibit significantly greater simplicity compared to their higher-loop counterparts. This inherent simplicity has been extensively studied in the context of symbol alphabets, canonical master integrals, and canonical differential equations for arbitrary one-loop integrals, employing various approaches [9, 27–38].

In the HHM representation at two loops, beyond the one-loop-like integral kernel, only two additional integrals remain. This observation suggests that if the one-loop-like integrals can be efficiently computed or directly expressed analytically, the entire two-loop integral problem can be reduced to a double integral. In [1], the authors implemented a computational scheme based on this concept, demonstrating remarkable efficiency. They introduced an exceptionally effective reduction scheme for the one-loop-like integrals and employed a relatively straightforward numerical differential equation method.

Motivated by this progress, in this paper we will investigate the properties of this one-loop-like integrals further: not only to enhance the computational efficiency of this approach but also to advance the understanding of the mathematical structures underlying higher-loop integrals. Consistent with the findings of [27], we observe that the one-loop-like integrals in the HHM representation exhibit properties under IBP and differential equations that closely resemble those of genuine one-loop integrals. Specifically, all of its canonical master integrals, as demonstrated in [27, 35], can be expressed using two remarkably simple formulas, applicable separately to cases with an odd or even number of propagators.

Moreover, similar to the results in [27], we identify a comparable canonical differential equation structure and symbol alphabet. In non-degenerate cases, we find that a given sector contains only a single master integral and that the corresponding canonical differential equations depend on subsectors with at most two propagators less. This suggests that, in the future, one might attempt to directly derive analytic expressions for arbitrary one-loop-like integrals or employ them for efficient numerical computations.

Although the canonical differential equations we present involve square roots, a simple diagonal transformation can be performed, if necessary, to obtain rational differential equations with respect to any chosen variable. This rational form enhances the efficiency of direct numerical differential equation methods. Additionally, we introduce an alternative reduction scheme for the one-loop-like integrals, which is expected to yield results comparable to those presented in [1], albeit with slight variations.

The arrangement of the paper is as follows. In section 2, we recall the Feynman parameterization of general loop integrals and how the one-loop-like integrals appear according to the proposal made in [1]. In section 3, we study the IBP reduction of these one-loop-like integrals. In section 4, we construct the canonical master integrals and their corresponding canonical differential equations. In section 5, some examples of degenerate bases have been discussed. Finally, a conclusion is given in section 6. A further technical point is clarified in appendix A. Appendix B provides a validation of our method at the one-loop level.

2 The Feynman parameterization

In this section, we review the Feynman parameterization form of general loop integrals. The purpose is to set up the framework of one-loop-like integrals of our focus on this paper. We will follow the line presented in [1].

The L-loop Feynman integral (FI) is

$$I = \int \prod_{r=1}^{L} \frac{d^D l_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{n} \frac{1}{D_j^{\nu_j}}, \tag{2.1}$$

where D denotes the spacetime dimension, D_j represents the propagators, n is the total number of propagators, and ν_j are positive exponents. Using Feynman parameterization to combine the denominators, we have

$$\frac{1}{\prod_{j=1}^{n} D_j^{\nu_j}} = \frac{\Gamma(\nu)}{\prod_{j=1}^{n}\Gamma(\nu_j)} \int d^n\boldsymbol{a}\ \delta(1-\sum_{j=1}^{n} a_j) \frac{\prod_j a_j^{\nu_j-1}}{(\sum_{j=1}^{n} a_j D_j)^{\nu}}\,, \tag{2.2}$$

where $\boldsymbol{a} \equiv (a_1, \ldots, a_n)$ is the list of the Feynman parameters, and $\nu = \nu_1 + \ldots + \nu_n$. Since the inverse propagators are $D_j = -q_j^2 + m_j^2$, where q_j is a linear combination of loop and external momenta, the denominator can then be expressed as

$$\sum_{j=1}^{n} a_j D_j = -\sum_{r=1}^{L}\sum_{s=1}^{L} l_r \boldsymbol{K}_{rs} l_s + \sum_{r=1}^{L} 2 l_r \boldsymbol{v}_r + J\,, \tag{2.3}$$

where $\boldsymbol{K}$ is an $L\times L$ matrix, $\boldsymbol{v}$ is an $L\times 1$ column matrix, and J is a scalar. Using (2.3), the Symanzik polynomials $\mathcal{U}$ and $\mathcal{F}$ define as

$$\mathcal{U} = \det(K) \equiv |\boldsymbol{K}|, \qquad \mathcal{F} = |\boldsymbol{K}|(J + \boldsymbol{v}^T\boldsymbol{K}^{-1}\boldsymbol{v})\,. \tag{2.4}$$

After integrating over the loop momentum, the FI can be written as

$$I = \frac{\Gamma(\nu - \frac{LD}{2})}{\prod_{j=1}^{n}\Gamma(\nu_j)} \int d^n\boldsymbol{a}\ \delta(1-\sum_{j=1}^{n} a_j)(\prod_{j=1}^{n} a_j^{\nu_j-1}) \frac{[\mathcal{U}(\boldsymbol{a})]^{\nu-\frac{(L+1)D}{2}}}{[\mathcal{F}(\boldsymbol{a})]^{\nu-\frac{LD}{2}}} \tag{2.5}$$

For general integrals with numerator $\mathcal{N}(l)$

$$I = \int \prod_{r=1}^{L} \frac{d^D l_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{n} \frac{\mathcal{N}(l)}{D_j^{\nu_j}}, \tag{2.6}$$

one can carry the similar procedure to arrive the similar form as (2.5) with $(\prod_{j=1}^{n} a_j^{\nu_j-1})$ replaced by more general polynomial of a and the change of powers of $\mathcal{U}$ and $\mathcal{F}$.[1]

In this paper, we will focus on the 2-loop integrals although our discussions can be straightforwardly generalized to higher loops. For general 2-loop Feynman diagrams, the topology is just the shape of sunset and propagators can be divided into 3 branches. Following the strategy of [1] we label them with different parameters. For propagators of the form $\frac{1}{-(l_1-p_{Li})^2+m_{Li}^2}$, we denote the corresponding Feynman parameters as $(x_1, \ldots, x_{n_x})$. For propagators of the form $\frac{1}{-(l_2-p_{Rj})^2+m_{Rj}^2}$, we denote the corresponding Feynman parameters as $(y_{n_x+1}, \ldots, y_{n_x+n_y})$. Finally for propagators of the form $\frac{1}{-(l_1+l_2-p_{Ml})^2+m_{Ml}^2}$, we denote the corresponding Feynman parameters as $(z_{n_x+n_y+1}, \ldots, z_{n_x+n_y+n_z})$. By introducing three δ-functions, $\delta(X - \sum_i x_i)$, $\delta(Y - \sum_i y_i)$, and $\delta(Z - \sum_i z_i)$, into (2.5), we get a new expression

$$I = \frac{\Gamma(\nu - D)}{\prod_{j=1}^{n}\Gamma(\nu_j)} \int dXdYdZ\ \delta(1-X-Y-Z) \int \widetilde{d^n\boldsymbol{a}}\ \mathcal{G}\,, \tag{2.7}$$

where

$$\widetilde{d^n\boldsymbol{a}} = d^n\boldsymbol{a}\ \delta(X - \sum_i x_i)\delta(Y - \sum_i y_i)\delta(Z - \sum_i z_i)\,,$$

$$\mathcal{G} = (\prod_{j=1}^{n} a_j^{\nu_j-1}) \frac{[\mathcal{U}(\boldsymbol{a})]^{\nu-\frac{3D}{2}}}{[\mathcal{F}(\boldsymbol{a})]^{\nu-D}}\,. \tag{2.8}$$

[1] More details of derivations can be found in [1, 39, 40].

The explicit expressions for $\mathcal{U}$ and $\mathcal{F}$ are

$$\boldsymbol{K} = \begin{pmatrix} X+Z & Z \\ Z & Y+Z \end{pmatrix}, \qquad \mathcal{U} = |\boldsymbol{K}| = XY + XZ + YZ \tag{2.9}$$

and

$$\begin{aligned}
\mathcal{F} = \mathcal{U}(&\sum_{i=1}^{n_x} x_i(m_{Li}^2 - p_{Li}^2) + \sum_{j=1}^{n_y} y_j(m_{Rj}^2 - p_{Rj}^2) + \sum_{l=1}^{n_z} z_l(m_{Ml}^2 - p_{Ml}^2)) \\
&+ (Y+Z)\sum_{i=1}^{n_x} x_i^2 p_{Li}^2 + (Y+Z)\sum_{1\leqslant i<j\leqslant n_x} x_i x_j (2p_{Li}\cdot p_{Lj}) \\
&+ (X+Z)\sum_{j=1}^{n_y} y_j^2 p_{Rj}^2 + (X+Z)\sum_{1\leqslant i<j\leqslant n_y} y_i y_j (2p_{Ri}\cdot p_{Rj}) \\
&+ (X+Y)\sum_{l=1}^{n_z} z_l^2 p_{Ml}^2 + (X+Y)\sum_{1\leqslant i<j\leqslant n_z} z_i z_j (2p_{Mi}\cdot p_{Mj}) \\
&+ (Y-Z)\sum_{i=1}^{n_x}\sum_{l=1}^{n_z} x_i z_l (p_{Li}\cdot p_{Ml}) + (X-Z)\sum_{j=1}^{n_y}\sum_{l=1}^{n_z} y_j z_l (p_{Rj}\cdot p_{Ml}) \\
&+ (-2Z)\sum_{i=1}^{n_x}\sum_{j=1}^{n_y} x_i y_j (p_{Li}\cdot p_{Rj})
\end{aligned} \tag{2.10}$$

In this paper, we will not use the explicit expression (2.10), but only the character that $\mathcal{F}$ is a polynomial of Feynman parameters (x, y, z) up to degree two. Since $\mathcal{U}$ does not depend on x, y, z we can write I in (2.7) as

$$I = \frac{\Gamma(\nu - D)}{\prod_{j=1}^{n}\Gamma(\nu_j)} \int dXdYdZ\ \delta(1-X-Y-Z)\mathcal{U}^{\eta} \int \widetilde{d^n\boldsymbol{a}}\ \mathcal{P}(x,y,z)\mathcal{F}^{\gamma}\,, \tag{2.11}$$

where $\mathcal{P}$ is polynomial and η, γ are general powers for general integrals with numerators. We will call the part $\int \widetilde{d^n\boldsymbol{a}}\ \mathcal{P}(x,y,z)\mathcal{F}^{\gamma}$ as the **one-loop-like integrals**, which will be the focus of the paper. We want to remark that the one-loop-like integrals is called **fixed-branch integrals** in [1, 26].

For latter convenience, we write

$$\begin{aligned}
\mathcal{F} &= \mathcal{C}_0 + \sum_i \mathcal{C}_i a_i + \sum_{i,j} \mathcal{C}_{i,j} a_i a_j \\
&= \frac{1}{2}\begin{pmatrix} 1 & \boldsymbol{a}^T \end{pmatrix} \begin{pmatrix} 2\mathcal{C}_0 & \boldsymbol{\mathcal{C}}^T \\ \boldsymbol{\mathcal{C}} & \boldsymbol{\mathcal{A}} \end{pmatrix} \begin{pmatrix} 1 \\ \boldsymbol{a} \end{pmatrix} \equiv \frac{1}{2}\begin{pmatrix} 1 & \boldsymbol{a}^T \end{pmatrix} \boldsymbol{\mathcal{M}} \begin{pmatrix} 1 \\ \boldsymbol{a} \end{pmatrix}
\end{aligned} \tag{2.12}$$

where

$$\boldsymbol{\mathcal{C}}_i = \mathcal{C}_i, \qquad \boldsymbol{\mathcal{A}}_{i,j} = 2\mathcal{C}_{i,j}\,. \tag{2.13}$$

It is worth to notice that $\boldsymbol{\mathcal{A}}$ is a symmetric matrix, i.e., $\mathcal{C}_{i,j} = \mathcal{C}_{j,i}$. When we consider the differential equation of master integrals over $\mathcal{C}$, we need to take care of this point.

3 Complete reduction of one-loop-like integrals

In this section, we will discuss the IBP reduction for one-loop-like integrals. First we discuss how to construct the IBP relations for integrands having the delta-functions. Secondly, we write down some useful IBP relations.

3.1 IBP in the parameterization

In this part, we will discuss the reduction of $\int \widetilde{d^n\boldsymbol{a}}\, \mathcal{P}(x,y,z)\mathcal{F}^\gamma$ in (2.11) for general polynomial. To deal with the presence of delta-function, we will take a slightly different approach compared to the one in [1]. First using the three delta-functions, we can solve one of x, for example, $x_{k_1} = X - \sum_{j\neq k_1} x_j$ and similarly $y_{k_2} = Y - \sum_{j\neq k_2} y_j$ and $z_{k_3} = Z - \sum_{j\neq k_3} z_j$. After that, the $\int \widetilde{d^n\boldsymbol{a}}$ part in (2.11) can be written as

$$I'(\gamma,\{\nu_1,\ldots,\nu_n\}) = \int \widetilde{d^n\boldsymbol{a}}\, (\prod_{j\neq k_1,k_2,k_3} a_j^{\nu_j-1})[F(\boldsymbol{a})]^\gamma \equiv \int \widetilde{d^n\boldsymbol{a}}\, G\,, \tag{3.1}$$

where $F(\boldsymbol{k})$ is obtained from $\mathcal{F}$ by substituting x_{k_1}, y_{k_2}, and z_{k_3}. We will denote it

$$\begin{aligned} F(\boldsymbol{k}) &= c_0 + \sum_i c_i a_i + \sum_{i,j} c_{i,j} a_i a_j = \frac{1}{2}\begin{pmatrix} 1 & \boldsymbol{a}_{\widehat{\boldsymbol{k}}}{}^T \end{pmatrix} \begin{pmatrix} 2c_0 & \boldsymbol{C}^T \\ \boldsymbol{C} & \boldsymbol{A} \end{pmatrix} \begin{pmatrix} 1 \\ \boldsymbol{a}_{\widehat{\boldsymbol{k}}} \end{pmatrix} \\ &\equiv \frac{1}{2}\begin{pmatrix} 1 & \boldsymbol{a}_{\widehat{\boldsymbol{k}}}{}^T \end{pmatrix} \boldsymbol{M} \begin{pmatrix} 1 \\ \boldsymbol{a}_{\widehat{\boldsymbol{k}}} \end{pmatrix}\,, \end{aligned} \tag{3.2}$$

where the $\widehat{\boldsymbol{k}}$ denotes the removal of the k_1-th, k_2-th, and k_3-th rows and columns. Different choices of $\widehat{\boldsymbol{k}}$ will end up different $F(\boldsymbol{k})$ from the same (2.12).

Now let us consider the IBP relation of x_i $(i \neq k_1)$

$$T = \int_0^\infty d^n\boldsymbol{a}\; \partial_{x_i}\Big(\delta(X - \sum_j x_j)\delta(Y - \sum_j y_j)\delta(Z - \sum_j z_j)G\Big)\,. \tag{3.3}$$

On the one hand, we have

$$T = \int \widetilde{d^{n-1}\boldsymbol{a}}\; G|_{x_i=\infty} - \int \widetilde{d^{n-1}\boldsymbol{a}}\; G|_{x_i=0}\,. \tag{3.4}$$

The presence of the delta function makes the boundary term at $x_i = \infty$ vanish, leaving

$$T = -\int \widetilde{d^{n-1}\boldsymbol{a}}\; G|_{x_i=0} = -\int \widetilde{d^n\boldsymbol{a}}\; \delta(x_i)G\,. \tag{3.5}$$

On the other hand, we have

$$T = \int d^n\boldsymbol{a}\; \Big(\partial_{x_i}\delta(X - \sum_j x_j)\Big)\delta(Y - \sum_j y_j)\delta(Z - \sum_j z_j)G + \int \widetilde{d^n\boldsymbol{a}}\; \partial_{x_i}G\,. \tag{3.6}$$

Combining these results, we obtain

$$\begin{aligned} -\int \widetilde{d^n\boldsymbol{a}}\; \delta(x_i)G &= \int d^n\boldsymbol{a}\; \Big(\partial_{x_i}\delta(X - \sum_j x_j)\Big)\delta(Y - \sum_j y_j)\delta(Z - \sum_j z_j)G \\ &\quad + \int \widetilde{d^n\boldsymbol{a}}\; \partial_{x_i}G\,. \end{aligned} \tag{3.7}$$

Next, we consider the IBP relation of x_{k_1}. Following similar computations, we obtain

$$-\int \widetilde{d^n\boldsymbol{a}}\ \delta(x_{k_1})G = \int d^n\boldsymbol{a}\ \Big(\partial x_{k_1}\delta(X - \sum_j x_j)\Big)\delta(Y - \sum_j y_j)\delta(Z - \sum_j z_j)G\,, \tag{3.8}$$

where we have used the property $\partial_{x_{k_1}} G = 0$ since x_{k_1} has been integrated out. Now coming to the key observation

$$\partial_{x_i}\delta(X - \sum_j x_j) = \partial_{x_{k_1}}\delta(X - \sum_j x_j)\,. \tag{3.9}$$

By subtraction we get

$$\int \widetilde{d^n\boldsymbol{a}}\ \partial_{x_i} G = \int \widetilde{d^n\boldsymbol{a}}\ G(\delta(x_{k_1}) - \delta(x_i))\,. \tag{3.10}$$

For simplicity, we can express the IBP relations as the following algebraic identities, with the understanding that these identities hold only when they are substituted back into the integration context:

$$\partial_{x_i} G = \big[\delta(x_{k_1}) - \delta(x_i)\big]G\,. \tag{3.11}$$

For later convenience, we will define a_j^I to be the corresponding a, which appears in the same delta-function and has been integrated out. For example, x_{k_1} in (3.11) will be denoted as x_i^I. Using this notation, we can rewrite (3.11) as

$$\partial_{a_i} G = \overline{\delta}(a_i)G, \qquad \overline{\delta}(a_i) \equiv \delta(a_i^I) - \delta(a_i) \tag{3.12}$$

Noticing that although originally there are n Feynman parameters, there are only $(n-3)$ IBP relations in the form (3.12).

3.2 Iterative IBP reduction relations

In this part, we will consider the IBP relation of the form

$$\sum_i \frac{\partial(P_i(a)QF^\gamma)}{\partial a_i} = \sum_i \overline{\delta}(a_i)P_i(a)QF^\gamma \tag{3.13}$$

on the left-hand side of (3.12), where F is given in (3.2), $P(a)$ are polynomials of a and

$$Q(\vec{\nu}) = \prod_i a_i^{\nu_i}\,. \tag{3.14}$$

With some nice choices of $P_i(a)$, we will get several useful IBP relations. Using these relations, we can find the master integrals and obtain reduction coefficients of any integrals to these master integrals. Using these results, we can derive the differential equations of master integrals in the next section.

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3.2.1 The first type of choices

Now let us consider the first type of choice. Fixing an index i_0, we take

$$P_{i_0} = F, \quad P_{j\neq i_0} = 0 \tag{3.15}$$

Putting it back to (3.13), it gives

$$\frac{\nu_{i_0}}{a_{i_0}} Q(\vec{\nu})F^{\gamma+1} + (\gamma+1)\frac{\partial F}{\partial a_{i_0}} Q(\vec{\nu})F^{\gamma} = \overline{\delta}(a_{i_0})Q(\vec{\nu})F^{\gamma+1} \tag{3.16}$$

Let us define following useful action[2]

$$\boldsymbol{i}_0^- Q(\vec{\nu}) \equiv \frac{\partial}{\partial a_i} Q(\vec{\nu}) = \frac{\nu_{i_0}}{a_{i_0}} Q(\vec{\nu}), \qquad \boldsymbol{i}_0^+ Q(\vec{\nu}) \equiv a_{i_0} Q(\vec{\nu}) \tag{3.17}$$

Using them, (3.16) can be written as

$$\boldsymbol{i}_0^- Q(\vec{\nu})F^{\gamma+1} + (\gamma+1)(c_{i_0} + 2\sum_j c_{i_0 j}\boldsymbol{j}^+)Q(\vec{\nu})F^{\gamma} = \overline{\delta}(a_{i_0})Q(\vec{\nu})F^{\gamma+1} \tag{3.18}$$

The relation (3.18) contains different powers of F, so it is like the relation connecting different dimensions. It will be useful when we consider the differential equation for master integrals. To see its usefulness, let us consider several applications of the formula:

(a) When taking $Q(\vec{\nu}) = 1$ and $i_0 = 1, \ldots, \overline{n}$ (for simplicity, we have written $\overline{n} = n-3$), (3.18) can be written in the matrix form

$$\boldsymbol{A}\cdot\boldsymbol{a}F^{\gamma} = -\boldsymbol{C}F^{\gamma} + \frac{1}{(\gamma+1)}\overline{\boldsymbol{\delta}}F^{\gamma+1} \tag{3.19}$$

where $\boldsymbol{A}, \boldsymbol{C}$ are given in (3.2) (especially, $A_{ij} = 2c_{ij}$) and $\boldsymbol{a}, \overline{\boldsymbol{\delta}}$ are row vectors

$$\boldsymbol{a}^T = (a_1, \ldots, a_{\overline{n}}); \qquad \overline{\boldsymbol{\delta}}^T = (\overline{\delta}(a_1), \ldots, \overline{\delta}(a_{\overline{n}})) \tag{3.20}$$

If the matrix $\boldsymbol{A}$ is non-degenerate, we can reduce the rank one integrals[3] to the scalar basis and subsectors

$$\boldsymbol{a}F^{\gamma} = \boldsymbol{A}^{-1}\cdot\left\{-\boldsymbol{C}F^{\gamma} + \frac{1}{(\gamma+1)}\overline{\boldsymbol{\delta}}F^{\gamma+1}\right\} \tag{3.21}$$

If matrix $\boldsymbol{A}$ is degenerate, (3.19) means the scalar basis F^{γ} is not a basis anymore and can be reduced to subsectors. One way to see it is that now there is an eigenvector of $\boldsymbol{A}$ with eigenvalue zero, i.e., $\boldsymbol{\alpha}\cdot\boldsymbol{A} = 0$. Multiplying at both sides we get[4]

$$F^{\gamma} = \frac{1}{(\gamma+1)\boldsymbol{\alpha}\cdot\boldsymbol{C}^T}\boldsymbol{\alpha}\cdot\overline{\boldsymbol{\delta}}F^{\gamma+1} \tag{3.22}$$

[2] It is important to notice that $\boldsymbol{i}^{\pm}$ act only on $Q(\vec{\nu})$, not on F.

[3] Here we define the rank of the integrals $Q(\vec{\nu})F^{\gamma+1}$ to the $|\nu| \equiv \sum_i \nu_i$. Also, we call $\delta(a_i)F^{\gamma}$ to be the subsector of F^{γ} since the number of Feynman parameters is reduced by one.

[4] There may be more than one eigenvector with eigenvalue zero; different choices of $\boldsymbol{\alpha}$ will lead to extra relations between $\overline{\delta}^T F^{\gamma+1}$, which means subsectors are not all independent, i.e., some subsectors will not be master integrals anymore.

(b) For the $Q(\vec{\nu})$ with rank one, we can write $Q(\vec{\nu}) = \boldsymbol{a}^T$ as a row vector. Using $\boldsymbol{i}^- \cdot \boldsymbol{a}^T = I_{\overline{n}\times\overline{n}}$, (3.18) becomes the matrix form

$$I_{\overline{n}\times\overline{n}}F^{\gamma+1} + (\gamma+1)(\boldsymbol{C} + \boldsymbol{A}\cdot\boldsymbol{a})\cdot\boldsymbol{a}^T F^{\gamma} = \overline{\boldsymbol{\delta}}\cdot\boldsymbol{a}^T F^{\gamma+1} \tag{3.23}$$

If we take the trace at the both side, we will get

$$\overline{n}F^{\gamma+1} + (\gamma+1)(2F - \boldsymbol{C}^T\cdot\boldsymbol{a} - 2c_0)F^{\gamma} = \{\delta(x_{k_1})X + \delta(y_{k_2})Y + \delta(z_{k_3})Z\}\,F^{\gamma+1}$$

where we have used the result

$$\begin{aligned}\int \widetilde{d^n\boldsymbol{a}}\ \mathrm{Tr}(\overline{\boldsymbol{\delta}}\cdot\boldsymbol{a}^T) &= \int \widetilde{d^n\boldsymbol{a}}\ \overline{\boldsymbol{\delta}}^T\cdot\boldsymbol{a} = \int \widetilde{d^n\boldsymbol{a}}\sum_i(\delta(a_i^I) - \delta(a_i))a_i \\ &= \int \widetilde{d^n\boldsymbol{a}}\sum_i \delta(a_i^I)a_i = \int \widetilde{d^n\boldsymbol{a}}\,\{\delta(x_{k_1})X + \delta(y_{k_2})Y + \delta(z_{k_3})Z\}\end{aligned} \tag{3.24}$$

Rearranging it we can write

$$(\overline{n} + 2(\gamma+1))F^{\gamma+1} = (\gamma+1)(\boldsymbol{C}\cdot\boldsymbol{a}^T + 2c_0)F^{\gamma} + \delta_{XYZ}F^{\gamma+1} \tag{3.25}$$

where for simplification, we have defined

$$\delta_{XYZ} \equiv \delta(x_{k_1})X + \delta(y_{k_2})Y + \delta(z_{k_3})Z \tag{3.26}$$

If the matrix $\boldsymbol{A}$ is non-degenerate, we can use (3.21) to simplify (3.25) further as

$$\begin{aligned}(\overline{n} + 2(\gamma+1))F^{\gamma+1} = &\left\{2(\gamma+1)c_0 - (\gamma+1)\boldsymbol{C}^T\cdot\boldsymbol{A}^{-1}\cdot\boldsymbol{C}\right\}F^{\gamma} \\ &+ \left\{\boldsymbol{C}^T\cdot\boldsymbol{A}^{-1}\cdot\overline{\boldsymbol{\delta}} + \delta_{XYZ}\right\}F^{\gamma+1}\end{aligned} \tag{3.27}$$

Now we can consider the reduction of rank two tensor using (3.23). Rewritting it as

$$\boldsymbol{A}\cdot\boldsymbol{a}\cdot\boldsymbol{a}^T F^{\gamma} = -\boldsymbol{C}\cdot\boldsymbol{a}^T F^{\gamma} + \frac{1}{(\gamma+1)}\left\{-I_{\overline{n}\times\overline{n}}F^{\gamma+1} + \overline{\boldsymbol{\delta}}\cdot\boldsymbol{a}^T F^{\gamma+1}\right\} \tag{3.28}$$

we can solve

$$\boldsymbol{a}\cdot\boldsymbol{a}^T F^{\gamma} = -\boldsymbol{A}^{-1}\cdot\boldsymbol{C}\cdot\boldsymbol{a}^T F^{\gamma} + \frac{1}{(\gamma+1)}\boldsymbol{A}^{-1}\cdot\left\{-I_{\overline{n}\times\overline{n}}F^{\gamma+1} + \overline{\boldsymbol{\delta}}\cdot\boldsymbol{a}^T F^{\gamma+1}\right\} \tag{3.29}$$

when the matrix $\boldsymbol{A}$ is non-degenerate. The $F^{\gamma+1}$ at the right-hand side can be simplified further using (3.27). If $\boldsymbol{A}$ is degenerate and there is zero eigenvector such that $\boldsymbol{\alpha}^T\cdot\boldsymbol{C} \neq 0$, we can solve

$$\boldsymbol{a}^T F^{\gamma} = \frac{1}{(\gamma+1)\boldsymbol{\alpha}^T\cdot\boldsymbol{C}}\boldsymbol{\alpha}^T\cdot\left\{-I_{\overline{n}\times\overline{n}}F^{\gamma+1} + \overline{\boldsymbol{\delta}}\cdot\boldsymbol{a}^T F^{\gamma+1}\right\} \tag{3.30}$$

which reduces the rank one tensor integrals.

3.2.2 The second type of choices

For this one, we take

$$P_{i\neq i_0}(\boldsymbol{a}) = a_i \frac{\partial F(\boldsymbol{a})}{\partial a_{i_0}}, \qquad P_{i_0}(\boldsymbol{a}) = a_{i_0}\frac{\partial F(\boldsymbol{a})}{\partial a_{i_0}} + 2c_0 + \sum_j c_j a_j. \tag{3.31}$$

Putting it to (3.13) and doing some algebraic simplifications, we can get

$$\begin{aligned}&\left\{(2c_0 + \sum_j c_j \boldsymbol{j}^+)\boldsymbol{i}_0^- + (2\gamma + \overline{n} + 1 + \sum_j \boldsymbol{j}^+\boldsymbol{j}^-)(c_{i_0} + 2\sum_j c_{i_0 j}\boldsymbol{j}^+)\right\} Q(\vec{\nu})F^\gamma \\ &= \left\{\delta_{XYZ}(c_{i_0} + 2\sum_j c_{i_0 j}\boldsymbol{j}^+) + \overline{\delta}(a_{i_0})(2c_0 + \sum_j c_j \boldsymbol{j}^+)\right\} Q(\vec{\nu})F^\gamma\end{aligned} \tag{3.32}$$

Defining $\nu_Q = \sum_i \nu_i$ for $Q(\vec{\nu})$ and rewriting (3.32) to matrix form, we have

$$\begin{aligned}&(2\gamma + \overline{n} + 1 + \nu_Q + 1)\boldsymbol{A}\cdot\boldsymbol{a}Q(\vec{\nu})F^\gamma \\ &= -(2\gamma + \overline{n} + 1 + \nu_Q)\boldsymbol{C}Q(\vec{\nu})F^\gamma - (2c_0 + \boldsymbol{C}^T\cdot\boldsymbol{a})(\boldsymbol{i}^-)Q(\vec{\nu})F^\gamma \\ &\quad + \left\{\delta_{XYZ}(\boldsymbol{C} + \boldsymbol{A}\cdot\boldsymbol{a}) + \overline{\boldsymbol{\delta}}(2c_0 + \boldsymbol{C}^T\cdot\boldsymbol{a})\right\} Q(\vec{\nu})F^\gamma\end{aligned} \tag{3.33}$$

A good point comparing to (3.18) is that the power of F is the same in (3.33). When $|\boldsymbol{A}| \neq 0$, we can use this formula to reduce the higher rank tensor integrals on the left-hand side to the lower rank tensor integrals as well as the subsectors on the right-hand side. When $|\boldsymbol{A}| = 0$, we can also use it to reduce some higher rank tensor integrals. To add extra relations to do the reduction, we can use the following strategy. Taking the zero eigenvector $\boldsymbol{\alpha}_i$ of $\boldsymbol{A}$, we multiply it at the both side and rewrite $Q(\vec{\nu}) \to \boldsymbol{a}Q(\vec{\nu})$ to get

$$\begin{aligned}&(2\gamma + \overline{n} + 1 + \nu_Q + 1)\boldsymbol{\alpha}_i^T\cdot\boldsymbol{C}\boldsymbol{a}Q(\vec{\nu})F^\gamma + (2c_0 + \boldsymbol{C}^T\cdot\boldsymbol{a})\boldsymbol{\alpha}_i^T\cdot(\boldsymbol{i}^-)\boldsymbol{a}Q(\vec{\nu})F^\gamma \\ &= \boldsymbol{\alpha}_i^T\cdot\left\{\delta_{XYZ}(\boldsymbol{C} + \boldsymbol{A}\cdot\boldsymbol{a}) + \overline{\boldsymbol{\delta}}(2c_0 + \boldsymbol{C}^T\cdot\boldsymbol{a})\right\}\boldsymbol{a}Q(\vec{\nu})F^\gamma\end{aligned} \tag{3.34}$$

Taking all $\boldsymbol{\alpha}_i$ we get the wanted extra relations to give the full reduction when combining with (3.33).

Now we present some simple applications of (3.33):

(a) When $Q(\vec{\nu}) = 1$, we get

$$\begin{aligned}(2\gamma + \overline{n} + 2)\boldsymbol{A}\cdot\boldsymbol{a}F^\gamma &= -(2\gamma + \overline{n} + 1)\boldsymbol{C}F^\gamma \\ &\quad + \left\{\delta_{XYZ}(\boldsymbol{C} + \boldsymbol{A}\cdot\boldsymbol{a}) + (2c_0 + \boldsymbol{C}^T\cdot\boldsymbol{a})\overline{\boldsymbol{\delta}}\right\} F^\gamma\end{aligned} \tag{3.35}$$

It is similar to (3.19). When combining them together, we can have extra relations to reduce $\overline{\boldsymbol{\delta}}F^{\gamma+1}$.

(b) When $Q(\vec{\nu}) = \boldsymbol{a}$, we get

$$\begin{aligned}&(2\gamma + \overline{n} + 3)\boldsymbol{A}\cdot(\boldsymbol{a}\cdot\boldsymbol{a}^T)F^\gamma \\ &= -(2\gamma + \overline{n} + 2)\boldsymbol{C}\cdot\boldsymbol{a}^T F^\gamma - (2c_0 + \boldsymbol{C}^T\cdot\boldsymbol{a})I_{\overline{n}\times\overline{n}}F^\gamma \\ &\quad + \left\{\delta_{XYZ}(\boldsymbol{C} + \boldsymbol{A}\cdot\boldsymbol{a}) + (2c_0 + \boldsymbol{C}^T\cdot\boldsymbol{a})\overline{\boldsymbol{\delta}}\right\}\cdot\boldsymbol{a}^T F^\gamma\end{aligned} \tag{3.36}$$

which can reduce the rank two tensor integrals.

3.2.3 The third type of choices

For this one, we take

$$P_i(\boldsymbol{a}) = a_i. \tag{3.37}$$

The IBP relation is

$$\begin{aligned}&\left(\sum_i \boldsymbol{i}^+\boldsymbol{i}^- + \overline{n} + 2\gamma\right) Q(\vec{\nu})F^\gamma - \gamma(2c_0 + \sum_j c_j \boldsymbol{j}^+)Q(\vec{\nu})F^{\gamma-1} \\ &= \left(\sum_i \overline{\delta}(a_i)a_i\right) Q(\vec{\nu})F^\gamma\end{aligned} \tag{3.38}$$

Using the result (3.24), it becomes

$$\left(\sum_i \boldsymbol{i}^+\boldsymbol{i}^- + \overline{n} + 2\gamma\right) Q(\vec{\nu})F^\gamma - \gamma(2c_0 + \sum_j c_j \boldsymbol{j}^+)Q(\vec{\nu})F^{\gamma-1} = \delta_{XYZ}Q(\vec{\nu})F^\gamma \tag{3.39}$$

Again, we see a few examples:

(a) When $Q(\vec{\nu}) = 1$, we have

$$(\overline{n} + 2\gamma)\, F^\gamma - \gamma(2c_0 + \boldsymbol{C}^T \cdot \boldsymbol{a})F^{\gamma-1} = \delta_{XYZ}F^\gamma \tag{3.40}$$

It is nothing, but the one (3.25).

(b) When $Q(\vec{\nu}) = \boldsymbol{a}$, we get

$$(1 + \overline{n} + 2\gamma)\, \boldsymbol{a}F^\gamma - \gamma(2c_0 + \boldsymbol{C}^T \cdot \boldsymbol{a})\boldsymbol{a}F^{\gamma-1} = \delta_{XYZ}\boldsymbol{a}F^\gamma \tag{3.41}$$

4 Master integrals and their differential equations

From above discussions, one can see that any integral $I'(\gamma, \{\nu_1, \ldots, \nu_n\})$ can be reduced to integrals $I'(\gamma, \boldsymbol{w}) \equiv I'(\gamma, \{w_1, \ldots, w_n\})$ with $w_i = 0, 1$. It is similar to the fact that any one-loop integral can be reduced to the scalar integrals. With this observation, the first natural choice is to take the master integrals to be $I'(\gamma + 1, \boldsymbol{w})$.

Now let us see the differential equations for these chosen master integrals. For $\frac{\partial I'(\gamma+1,\boldsymbol{w})}{\partial c_0}$, we have,[5]

$$\frac{\partial F^{\gamma+1}}{\partial c_0} = (\gamma+1)F^\gamma = -\frac{2\gamma + n - 1}{\mathcal{D}} F^{\gamma+1} + \frac{\boldsymbol{C}^T \cdot \boldsymbol{A}^{-1} \cdot \overline{\boldsymbol{\delta}} + \delta_{XYZ}}{\mathcal{D}} F^{\gamma+1} \tag{4.1}$$

where the second equation has used the result (3.27) and

$$\mathcal{D} = \boldsymbol{C}^T \cdot \boldsymbol{A}^{-1} \cdot \boldsymbol{C} - 2c_0\,. \tag{4.2}$$

[5]Again, equation (4.1) holds only under the integration. However, since $\frac{\mathrm{d}}{\mathrm{d}c_0}$ commutes with integration, we can write it as the algebraic relation.

Here and in the later part of this section, we have assumed $\boldsymbol{A}$ is non-degenerate. The result (4.1) is nice since a given master integral in the differential equation depends only on itself and the nearest subsector.

Now we consider the differential equations for $\frac{\partial F^{\gamma+1}}{\partial c_i}$. Similarly we have

$$\frac{\partial F^{\gamma+1}}{\partial c_i} = (\gamma+1)a_i F^\gamma = -(\gamma+1)(\boldsymbol{A}^{-1}\cdot\boldsymbol{C})_i F^\gamma + (\boldsymbol{A}^{-1}\cdot\overline{\boldsymbol{\delta}})_i F^{\gamma+1} \tag{4.3}$$

where (3.21) has been used. Using (4.1) again, we get

$$\begin{aligned}\frac{\partial F^{\gamma+1}}{\partial c_i} &= (\gamma+1)a_i F^\gamma \\ &= -(\boldsymbol{A}^{-1}\cdot\boldsymbol{C})_i\left(-\frac{2\gamma+n-1}{\mathcal{D}}F^{\gamma+1}+\frac{\boldsymbol{C}^T\cdot\boldsymbol{A}^{-1}\cdot\overline{\boldsymbol{\delta}}+\delta_{XYZ}}{\mathcal{D}}F^{\gamma+1}\right) \\ &\quad+(\boldsymbol{A}^{-1}\cdot\overline{\boldsymbol{\delta}})_i F^{\gamma+1}\end{aligned} \tag{4.4}$$

Again we see a given master integral in the differential equation depends only on itself and the nearest subsector.

Finally we consider the differential equations for $\frac{\partial F^{\gamma+1}}{\partial c_{i,j}}$.[6] We have

$$\begin{aligned}\frac{\partial F^{\gamma+1}}{\partial c_{i,j}} &= (\gamma+1)a_i a_j F^\gamma \\ &= -(\gamma+1)(\boldsymbol{A}^{-1}\cdot\boldsymbol{C})_i a_j F^\gamma - (\boldsymbol{A}^{-1})_{ij}F^{\gamma+1} + (\boldsymbol{A}^{-1}\cdot\overline{\boldsymbol{\delta}})_i a_j F^{\gamma+1},\end{aligned} \tag{4.5}$$

where (3.29) has been used. To reduce to the chosen master integrals, the first term in the second line of (4.5) should be replaced by the result in (4.4) and we see the dependence of itself and the nearest subsector. The trouble part is the third term $a_j F^{\gamma+1}$ of the nearest subsector. Reducing this term will produce dependence in all subsectors, which makes the pattern of differential equation complicated. There is another disadvantage, i.e., the differential equations (4.1) and (4.4) are not canonical.

Now we want to construct the canonical basis from the natural scalar basis, i.e., looking at the basis of the form $g(c)F^{\gamma+1}$. Considering the action $\partial_{c_{ij}}$, we have

$$\begin{aligned}\frac{\partial g(c)F^{\gamma+1}}{\partial c_{i,j}} &= \frac{\partial g(c)}{\partial c_{i,j}}F^{\gamma+1} - (\boldsymbol{A}^{-1})_{ij}g(c)F^{\gamma+1} \\ &\quad - (\gamma+1)(\boldsymbol{A}^{-1}\cdot\boldsymbol{C})_i a_j g(c)F^\gamma + (\boldsymbol{A}^{-1}\cdot\overline{\boldsymbol{\delta}})_i a_j g(c)F^{\gamma+1},\end{aligned} \tag{4.6}$$

We can eliminate the first line by demanding

$$0 = \frac{\partial g}{\partial c_{i,j}} - g(\boldsymbol{A}^{-1})_{i,j} = \frac{\partial g}{\partial c_{i,j}} - g\frac{\partial_{\boldsymbol{A}_{i,j}}|\boldsymbol{A}|}{|\boldsymbol{A}|} = \frac{\partial g}{\partial c_{i,j}} - g\frac{\partial c_{i,j}}{\partial \boldsymbol{A}_{i,j}}\frac{\partial_{c_{i,j}}|\boldsymbol{A}|}{|\boldsymbol{A}|}\,. \tag{4.7}$$

Using $\boldsymbol{A}_{i,j} = 2c_{i,j}$, the solution is

$$g = \overline{g}(c_0, c_i)|\boldsymbol{A}|^{1/2}\,, \tag{4.8}$$

where $\overline{g}$ depends only on c_0, c_i.

[6]When we take the derivative over c_{ij} in (4.5), we have assumed c_{ij} to be independent. If one insistent the condition $c_{ij} = c_{ji}$, one just need to add up $\frac{\partial F^{\gamma+1}}{\partial c_{i,j}}$ and $\frac{\partial F^{\gamma+1}}{\partial c_{ji}}$.

Next, we want the appearance of the overall ϵ factor in the differential equation. To get the hind, let us look equation (4.1). For scalar integrals, from (2.5), we can see that

$$\gamma = \frac{LD}{2} - n = \frac{L(d-2\epsilon)}{2} - n \tag{4.9}$$

where n is the number of propagators of this sector. Thus

$$2\gamma + n - 1 = 2(d-2\epsilon) - n - 1 = 2d - n - 1 - 4\epsilon \tag{4.10}$$

where we have set $L = 2$ for our case. When n is odd, we can take the space-time dimension to be $d = \frac{n+1}{2}$ and $2\gamma + n - 1 = -4\epsilon$. However, when n is even, we can only take $d = \frac{n}{2}$ and now $2\gamma + n - 1 = -1 - 4\epsilon$. To cure this point, we need another nontrivial factor $\overline{g}(c)$ in the definition of canonical basis in (4.8). With the above explanation, now we define the canonical basis:

$$\mathcal{I}_{2m} = \epsilon^m \Gamma(m-1+2\epsilon) |\boldsymbol{A}|^{1/2} \mathcal{D}^{1/2} F_{2m}^{1-m-2\epsilon} \tag{4.11}$$

$$\mathcal{I}_{2m+1} = \epsilon^{m+1} \Gamma(m-1+2\epsilon) |\boldsymbol{A}|^{1/2} F_{2m+1}^{1-m-2\epsilon} \tag{4.12}$$

Here the subscript gives the number of Feynman parameters before integrating out three a using the delta-functions, thus for odd case, $m \geqslant 1$ and for even case $m \geqslant 2$.

Now we present canonical differential equation according to the value of n.

4.1 The case of $n = 2m$

By combining (4.1), (4.3) and (4.5), we have

$$d\mathcal{I}_{2m} = c_{2m\to 2m}\mathcal{I}_{2m} + \sum_i c_{2m\to 2m-1;i}\mathcal{I}_{2m-1}^{(i)} + \sum_{i\neq j} c_{2m\to 2m-2;ij}\mathcal{I}_{2m-2}^{(ij)} \tag{4.13}$$

In (4.13) the summation of i is $i = 1, \ldots, n$, where the three integrated indices should also be included. For the summation $\sum_{i\neq j}$ similar understanding should be taken. An important observation of (4.13) is that the right-hand side is up to sub-sub-sectors only. Now we give the expressions of coefficients c:

(a) For $c_{2m\to 2m}$, it is easy to find

$$\begin{aligned} c_{2m\to 2m} &= \frac{4\epsilon}{\mathcal{D}} \left\{ dc_0 - (\boldsymbol{A}^{-1}\boldsymbol{C})_i dc_i - (\boldsymbol{A}^{-1}\boldsymbol{C})_i (\boldsymbol{A}^{-1}\boldsymbol{C})_j dc_{i,j} \right\} \\ &= -2\epsilon d\log\mathcal{D} \end{aligned} \tag{4.14}$$

(b) For the coefficients $c_{2m\to 2m-1;i}$, using (4.1), (4.3) and (4.5) we will get the combination $|\boldsymbol{A}_{\widehat{i}}|^{1/2}\delta(a_i)F_{2m}^{2-m-2\epsilon}$ where $\boldsymbol{A}_{\widehat{i}}$ is the matrix obtained from $\boldsymbol{A}$ by removing the i-th row and column. However, $\delta(a_i)F_{2m}^{2-m-2\epsilon} = F_{2m-1}^{2-m-2\epsilon}$ which gives the basis (4.12). With this clarification, we find that when $i \notin (k_1, k_2, k_3)$,

$$\begin{aligned} c_{2m\to 2m-1;i} &= \frac{-2\epsilon(\boldsymbol{C}^T\boldsymbol{A}^{-1})_i|\boldsymbol{A}|^{1/2}}{\sqrt{(\mathcal{D}-\mathcal{D}_{\widehat{i}})}|\boldsymbol{A}_{\widehat{i}}|^{1/2}} d\log\left(\frac{(\boldsymbol{C}^T\boldsymbol{A}^{-1})_i|\boldsymbol{A}|^{1/2}/|\boldsymbol{A}_{\widehat{i}}|^{1/2} - \sqrt{\mathcal{D}}}{(\boldsymbol{C}^T\boldsymbol{A}^{-1})_i|\boldsymbol{A}|^{1/2}/|\boldsymbol{A}_{\widehat{i}}|^{1/2} + \sqrt{\mathcal{D}}}\right) \\ &= -2\epsilon d\log\left(\frac{(\boldsymbol{C}^T\boldsymbol{A}^{-1})_i|\boldsymbol{A}|^{1/2}/|\boldsymbol{A}_{\widehat{i}}|^{1/2} - \sqrt{\mathcal{D}}}{(\boldsymbol{C}^T\boldsymbol{A}^{-1})_i|\boldsymbol{A}|^{1/2}/|\boldsymbol{A}_{\widehat{i}}|^{1/2} + \sqrt{\mathcal{D}}}\right) \end{aligned} \tag{4.15}$$

where $\mathcal{D}_{\widehat{i}} = \boldsymbol{C}^T_{\widehat{i}} \cdot (\boldsymbol{A}_{\widehat{i}})^{-1} \cdot \boldsymbol{C}_{\widehat{i}} - 2c_0$. Later we will meet $\boldsymbol{A}_{\widehat{ij}}$ and $\mathcal{D}_{\widehat{ij}}$ and the similar understanding should be taken. The proof from the first line to second line is as follows. Take $i = 1$ as an example, expanding $\mathcal{D} - \mathcal{D}_{\widehat{1}}$ we will get

$$T = c_1^2 \boldsymbol{A}^{-1}_{1,1} + 2c_1 \sum_{j\neq 1} \boldsymbol{A}^{-1}_{1,j} c_j + \sum_{k,k\neq 1} [c_k^2(\boldsymbol{A}^{-1}_{k,k} - \boldsymbol{A}^{-1}_{\widehat{1};k,k}) + 2c_k \sum_{j\neq 1,k} (\boldsymbol{A}^{-1}_{k,j} - \boldsymbol{A}^{-1}_{\widehat{1};k,j}) c_j]$$

To continue, we need to use the Jacobi's identity (see [41] Lemma A.1 (e))

$$|\boldsymbol{A}^I_J| = (-)^{I+J} |\boldsymbol{A}| |(\boldsymbol{A}^{-1})^{\overline{I}}_{\overline{J}}| \tag{4.16}$$

where $\boldsymbol{A}^I_J$ is the submatrix constructed from the elements $A_{ij}, i \in I, j \in J$ and $\boldsymbol{A}^{\overline{I}}_{\overline{J}}$ is the submatrix constructed from the elements $A_{ij}, i \notin I, j \notin J$. The factor $(-)^{I+J}$ is $(-)^{\sum_{i\in I} i + \sum_{j\in J} j}$. Using (4.16), we have

$$\begin{aligned} \boldsymbol{A}^{-1}_{\widehat{1};k,j} &= \frac{\left(\mathrm{adj}\boldsymbol{A}_{\widehat{1}}\right)_{k,j}}{|\boldsymbol{A}_{\widehat{1}}|} = \frac{(-)^{k+j} |\boldsymbol{A}^{\overline{(1k)}}_{\overline{(1j)}}|}{|\boldsymbol{A}_{\widehat{1}}|} \\ &= \frac{|\boldsymbol{A}||(\boldsymbol{A}^{-1})^{1k}_{1j}|}{|\boldsymbol{A}_{\widehat{1}}|} = \frac{|\boldsymbol{A}|(\boldsymbol{A}^{-1}_{1,1}\boldsymbol{A}^{-1}_{k,j} - \boldsymbol{A}^{-1}_{1,k}\boldsymbol{A}^{-1}_{1,j})}{|\boldsymbol{A}|\boldsymbol{A}^{-1}_{1,1}} \end{aligned} \tag{4.17}$$

thus

$$\begin{aligned} T =& c_1^2 \boldsymbol{A}^{-1}_{1,1} + 2c_1 \sum_{j\neq 1} \boldsymbol{A}^{-1}_{1,j} c_j + \sum_{k,k\neq 1} [c_k^2 (\boldsymbol{A}^{-1}_{k,k} - \frac{|\boldsymbol{A}|(\boldsymbol{A}^{-1}_{1,1}\boldsymbol{A}^{-1}_{k,k} - (\boldsymbol{A}^{-1}_{1,k})^2)}{|\boldsymbol{A}|\boldsymbol{A}^{-1}_{1,1}}) \\ &+ 2c_k \sum_{j\neq 1,k} (\boldsymbol{A}^{-1}_{k,j} - \frac{|\boldsymbol{A}|(\boldsymbol{A}^{-1}_{1,1}\boldsymbol{A}^{-1}_{k,j} - \boldsymbol{A}^{-1}_{1,k}\boldsymbol{A}^{-1}_{1,j})}{|\boldsymbol{A}|\boldsymbol{A}^{-1}_{1,1}}) c_j] \\ =& \frac{((\boldsymbol{C}^T\boldsymbol{A}^{-1})_1)^2 |\boldsymbol{A}|}{|\boldsymbol{A}_{\widehat{1}}|} \,. \end{aligned} \tag{4.18}$$

When $i \in (k_1, k_2, k_3)$, for example, $i = k_1$, we find

$$\begin{aligned} c_{2m\to 2m-1;k_1} &= -\frac{2\epsilon |\boldsymbol{A}|^{1/2} \sum_{i=1, i\neq k_1}^{n_x} (\boldsymbol{C}^T\boldsymbol{A}^{-1})_i}{\sqrt{(\mathcal{D}-\mathcal{D}')}|\boldsymbol{A}'|^{1/2}} d\log\left(\frac{\sqrt{\mathcal{D}-\mathcal{D}'} - \sqrt{\mathcal{D}}}{\sqrt{\mathcal{D}-\mathcal{D}'} + \sqrt{\mathcal{D}}}\right), \\ &= -2\epsilon d\log\left(\frac{\sqrt{\mathcal{D}-\mathcal{D}'} - \sqrt{\mathcal{D}}}{\sqrt{\mathcal{D}-\mathcal{D}'} + \sqrt{\mathcal{D}}}\right) \end{aligned} \tag{4.19}$$

The $\mathcal{D}'$ and $\boldsymbol{A}'$ are obtained as follows. First we replace $x_{k_1'} = X - \sum_{i=1, i\neq k_1, k_1'}^{n_x}$ in the F (see (3.2)) and get the new $c_0', \boldsymbol{A}', \boldsymbol{C}'$. Then we construct $\mathcal{D}'$ using (4.2). In the appendix, we will prove that (4.19) can be expressed in the same form as (4.15) with the understanding that now $c_0, \boldsymbol{A}, \boldsymbol{C}$ are read out from the F, which is obtained from $\mathcal{F}$ (see (2.12)) by integrating out $x_{k_1'}$ instead of x_{k_1}.

(c) For coefficients $c_{2m\to 2m-2;ij}$, we again encounter the combination $\delta(a_i)\delta(a_j)F_{2m}^{2-m-2\epsilon} = F_{2m-2}^{1-(m-1)-2\epsilon}$, which is the basis (4.11). When $i \notin (k_1, k_2, k_3)$ and $j \notin (k_1, k_2, k_3)$, direct computation gives

$$c_{2m\to 2m-2;ij} = \frac{-\epsilon N}{2} d\log\left(\frac{\sqrt{(\mathcal{D}_{\widehat{i}}-\mathcal{D})\mathcal{D}_{\widehat{i,j}}} - \sqrt{(\mathcal{D}_{\widehat{i}}-\mathcal{D}_{\widehat{i,j}})\mathcal{D}}}{\sqrt{(\mathcal{D}_{\widehat{i}}-\mathcal{D})\mathcal{D}_{\widehat{i,j}}} + \sqrt{(\mathcal{D}_{\widehat{i}}-\mathcal{D}_{\widehat{i,j}})\mathcal{D}}}\right) + (i \leftrightarrow j)\,, \tag{4.20}$$

where coefficient

$$N = \frac{(\boldsymbol{C}^T\boldsymbol{A}^{-1})_{\widehat{i};j-\theta(j-i)}}{\sqrt{|\boldsymbol{A}_{\widehat{i,j}}|}} \frac{\sqrt{|\boldsymbol{A}|}(\boldsymbol{C}^T\boldsymbol{A}^{-1})_i}{\sqrt{(\mathcal{D}_{\widehat{i}}-\mathcal{D})(\mathcal{D}_{\widehat{i}}-\mathcal{D}_{\widehat{i,j}})}} = -\sqrt{-1}\,, \tag{4.21}$$

after using (4.18). When $i \in (k_1, k_2, k_3)$ or/and $j \in (k_1, k_2, k_3)$, we will have the similar expression with the understanding that now $c_0, \boldsymbol{A}, \boldsymbol{C}$ are read out from the F, which is obtained from $\mathcal{F}$ (see (2.12)) by integrating out proper variables, for example, when $i = k_1$, integrating out $x_{k_1'}$ instead of x_{k_1}.

4.2 The case of $n = 2m + 1$

By combining (4.1), (4.3) and (4.5), we have

$$d\mathcal{I}_{2m+1} = c_{2m+1\to 2m+1}\mathcal{I}_{2m+1} + \sum_i c_{2m+1\to 2m;i}\mathcal{I}_{2m}^{(i)} + \sum_{i\neq j} c_{2m+1\to 2m-1;ij}\mathcal{I}_{2m-1}^{(ij)} \tag{4.22}$$

The situation is similar to the case $n = 2m$, so we will be brief. For coefficient $c_{2m+1\to 2m+1}$ we have

$$c_{2m+1\to 2m+1} = -2\epsilon d\log\mathcal{D} \tag{4.23}$$

For coefficients to sub-sectors we have

$$\begin{aligned} c_{2m+1\to 2m;i} &= \frac{-\epsilon(\boldsymbol{C}^T\boldsymbol{A}^{-1})_i|\boldsymbol{A}|^{1/2}}{2\sqrt{(\mathcal{D}_{\widehat{i}}-\mathcal{D})}|\boldsymbol{A}_{\widehat{i}}|^{1/2}} d\log\left(\frac{\sqrt{-1}(\boldsymbol{C}^T\boldsymbol{A}^{-1})_i|\boldsymbol{A}|^{1/2}/|\boldsymbol{A}_{\widehat{i}}|^{1/2} - \sqrt{\mathcal{D}_{\widehat{i}}}}{\sqrt{-1}(\boldsymbol{C}^T\boldsymbol{A}^{-1})_i|\boldsymbol{A}|^{1/2}/|\boldsymbol{A}_{\widehat{i}}|^{1/2} + \sqrt{\mathcal{D}_{\widehat{i}}}}\right) \\ &= -\frac{\epsilon\sqrt{-1}}{2} d\log\left(\frac{\sqrt{-1}(\boldsymbol{C}^T\boldsymbol{A}^{-1})_i|\boldsymbol{A}|^{1/2}/|\boldsymbol{A}_{\widehat{i}}|^{1/2} - \sqrt{\mathcal{D}_{\widehat{i}}}}{\sqrt{-1}(\boldsymbol{C}^T\boldsymbol{A}^{-1})_i|\boldsymbol{A}|^{1/2}/|\boldsymbol{A}_{\widehat{i}}|^{1/2} + \sqrt{\mathcal{D}_{\widehat{i}}}}\right)\,, \end{aligned} \tag{4.24}$$

where the case $i \in (k_1, k_2, k_3)$ should be understood similarly. For coefficients to sub-sub-sectors we have

$$c_{2m+1\to 2m-1;ij} = \frac{\sqrt{-1}}{2}\epsilon d\log\left(\frac{\sqrt{-1}(\mathrm{adj}\boldsymbol{A})_{i,j} - \sqrt{|\boldsymbol{A}_{\widehat{i,j}}|}\sqrt{|\boldsymbol{A}|}}{\sqrt{-1}(\mathrm{adj}\boldsymbol{A})_{i,j} + \sqrt{|\boldsymbol{A}_{\widehat{i,j}}|}\sqrt{|\boldsymbol{A}|}}\right) \tag{4.25}$$

where $\mathrm{adj}\boldsymbol{A}$ denotes adjugate matrix of $\boldsymbol{A}$.

5 Degenerate examples

In the above sections, we have discussed the non-degenerate case systematically. In this section, we will briefly discuss the degenerate cases. There are two degenerate situations:

1) $|\boldsymbol{A}| = 0$.

Multiplying both sides of the equation (4.1) by $|\boldsymbol{A}|\mathcal{D}$ yields

$$(\gamma+2)|\boldsymbol{A}|\mathcal{D}F^{\gamma+1} = \boldsymbol{C}^T(\mathrm{adj}\boldsymbol{A})\bar{\boldsymbol{\delta}}F^{\gamma+2}\,. \tag{5.1}$$

Although $|\boldsymbol{A}| = 0$, while $|\boldsymbol{A}|\mathcal{D} = \boldsymbol{C}^T(\mathrm{adj}\boldsymbol{A})\boldsymbol{C} - |\boldsymbol{A}|2c_0 = \boldsymbol{C}^T(\mathrm{adj}\boldsymbol{A})\boldsymbol{C}$ remains non-vanishing. Consequently

$$(\gamma+2)F^{\gamma+1} = \frac{\boldsymbol{C}^T(\mathrm{adj}\boldsymbol{A})\bar{\boldsymbol{\delta}}}{\boldsymbol{C}^T(\mathrm{adj}\boldsymbol{A})\boldsymbol{C}}F^{\gamma+2}\,. \tag{5.2}$$

2) $|\mathcal{D}| = 0$.

Multiplying both sides of the equation (4.1) by $|\mathcal{D}|$ yields

$$0 = -(2\gamma+n-1)F^{\gamma+1} + \boldsymbol{C}^T\boldsymbol{A}^{-1}\bar{\boldsymbol{\delta}}F^{\gamma+1} + \delta_{XYZ}F^{\gamma+1}\,. \tag{5.3}$$

Thus,

$$F^{\gamma+1} = \frac{\boldsymbol{C}^T\boldsymbol{A}^{-1}\bar{\boldsymbol{\delta}}}{2\gamma+n-1}F^{\gamma+1} + \frac{\delta_{XYZ}}{2\gamma+n-1}F^{\gamma+1}\,. \tag{5.4}$$

In this section, we present several examples to illustrate how to handle degenerate cases. For convenience, we use (a,b,c) to denote a diagram in which one branch contains a propagators, while the other two branches contain b and c propagators, respectively.

5.1 (3,1,1)

Without loss of generality, we choose $a_{k_1} = x_3, a_{k_2} = y_4$ and $a_{k_3} = z_5$. In this case,

$$\begin{aligned}
&\gamma = -2-2\epsilon, \qquad \boldsymbol{A} = \begin{pmatrix} 2c_{1,1} & 2c_{1,2} \\ 2c_{1,2} & 2c_{2,2} \end{pmatrix} \\
&\mathcal{D} = -\frac{c_1^2c_{2,2} - 2c_2c_1c_{1,2} + c_2^2c_{1,1} + 4c_0\left(c_{1,2}^2 - c_{1,1}c_{2,2}\right)}{2c_{1,2}^2 - 2c_{1,1}c_{2,2}}\,.
\end{aligned} \tag{5.5}$$

If $|\boldsymbol{A}_{\hat{1}}| = 2c_{2,2} = 0$, then using (5.2), we obtain:

$$\begin{aligned}
-2\epsilon\delta(x_1)F^{-1-2\epsilon} &= \frac{(\boldsymbol{C}_{\hat{1}}^T(\mathrm{adj}\boldsymbol{A}_{\hat{1}}))_1}{\boldsymbol{C}_{\hat{1}}^T(\mathrm{adj}\boldsymbol{A}_{\hat{1}})\boldsymbol{C}_{\hat{1}}}\delta(x_1)(\delta(x_3)-\delta(x_2))F^{-2\epsilon} \\
&= \frac{1}{c_2}\delta(x_1)(\delta(x_3)-\delta(x_2))F^{-2\epsilon}\,.
\end{aligned} \tag{5.6}$$

Using the master integrals definition (4.11) and (4.12), we can get

$$\mathcal{I}_4^{(1)} = \frac{-1}{c_2}(\boldsymbol{C}_{\hat{1}}^T(\mathrm{adj}\boldsymbol{A}_{\hat{1}})\boldsymbol{C}_{\hat{1}})^{1/2}(\mathcal{I}_3^{(13)} - \mathcal{I}_3^{(12)}) = -\mathcal{I}_3^{(13)} + \mathcal{I}_3^{(12)}\,. \tag{5.7}$$

We can directly apply the differential equations (4.1), (4.3) and (4.5) to obtain the CDEs in degenerate cases. Since only $\mathcal{I}_3^{(13)}$ and $\mathcal{I}_3^{(12)}$ are affected, we include only the $\mathcal{I}_3^{(12)}$ term in the differential equations for simplicity (we do not explicitly write both terms, as their treatment follows a similar approach). From (4.1), we can get

$$\begin{aligned}\partial_{c_0}(\epsilon^3(2\epsilon)|\boldsymbol{A}|^{1/2}F^{-1-2\epsilon}) &= -2\epsilon^4\frac{(\boldsymbol{C}^T\boldsymbol{A}^{-1})_1|\boldsymbol{A}|^{1/2}}{\mathcal{D}}\delta(x_1)F^{-2\epsilon}\\ &= \frac{2\epsilon^3\sqrt{-1}c_{1,2}^2}{c_2^2c_{1,1}-2c_1c_2c_{1,2}+4c_0c_{1,2}^2}\delta(x_1)\delta(x_2)F^{-2\epsilon}\end{aligned} \tag{5.8}$$

where to get the second line, we have used (5.6). Analogously, we can get

$$\begin{aligned}\partial_{c_1}(\epsilon^3(2\epsilon)|\boldsymbol{A}|^{1/2}F^{-1-2\epsilon}) &= \frac{-\epsilon^3\sqrt{-1}c_2c_{1,2}}{c_2^2c_{1,1}-2c_1c_2c_{1,2}+4c_0c_{1,2}^2}\delta(x_1)\delta(x_2)F^{-2\epsilon}\\ \partial_{c_2}(\epsilon^3(2\epsilon)|\boldsymbol{A}|^{1/2}F^{-1-2\epsilon}) &= \frac{\epsilon^3\sqrt{-1}c_{1,2}(c_1c_2-4c_0c_{1,2})}{c_2\left(4c_0c_{1,2}^2+c_2(c_{1,1}c_2-2c_1c_{1,2})\right)}\delta(x_1)\delta(x_2)F^{-2\epsilon}\\ \partial_{c_{1,1}}(\epsilon^3(2\epsilon)|\boldsymbol{A}|^{1/2}F^{-1-2\epsilon}) &= \frac{\epsilon^3\sqrt{-1}c_{1,2}(c_1c_2-2c_0c_{1,2})}{c_{1,1}\left(4c_0c_{1,2}^2+c_2(c_{1,1}c_2-2c_1c_{1,2})\right)}\delta(x_1)\delta(x_2)F^{-2\epsilon}\\ \partial_{c_{1,2}}(\epsilon^3(2\epsilon)|\boldsymbol{A}|^{1/2}F^{-1-2\epsilon}) &= \frac{\epsilon^3\sqrt{-1}(c_1c_2-4c_0c_{1,2})}{2\left(4c_0c_{1,2}^2+c_2(c_{1,1}c_2-2c_1c_{1,2})\right)}\delta(x_1)\delta(x_2)F^{-2\epsilon}\,.\end{aligned} \tag{5.9}$$

Recall that $\mathcal{I}_5 = \epsilon^3(2\epsilon)|\boldsymbol{A}|^{1/2}F^{-1-2\epsilon}$ and $\mathcal{I}_3^{(12)} = \epsilon^2\delta(x_1)\delta(x_2)F^{-2\epsilon}$. From above equations, we can easily get the coefficient $c_{5\to3;12}$ in the degenerate case is

$$\frac{\sqrt{-1}\epsilon}{2}d\log\left(\frac{c_2^2c_{1,1}-2c_1c_2c_{1,2}+4c_0c_{1,2}^2}{c_2^2c_{1,1}}\right)\,. \tag{5.10}$$

Alternatively, we can attempt to derive the degenerate CDEs from the non-degenerate ones. From (5.7), we know that in the degenerate case, $\mathcal{I}_4^{(1)}$ decomposes into $\mathcal{I}_3^{(12)}$ and $\mathcal{I}_3^{(13)}$. Consequently, there are two contributions for the coefficients of $\mathcal{I}_3^{(12)}$: 1) A contribution already existed for the non-degenerate case; 2) the contribution originally came from the $\mathcal{I}_4^{(1)}$, which is further reduced to $\mathcal{I}_3^{(12)}$. Thus, the coefficient for $\mathcal{I}_5\to\mathcal{I}_3^{(12)}$ is given by

$$\begin{aligned}&c_{5\to3;12}+c_{5\to4;1}c_{4;1\to3;12}\\ &=\frac{\sqrt{-1}\epsilon}{2}d\log\left(\frac{-\sqrt{-4}c_{1,2}-\sqrt{4c_{1,1}c_{2,2}-4c_{1,2}^2}}{-\sqrt{-4}c_{1,2}+\sqrt{4c_{1,1}c_{2,2}-4c_{1,2}^2}}\right)\\ &-\frac{\sqrt{-1}\epsilon}{2}d\log\left(\frac{\sqrt{-1}(\boldsymbol{C}^T\boldsymbol{A}^{-1})_1\sqrt{|\boldsymbol{A}|}/\sqrt{|\boldsymbol{A}_{\hat{1}}|}-\sqrt{\mathcal{D}_{\hat{1}}}}{\sqrt{-1}(\boldsymbol{C}^T\boldsymbol{A}^{-1})_1\sqrt{|\boldsymbol{A}|}/\sqrt{|\boldsymbol{A}_{\hat{1}}|}+\sqrt{\mathcal{D}_{\hat{1}}}}\right)(1)\\ &=\frac{\sqrt{-1}\epsilon}{2}d\log\left(\frac{\sqrt{-1}c_{1,2}+\sqrt{c_{1,1}c_{2,2}-c_{1,2}^2}}{\sqrt{-1}c_{1,2}-\sqrt{c_{1,1}c_{2,2}-c_{1,2}^2}}\frac{\sqrt{-1}(\boldsymbol{C}^T\boldsymbol{A}^{-1})_1\sqrt{|\boldsymbol{A}|}/\sqrt{|\boldsymbol{A}_{\hat{1}}|}+\sqrt{\mathcal{D}_{\hat{1}}}}{\sqrt{-1}(\boldsymbol{C}^T\boldsymbol{A}^{-1})_1\sqrt{|\boldsymbol{A}|}/\sqrt{|\boldsymbol{A}_{\hat{1}}|}-\sqrt{\mathcal{D}_{\hat{1}}}}\right)\\ &=\frac{\sqrt{-1}\epsilon}{2}d\log\left(\frac{c_2^2c_{1,1}-2c_1c_2c_{1,2}+4c_0c_{1,2}^2}{c_2^2c_{1,1}}\right)\,.\end{aligned} \tag{5.11}$$

To obtain the third equation, expand the terms inside the parentheses as a power series in c_{22} and extract the coefficient of the constant term. The result is consistent with (5.10).

If $\mathcal{D}_{\hat{i}} = 0$, $\mathcal{I}_{2m}^{(i)} = 0$ follows from (4.11), and $c_{2m+1\to 2m;i} = 0$ follows from (4.24). This implies that $\mathcal{I}_{2m}^{(i)}$ can be omitted, while the other terms in the canonical differential equations remain unaffected.

5.2 (4,1,1)

In $(4,1,1)$ diagram, $\mathcal{I}_6 = 2\epsilon^4(1+2\epsilon)|\boldsymbol{A}|^{1/2}\mathcal{D}^{1/2}F_6^{-2-2\epsilon}$, $\mathcal{I}_5^{(1)} = 2\epsilon^4|\boldsymbol{A}_{\hat{1}}|^{1/2}F_5^{-1-2\epsilon}$ and $\mathcal{I}_4^{(12)} = 2\epsilon^3|\boldsymbol{A}_{\widehat{1,2}}|^{1/2}\mathcal{D}_{\widehat{1,2}}F_4^{-1-2\epsilon}$. Without loss of generality, we choose $a_{k_1} = x_4, a_{k_2} = y_4$ and $a_{k_3} = z_5$. We derive the degenerate CDEs from the non-degenerate ones. For $\mathcal{D}_{\hat{1}} = 0$, from (5.4) we can obtain

$$2\epsilon^4\delta(x_1)F^{\gamma+2} = \frac{1}{4}(2\epsilon^3)(\boldsymbol{C}_{\hat{1}}^T\boldsymbol{A}_{\hat{1}}^{-1})_1\delta(x_1)\delta(x_2)F^{\gamma+2} + \dots \tag{5.12}$$

Consequently,

$$\mathcal{I}_5^{(1)} = \frac{1}{4}\frac{|\boldsymbol{A}_{\hat{1}}|^{1/2}(\boldsymbol{C}_{\hat{1}}^T\boldsymbol{A}_{\hat{1}}^{-1})_1}{|\boldsymbol{A}_{\widehat{1,2}}|^{1/2}(\mathcal{D}_{\widehat{1,2}})^{1/2}}\mathcal{I}_4^{(12)} + \dots = \frac{\sqrt{-1}}{4}\mathcal{I}_4^{(12)} + \dots \tag{5.13}$$

Here, for simplicity, we omit other sub-sub-sector master integrals on the right-hand side of the above two equations, as their treatment follows a similar approach. Analogous to (5.11), the coefficient for $\mathcal{I}_6 \to \mathcal{I}_4^{(12)}$ is given by

$$\begin{aligned}
&c_{6\to 4;12} + c_{6\to 5;1}c_{5;1\to 4;12}\\
&= \frac{\epsilon\sqrt{-1}}{2}d\log\frac{\sqrt{(\mathcal{D}_{\hat{1}}-\mathcal{D})\mathcal{D}_{\widehat{1,2}}} - \sqrt{(\mathcal{D}_{\hat{1}}-\mathcal{D}_{\widehat{1,2}})\mathcal{D}}}{\sqrt{(\mathcal{D}_{\hat{1}}-\mathcal{D})\mathcal{D}_{\widehat{1,2}}} + \sqrt{(\mathcal{D}_{\hat{1}}-\mathcal{D}_{\widehat{1,2}})\mathcal{D}}} + (1\leftrightarrow 2)\\
&\quad - \frac{\epsilon 2\sqrt{-1}}{4}d\log(\frac{\sqrt{\mathcal{D}-\mathcal{D}_{\hat{1}}} - \sqrt{\mathcal{D}}}{\sqrt{\mathcal{D}-\mathcal{D}_{\hat{1}}} + \sqrt{\mathcal{D}}})\\
&= -\frac{\epsilon\sqrt{-1}}{2}d\log\frac{\mathcal{D}_{\widehat{1,2}}}{\mathcal{D}_{\widehat{1,2}}-\mathcal{D}} + \frac{\epsilon\sqrt{-1}}{2}d\log\frac{\sqrt{(\mathcal{D}_{\hat{2}}-\mathcal{D})\mathcal{D}_{\widehat{1,2}}} - \sqrt{(\mathcal{D}_{\hat{2}}-\mathcal{D}_{\widehat{1,2}})\mathcal{D}}}{\sqrt{(\mathcal{D}_{\hat{2}}-\mathcal{D})\mathcal{D}_{\widehat{1,2}}} + \sqrt{(\mathcal{D}_{\hat{2}}-\mathcal{D}_{\widehat{1,2}})\mathcal{D}}},
\end{aligned} \tag{5.14}$$

where

$$\mathcal{D} = \boldsymbol{C}^T\boldsymbol{A}^{-1}\boldsymbol{C} - 2c_0\,, \tag{5.15}$$

with

$$\boldsymbol{A} = \begin{pmatrix} 2c_{1,1} & 2c_{1,2} & 2c_{1,3}\\ 2c_{1,2} & 2c_{2,2} & 2c_{2,3}\\ 2c_{1,3} & 2c_{2,3} & 2c_{3,3} \end{pmatrix} \tag{5.16}$$

If $|\boldsymbol{A}_{\hat{i}}| = 0$, $\mathcal{I}_{2m-1}^{(i)} = 0$ follows from (4.12), and $c_{2m\to 2m-1;i} = 0$ follows from (4.15). This implies that $\mathcal{I}_{2m-1}^{(i)}$ can be omitted, while the other terms in the canonical differential equations remain unaffected.

5.3 (5,1,1)

Without loss of generality, we choose $a_{k_1} = x_5, a_{k_2} = y_4$ and $a_{k_3} = z_5$. Following the steps outlined in the previous two subsections, the first step in deriving the canonical differential equations (CDEs) for the degenerate case is to determine the coefficient for the transition $\mathcal{I}_6^{(i)} \to \mathcal{I}_5^{(ij)}$.

For $|\boldsymbol{A}_{\widehat{1}}| = 0$, then using (5.2), we obtain:

$$\mathcal{I}_6^{(1)} = \mathcal{I}_5^{(12)} + \dots\,, \tag{5.17}$$

where

$$\mathcal{I}_6^{(1)} = (1+2\epsilon)2\epsilon^2\delta(x_1)|\boldsymbol{A}_{\widehat{1}}|^{1/2}\mathcal{D}_{\widehat{1}}^{1/2}F^{-2-2\epsilon}, \quad \mathcal{I}_5^{(12)} = \delta(x_1)\delta(x_2)(2\epsilon^4)|\boldsymbol{A}_{\widehat{1,2}}|^{1/2}F^{-2\epsilon}. \tag{5.18}$$

Analogous to (5.11), the coefficient for $\mathcal{I}_7 \to \mathcal{I}_5^{(12)}$ is given by

$$\begin{aligned}
&c_{7\to5;12} + c_{7\to6;1}c_{6;1\to5;12} \\
&= \frac{\epsilon\sqrt{-1}}{2} d\log\left(\frac{\sqrt{-1}(\mathrm{adj}\boldsymbol{A})_{1,2} - \sqrt{|\boldsymbol{A}_{\widehat{1,2}}|}\sqrt{|\boldsymbol{A}|}}{\sqrt{-1}(\mathrm{adj}\boldsymbol{A})_{1,2} + \sqrt{|\boldsymbol{A}_{\widehat{1,2}}|}\sqrt{|\boldsymbol{A}|}}\right) \\
&\quad - \frac{\epsilon\sqrt{-1}}{2} d\log\left(\frac{\sqrt{-1}(\boldsymbol{C}^T\boldsymbol{A}^{-1})_1\sqrt{|\boldsymbol{A}|}/\sqrt{|\boldsymbol{A}_{\widehat{1}}|} - \sqrt{\mathcal{D}_{\widehat{1}}}}{\sqrt{-1}(\boldsymbol{C}^T\boldsymbol{A}^{-1})_1\sqrt{|\boldsymbol{A}|}/\sqrt{|\boldsymbol{A}_{\widehat{1}}|} + \sqrt{\mathcal{D}_{\widehat{1}}}}\right) \\
&= \frac{\epsilon\sqrt{-1}}{2} d\log\left(\frac{-|\boldsymbol{A}_{\widehat{2}}|\boldsymbol{C}_{\widehat{1}}^T(\mathrm{adj}\boldsymbol{A}_{\widehat{1}})\boldsymbol{C}_{\widehat{1}}}{((\mathrm{adj}\boldsymbol{A})_{1,2})^2\mathcal{D}}\right)\,,
\end{aligned} \tag{5.19}$$

where

$$\mathcal{D} = \boldsymbol{C}^T\boldsymbol{A}^{-1}\boldsymbol{C} - 2c_0\,, \tag{5.20}$$

with

$$\boldsymbol{A} = \begin{pmatrix} 2c_{1,1} & 2c_{1,2} & 2c_{1,3} & 2c_{1,4} \\ 2c_{1,2} & 2c_{2,2} & 2c_{2,3} & 2c_{2,4} \\ 2c_{1,3} & 2c_{2,3} & 2c_{3,3} & 2c_{3,4} \\ 2c_{1,4} & 2c_{2,4} & 2c_{3,4} & 2c_{4,4} \end{pmatrix} \tag{5.21}$$

6 Conclusion

We systematically analyzed the properties of the one-loop-like integrals under the newly proposed HHM representation of Feynman integrals. This includes providing an alternative iterative reduction scheme equivalent to that in [1], based on which we derived the canonical basis and canonical differential equations for this function family. We found that its properties are remarkably similar to those of traditional one-loop integrals. Both cases require only two formulas, distinguished by whether the number of propagators is even or odd, to express all canonical master integrals, as given in (4.11) and (4.12) of our paper. In the non-degenerate

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case, their canonical differential equations depend on at most two fewer master integrals, making a complete and systematic discussion feasible. The corresponding matrix elements and symbol structures are provided in (4.14), (4.15), and (4.20), as well as in (4.23), (4.24) and (4.25). The symbol structure of the one-loop-like integrals in the HHM representation also closely resembles that of traditional one-loop diagrams (see, for example, [27]). Additionally, we presented several examples of degenerate cases for the canonical differential equations.

Since the one-loop-like integral family we studied is the first step that needs to be computed for efficient two-loop calculations under the HHM representation [1], providing its canonical differential equations is crucial for directly obtaining its analytic results, performing fast numerical computations, and systematically analyzing its singularity structure. This could further facilitate more efficient two-loop and even higher-loop computations under the HHM representation.

We also observe that the differential equations of one-loop-like integrals can be used to establish IBP relations for the full two-loop integrals, i.e., it effectively transforms the problem of multi-variable IBP reduction for arbitrary two-loop integrals into a two-variable reduction problem (where three variables X, Y, Z are reduced to two after integrating out a delta function).

As a result, an interesting direction for future exploration is whether our differential equations for the one-loop-like integrals can aid in studying the reduction properties of complete two-loop or higher-loop integrals and help uncover the iterative structure of IBP relations (see, for example, [42]). Furthermore, integrating this representation with modern mathematical tools such as computational algebraic geometry [45–47], intersection theory [48, 49] and generating function [37] are also worth considering in future research.

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A Independence of the choice of $\boldsymbol{k}$

In the paper, we derive the canonical basis and canonical differential equation using the F, which is obtained from $\mathcal{F}$ after integrating out three a_i's. Since different choices of a_i's will give different expressions of F, it is not obvious that the canonical basis and canonical differential equation will be independent of these choices. In this part, we will prove this point.

The procedure of integrated out can be represented by an $(n+1)\times(n-2)$ transformation matrix as

$$(1, a_1, \ldots, a_n)^T = \boldsymbol{b}(\boldsymbol{k})(1, \boldsymbol{x}_{\widehat{k_1}}, \boldsymbol{y}_{\widehat{k_2}}, \boldsymbol{z}_{\widehat{k_3}})^T\,, \tag{A.1}$$

where the $\boldsymbol{x}_{\widehat{k_1}}$ represents the remaining parameters along the X-branch, i.e., $\widehat{k_1}$ means the x_{k_1} has been removed. Thus from (2.12) and (3.2) we get

$$\boldsymbol{M} = \boldsymbol{b}(\boldsymbol{k})^T \boldsymbol{\mathcal{M}} \boldsymbol{b}(\boldsymbol{k}) \tag{A.2}$$

Since matrix $\boldsymbol{b}(\boldsymbol{k})$ is not a square matrix, we want to rewrite (A.1) as

$$(1, a_1, \ldots, a_n)^T = \boldsymbol{B}(\boldsymbol{k})(1, \boldsymbol{x}_{\widetilde{k_1}}, \boldsymbol{y}_{\widetilde{k_2}}, \boldsymbol{z}_{\widetilde{k_3}})^T\,, \tag{A.3}$$

where the $\boldsymbol{x}_{\widetilde{k_1}}$ represents the same X-branch, but the x_{k_1} has been set to zero. The elements of the sqaure $(n+1) \times (n+1)$ matrix $\boldsymbol{B}(\boldsymbol{k})$ are

$$\boldsymbol{B}(\boldsymbol{k})_{i,j} = \begin{cases} \delta_{jk}, & i \neq k_1+1, k_2+1, k_3+1 \\ X, & i = k_1+1, j=1 \\ Y, & i = k_2+1, j=1 \\ Z, & i = k_3+1, j=1 \\ -1, & i = k_1+1, j = 2, \ldots, n_x+1 \\ -1, & i = k_2+1, j = n_x+2, \ldots, n_x+n_y+1 \\ -1, & i = k_3+1, j = n_x+n_y+2, \ldots, n_x+n_y+n_z+1 \end{cases} \tag{A.4}$$

There are some properties about $\boldsymbol{B}(\boldsymbol{k})$,

$$\boldsymbol{B}(\boldsymbol{k})^{-1} = \boldsymbol{B}(\boldsymbol{k}), \quad |\boldsymbol{B}(\boldsymbol{k})| = -1\,. \tag{A.5}$$

Similarly, we get

$$\boldsymbol{\mathcal{M}}(\boldsymbol{k}) = \boldsymbol{B}(\boldsymbol{k})^T \boldsymbol{\mathcal{M}} \boldsymbol{B}(\boldsymbol{k})\,. \tag{A.6}$$

Matrices $\boldsymbol{b}(\boldsymbol{k})$ and $\boldsymbol{B}(\boldsymbol{k})$ have following relation: $\boldsymbol{B}(\boldsymbol{k})$ includes three additional columns, specifically the k_1+1-th, k_2+1-th, and k_3+1-th column. Thus $\boldsymbol{M} = \boldsymbol{\mathcal{M}}(\boldsymbol{k})_{\widehat{\boldsymbol{k+1}}}$, i.e., three corresponding rows and three columns have been removed from $\boldsymbol{\mathcal{M}}(\boldsymbol{k})$. Utilizing Jacobi's identity (4.16) we have

$$|\boldsymbol{M}| = |\boldsymbol{\mathcal{M}}(\boldsymbol{k})_{\widehat{\boldsymbol{k+1}}}| = |\boldsymbol{\mathcal{M}}(\boldsymbol{k})||\boldsymbol{\mathcal{M}}(\boldsymbol{k})^{-1}_{\boldsymbol{k+1},\boldsymbol{k+1}}|\,, \tag{A.7}$$

where

$$|\boldsymbol{\mathcal{M}}(\boldsymbol{k})| = |\boldsymbol{B}(\boldsymbol{k})^T \boldsymbol{\mathcal{M}} \boldsymbol{B}(\boldsymbol{k})| = |\boldsymbol{\mathcal{M}}| \tag{A.8}$$

which is independent of $\boldsymbol{k}$ and

$$|\boldsymbol{\mathcal{M}}(\boldsymbol{k})^{-1}_{\boldsymbol{k+1},\boldsymbol{k+1}}| = \begin{vmatrix} \boldsymbol{\mathcal{M}}(\boldsymbol{k})^{-1}_{k_1+1,k_1+1} & \boldsymbol{\mathcal{M}}(\boldsymbol{k})^{-1}_{k_1+1,k_2+1} & \boldsymbol{\mathcal{M}}(\boldsymbol{k})^{-1}_{k_1+1,k_3+1} \\ \boldsymbol{\mathcal{M}}(\boldsymbol{k})^{-1}_{k_1+1,k_2+1} & \boldsymbol{\mathcal{M}}(\boldsymbol{k})^{-1}_{k_2+1,k_2+1} & \boldsymbol{\mathcal{M}}(\boldsymbol{k})^{-1}_{k_2+1,k_3+1} \\ \boldsymbol{\mathcal{M}}(\boldsymbol{k})^{-1}_{k_3+1,k_1+1} & \boldsymbol{\mathcal{M}}(\boldsymbol{k})^{-1}_{k_3+1,k_2+1} & \boldsymbol{\mathcal{M}}(\boldsymbol{k})^{-1}_{k_3+1,k_3+1} \end{vmatrix} \tag{A.9}$$

One can easily discover $\boldsymbol{\mathcal{M}}(\boldsymbol{k})^{-1}_{\boldsymbol{k+1},\boldsymbol{k+1}}$ is independent of $\boldsymbol{k}$. For example

$$\begin{aligned} \boldsymbol{\mathcal{M}}(\boldsymbol{k})^{-1}_{k_1+1,k_1+1} &= \sum_{js} \boldsymbol{B}(\boldsymbol{k})_{k_1+1,j} \boldsymbol{\mathcal{M}}^{-1}_{j,s} \boldsymbol{B}(\boldsymbol{k})^T_{s,k_1+1} \\ &= X^2 \boldsymbol{\mathcal{M}}^{-1}_{11} - 2X \sum_{j=2}^{n_x+1} \boldsymbol{\mathcal{M}}^{-1}_{1j} + \sum_{j,s=2}^{n_x+1} \boldsymbol{\mathcal{M}}^{-1}_{js} \end{aligned} \tag{A.10}$$

which is independent of the explicit choice x_{k_1}. Analogously, $|\boldsymbol{A}| = |\boldsymbol{M}_{\widehat{1}}|$ is also independent of $\boldsymbol{k}$. Since $|\boldsymbol{M}| = |\boldsymbol{A}|(-\mathcal{D})$, consequently, $\mathcal{D}$ is also independent of $\boldsymbol{k}$. These arguments have proved results, such as (4.15), are independent of $\boldsymbol{k}$.

Now we prove (4.19) can be written into the form (4.15), which is equal to the proof of $\mathcal{D}' = \mathcal{D}(k_1')_{\widehat{k_1}}$, where $\mathcal{D}(k_1')$ is obtained from the replacement of $x_{k_1'}$ rather than x_{k_1}. Recall that $\mathcal{D}'$ is obtained through the simultaneous replacement of $x_{k_1}, x_{k_2}, x_{k_3}, x_{k_1'}$. Since the order of replacements does not affect the result, we can conceptualize $\mathcal{D}'$ as being first obtained from the replacement of $x'_{k_1}, x_{k_2}, x_{k_3}$, followed by the replacement of x_{k_1}. Thus

$$\mathcal{D}' = -\frac{|\boldsymbol{\mathcal{M}}'(k_1)_{\widehat{k_1+1}}|}{|\boldsymbol{\mathcal{A}}'(k_1)_{\widehat{k_1}}|}\,, \tag{A.11}$$

where $\boldsymbol{\mathcal{M}}' = \boldsymbol{\mathcal{M}}(k_1', k_2, k_3)$ and $\boldsymbol{\mathcal{A}}' = \boldsymbol{\mathcal{A}}(k_1', k_2, k_3)$. Thus we have $\mathcal{D}' = \mathcal{D}(k_1')_{\widehat{k_1}}$.

B One-loop check

In the appendix, we will validate our main results at the one-loop level by examining examples of IBP relations and symbols presented in sections 3 and 4. This validation will cover one-loop integrals up to the pentagon. For the sake of simplicity, this check will be performed numerically using the following mass and kinematic parameters:

$$\begin{aligned}\{m_1^2, m_2^2, m_3^2, m_4^2, m_5^2\} &= \{31, 37, 41, 43, 47\}\\ \{s_{11}, s_{12}, s_{13}, s_{22}, s_{23}, s_{33}, s_{14}, s_{24}, s_{34}, s_{44}\} &= \left\{\frac{1}{2}, \frac{3}{5}, \frac{7}{11}, \frac{13}{17}, \frac{19}{23}, \frac{27}{29}, \frac{53}{59}, \frac{61}{67}, \frac{71}{73}, \frac{79}{83}\right\}\end{aligned} \tag{B.1}$$

where $s_{ij} = p_i \cdot p_j$. We define the following notations for simplicity:

$$\begin{aligned}\mathcal{K}_n &= \det G(p_1, \ldots, p_{n-1}), \quad \mathcal{B}_n(z_1, \ldots, z_n) = \det G(\ell, p_1, \ldots, p_{n-1})\,,\\ \mathcal{B}_n'(z_1, \ldots, z_n) &= \det G(\ell, p_1, \ldots, p_{n-2}; p_{n-1}, p_1, \ldots, p_{n-2})\,,\\ \mathcal{B}_n''(z_1, \ldots, z_n) &= \det G(\ell, p_1, \ldots, p_{n-2}; \ell, p_1, \ldots, p_{n-3}, p_{n-2}+p_{n-1})\,,\\ \mathcal{K}_n' &= \det G(p_1, \ldots, p_{n-2}; p_1, \ldots, p_{n-3}, p_{n-2}+p_{n-1}),\end{aligned} \tag{B.2}$$

where $G(a_1, \ldots, a_n; b_1, \ldots, b_n)$ represents the Gram matrix with entries $G_{i,j} = a_i \cdot b_j$, and $G(a_1, \ldots, a_n) \equiv G(a_1, \ldots, a_n; a_1, \ldots, a_n)$

The one-loop FI in Feynman parameterization is given by:

$$I(\nu_1, \ldots, \nu_n; D) = \frac{\Gamma(\nu - D/2)}{\prod_{j=1}^n \Gamma(\nu_j)} \int d^n\boldsymbol{a}\, \delta(1 - \sum_{j=1}^n a_j^{\nu_j})\, F^{D/2-\nu}\,, \tag{B.3}$$

where a_i in F has been substituted with $(1 - \sum_{j\neq i} a_j)$. Without loss of generality, we set $a_i = a_1$ hereafter. Consequently, in the one-loop case, equation (4.1) can be reformulated as[7]

$$I(D+2) = \frac{\mathcal{D}}{D-n+1} I(D) - \frac{(\boldsymbol{C}^T \cdot \boldsymbol{A}^{-1})_i}{D-n+1} I_{\widehat{i+1}}(D) + \frac{1+\sum_i (\boldsymbol{C}^T \cdot \boldsymbol{A}^{-1})_i}{D-n+1} I_{\widehat{1}}(D)\,. \tag{B.4}$$

[7]We remind the reader that our definition of F in the differential equations does not include the prefactor gamma function.

For notation simplicity, $I(D) \equiv I(1,\ldots,1;D)$. Using the Baikov representation, the dimension shift relation can be derived (see [50]),

$$I(\nu_1,\ldots,\nu_n;D+2) = \frac{-2}{\mathcal{K}_n(D-n+1)}\mathcal{B}_n(b_1,\ldots,b_n)I(\nu_1,\ldots,\nu_n;D)\,, \tag{B.5}$$

where $\mathcal{B}$ is obtained by replacing the arguments z_i of Baikov polynomial $\mathcal{B}_n(z_1,\ldots,z_n)$ with operators b_i.[8] These operators lower the value of the exponent ν_i, according to:

$$b_i I(\nu_1,\ldots,\nu_n;D) = I(\nu_1,\ldots,\nu_i-1,\ldots,\nu_n;D)\,. \tag{B.6}$$

Take the box diagram as an example, we have:

$$\begin{aligned}
\mathcal{K}_4 =& \frac{35143}{46406525}\,,\\
\mathcal{B}_4(b_1,\ldots,b_4) =& -\frac{3853b_1^2}{521594}+\frac{15451b_1b_2}{494615}-\frac{13081b_1b_3}{623645}+\frac{97b_1b_4}{21505}-\frac{1151309b_1}{28687670}-\frac{49216817b_2^2}{1262257480}\\
&+\frac{458714b_2b_3}{6860095}-\frac{1731b_2b_4}{86020}+\frac{62050463b_2}{1577821850}-\frac{4985989b_3^2}{137201900}+\frac{5761b_3b_4}{215050}\\
&+\frac{1990892b_3}{71719175}-\frac{19b_4^2}{3400}-\frac{86409b_4}{3118225}-\frac{4047481551}{36605466920}
\end{aligned} \tag{B.7}$$

Plugging (B.7) and into (B.5) yields:

$$\begin{aligned}
I(D+2) =& \frac{20237407755}{69301996(D-3)}I(D)+\frac{63321995}{597431(D-3)}I_{\widehat{1}}(D)-\frac{62050463}{597431(D-3)}I_{\widehat{2}}(D)\\
&-\frac{43799624}{597431(D-3)}I_{\widehat{3}}(D)+\frac{43722954}{597431(D-3)}I_{\widehat{4}}(D)+\frac{11655325}{597431(D-3)}I(-1,1,1,1;D)\\
&+\frac{246084085}{2389724(D-3)}I(1,-1,1,1;D)+\frac{114677747}{1194862(D-3)}I(1,1,-1,1;D)\\
&+\frac{35268959}{2389724(D-3)}I(1,1,1,-1;D)-\frac{49288690}{597431(D-3)}I_{\widehat{1,2}}(D)+\frac{33094930}{597431(D-3)}I_{\widehat{1,3}}(D)\\
&-\frac{7116890}{597431(D-3)}I_{\widehat{1,4}}(D)-\frac{105504220}{597431(D-3)}I_{\widehat{2,3}}(D)+\frac{63501735}{1194862(D-3)}I_{\widehat{2,4}}(D)\\
&-\frac{42268457}{597431(D-3)}I_{\widehat{3,4}}(D)
\end{aligned} \tag{B.8}$$

Using FIRE6 [51], we obtain the following relations:

$$\begin{aligned}
I(-1,1,1,1;D) =& \frac{448079}{211915}I_{\widehat{1,2}}(D)+\frac{64699}{211915}I_{\widehat{1,4}}(D)-\frac{300863}{211915}I_{\widehat{1,3}}(D)-\frac{1151309}{423830}I_{\widehat{1}}(D)\\
I(1,-1,1,1;D) =& \frac{19715476}{49216817}I_{\widehat{1,2}}(D)+\frac{42201688}{49216817}I_{\widehat{2,3}}(D)-\frac{12700347}{49216817}I_{\widehat{2,4}}(D)+\frac{124100926}{246084085}I_{\widehat{2}}(D)\\
I(1,1,-1,1;D) =& -\frac{1438910}{4985989}I_{\widehat{1,3}}(D)+\frac{4587140}{4985989}I_{\widehat{2,3}}(D)+\frac{1837759}{4985989}I_{\widehat{3,4}}(D)+\frac{43799624}{114677747}I_{\widehat{3}}(D)\\
I(1,1,1,-1;D) =& \frac{1940}{4807}I_{\widehat{1,4}}(D)+\frac{11522}{4807}I_{\widehat{3,4}}(D)-\frac{8655}{4807}I_{\widehat{2,4}}(D)-\frac{345636}{139403}I_{\widehat{4}}(D)\,.
\end{aligned} \tag{B.9}$$

[8] As the operators b_i are commute, the $\mathcal{B}_n(b_1,\ldots,b_n)$ is well-defined.

Combine all, we get

$$
\begin{aligned}
I(D+2) = {} & \frac{20237407755}{69301996(D-3)} I(D) + \frac{63321995}{1194862(D-3)} I_{\widehat{1}}(D) - \frac{62050463}{1194862(D-3)} I_{\widehat{2}}(D) \\
& - \frac{21899812}{597431(D-3)} I_{\widehat{3}}(D) + \frac{21861477}{597431(D-3)} I_{\widehat{4}}(D)
\end{aligned}
\tag{B.10}
$$

Subsequently, we employ our formula (B.4) to compute the reduction coefficients. The relevant matrices are:

$$
\boldsymbol{C} = \begin{pmatrix} \frac{11}{2} & \frac{1281}{170} & \frac{7083807}{1247290} \end{pmatrix}, \qquad \boldsymbol{A} = \begin{pmatrix} 1 & \frac{11}{5} & \frac{191}{55} \\ \frac{11}{5} & \frac{419}{85} & \frac{168907}{21505} \\ \frac{191}{55} & \frac{168907}{21505} & \frac{7883673}{623645} \end{pmatrix}
\tag{B.11}
$$

Thus,

$$
\boldsymbol{C}^T \cdot \boldsymbol{A}^{-1} = \begin{pmatrix} \frac{62050463}{1194862} & \frac{21899812}{597431} & -\frac{21861477}{597431} \end{pmatrix}, \qquad \mathcal{D} = \frac{20237407755}{69301996}
\tag{B.12}
$$

Plugging (B.12) back into (B.4), the result is consistent with (B.10). For brevity, the detailed derivation is omitted; however, (4.3) and (4.5) are also consistent with the results obtained from FIRE6.

Finally, we present the symbol letters. For an odd number of propagators, taking $n = 5$ as an example, the results from [27] are

$$
\begin{aligned}
M_5 &= d\log\left(\frac{-\mathcal{K}_5}{\mathcal{B}_5(0)}\right) = d\log\frac{237922359363012971055400}{861926378528471209482605 7}, \\
M_{5,4} &= \tfrac{1}{2} d\log\frac{\mathcal{B}_5'(0)-\sqrt{\mathcal{B}_4(0)\mathcal{K}_5}}{\mathcal{B}_5'(0)-\sqrt{\mathcal{B}_4(0)\mathcal{K}_5}} = \tfrac{1}{2} d\log\frac{5\sqrt{14161564119002751648829830866 55}-5159419877130114}{-5\sqrt{1416156411900275164882983086655}-5159419877130114}, \\
M_{5,3} &= \tfrac{i}{4} d\log\frac{\mathcal{B}_5''-\sqrt{-\mathcal{K}_5\mathcal{K}_3}}{\mathcal{B}_5''+\sqrt{-\mathcal{K}_5\mathcal{K}_3}} = \tfrac{i}{4} d\log\frac{100106342929340\sqrt{13295661258521313088 39}-49135646403084277961164 63}{32892496391777479096643 13},
\end{aligned}
\tag{B.13}
$$

where $\mathcal{B}_5(0) \equiv \mathcal{B}_5(0,0,0,0,0)$. Our results, obtained using equations (4.23), (4.24), and (4.25), are:

$$
\begin{aligned}
c_{5\to5} &= -\epsilon\, d\log\frac{8619263785284712094826057}{1189611796815064855277 00} \\
c_{5\to4;5} &= -\epsilon\, \frac{i}{2} d\log\frac{-5\sqrt{1416156411900275164882983086655}-5159419877130114}{5\sqrt{1416156411900275164882983086655}-5159419877130114}, \\
c_{5\to3;4,5} &= \epsilon\, \tfrac{-i}{2} d\log\frac{-100106342929340\sqrt{1329566125852131308839}-4913564640308427796116463}{3289249639177747909664313}.
\end{aligned}
\tag{B.14}
$$

The relation between our MIs $\mathcal{I}_n$ and g_n in [27] is

$$
\begin{pmatrix} \mathcal{I}_5 \\ \mathcal{I}_{4;5} \\ \mathcal{I}_{3;45} \\ \mathcal{I}_{2;345} \\ \mathcal{I}_{1;2345} \end{pmatrix} = T_5 \begin{pmatrix} g_5 \\ g_4 \\ g_3 \\ g_2 \\ g_1 \end{pmatrix}
\tag{B.15}
$$

where $T_5 = \mathrm{diag}(4, -4i, 2, 2i, 1)$. The transformation relation between the prefactor matrix L of $d\log$ in our results and L' in [27] is:

$$T_5^{-1} L T_5 = L' \,. \tag{B.16}$$

This leads to the following consistency checks:

$$\epsilon\, M_5 = c_{5\to 5}, \quad \epsilon\, M_{5,4} = -i\, c_{5\to 4;5}, \quad \epsilon\, M_{5,3} = \frac{1}{2} c_{5\to 3;45}. \tag{B.17}$$

Analogously, for an even number of propagators, the results for $n = 4$ in [27]

$$\begin{aligned} M_4 &= d\log(\frac{-\mathcal{K}_4}{\mathcal{B}_4(0)}) = d\log\frac{138603992}{20237407755}\,, \\ M_{4,3} &= \frac{i}{2}\, d\log\frac{\mathcal{B}_4'(0) - \sqrt{-\mathcal{B}_4(0)\mathcal{K}_3}}{\mathcal{B}_4'(0) - \sqrt{-\mathcal{B}_4(0)\mathcal{K}_3}} = \frac{i}{2}\, d\log\frac{230424\sqrt{384510747345} - 167991663947}{88348834283}\,, \\ M_{4,2} &= \ \tfrac{i}{4} d\log\tfrac{\mathcal{K}_4' - \sqrt{-\mathcal{B}_4(0)\mathcal{B}_2(0)}}{\mathcal{K}_4' + \sqrt{-\mathcal{B}_4(0)\mathcal{B}_2(0)}} = \tfrac{i}{4} d\log\tfrac{163582377430\sqrt{87385126686090} + 5660768306333952421}{5450315792898285671} \end{aligned} \tag{B.18}$$

Using (4.14), (4.15), and (4.20), our corresponding results are:

$$\begin{aligned} c_{4\to 4} &= -\epsilon\, d\log\frac{20237407755}{69301996} \\ c_{4\to 3;4} &= -\epsilon\, d\log\frac{-230424\sqrt{384510747345} - 167991663947}{88348834283}\,, \\ c_{4\to 2;3,4} &= \epsilon\, \frac{i}{2} d\log\frac{-1511810\sqrt{10230953817551710414028 10} + 5660768306333952421}{5450315792898285671} \end{aligned} \tag{B.19}$$

The consistency relations are:

$$\epsilon\, M_4 = c_{4\to 4}, \quad \epsilon\, M_{4,3} = \frac{i}{2}\, c_{4\to 3;4}, \quad \epsilon\, M_{4,2} = -\frac{1}{2} c_{4\to 2;34} \tag{B.20}$$

Our results for both $n = 5$ and $n = 4$ are consistent with those presented in [27].

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Multivariate hypergeometric solutions of cosmological (dS) correlators by d log-form differential equations

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Abstract: In this paper, we give the analytic expression for the homogeneous part of solutions of arbitrary tree-level cosmological correlators, including massive propagators and time-derivative interaction cases. The solutions are given in the form of multivariate hypergeometric functions. It is achieved by two steps. Firstly, we indicate the factorization of the homogeneous part of solutions, i.e., the homogeneous part of solutions of multiple vertices is the product of the solutions of the single vertex. Secondly, we give the solution to the d log-form differential equations of arbitrary single vertex integral family. We also show how to determine the boundary conditions for the differential equations. There are two techniques we developed for the computation. Firstly, we analytically solve d log-form differential equations via power series expansion. Secondly, we handle degenerate multivariate poles in power series expansion of differential equations by blow-up. They could also be useful in the evaluation of multi-loop Feynman integrals in flat spacetime.

Keywords: de Sitter space, Early Universe Particle Physics, Scattering Amplitudes

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Contents

1 Introduction

Anti-de Sitter (AdS) and de Sitter (dS) space as the simplest curved spacetime, quantum field theory (QFT) in them should be the first step for people to understand QFT in curved space. Meanwhile, QFT in dS space also catches people's interest due to its phenomenological application in cosmology, especially inflation physics. Inflation has been widely accepted as a period of the evolution of our early universe. During inflation, the background spacetime can be regarded as approximately a dS spacetime. The quantum fluctuation of all particles in the early universe gave rise to cosmic microwave background (CMB) and produced the Large Scale Structure (LSS) we can observe today. In order to obtain more information contained in the CMB and LSS, we need to analyze the cosmological correlators that are related to the Cosmological Collider signals [1–5].

Cosmological correlators can be calculated by wavefunction coefficients or in-in Feynman rules. The former regards the cosmological correlator as inserting external fields in the field integral of the squared norm of the Hartle-Hawking wavefunction. The wavefunction coefficients, which encode all information of the wavefunction, are equivalent to the AdS amplitudes in the momentum space up to an analytic continuation. The latter is based on in-in formalism [6–9]. On the one side, many techniques analog to these methods in flat amplitudes and CFT correlators are developed in the calculation, including cosmological bootstrap [10–21] which involves some singularity behaviors and weight-shifting operators, off-shell methods [22–25], family-tree decomposition [26] which could give power series solutions of arbitrary tree-level amplitude in conformal coupled case,[1] Mellin amplitudes [28–30], summation-by-parts relations in Mellin space [31], bootstrap equation [32, 33], (partial) Mellin-Barnes integration [34–39], spectral decomposition [40–42], Integrate-By-Part (IBP) [43] and the IBP-based differential equations [44–47] for conformal coupled case [48–51] and general case [52] of dS background, and so on [53–59].

This paper aims to show a systematic, powerful, and user-friendly method to evaluate perturbative QFT in dS space, thus cosmological correlators as well, and show the elegant structures of tree-level cosmological correlators. The method is mainly based on IBP and differential equations of general dS case [52], including massive propagators and time derivative interaction. This case is more non-trivial because people need to generalize IBP from the polynomial integrand case of flat or conformal coupled dS cases to the Hankel integrand case. Moreover, [52] directly gives the uniform formulas of iterative IBP reduction and $d\log$-form differential equations of arbitrary tree-level cosmological correlators, as we will review them and give the notations of this paper in section 2. Once we have differential equations, the next step obviously is to solve them. In section 3 We further introduce the generalized power series expansion method [60] of flat amplitudes to solve this problem. Slightly unlike in [60], we directly perform just power series expansions (while "generalized" is found to be not necessary here) on the first-order differential equations, rather than deriving the higher-order differential equation for each master integral first. We also find several boundaries whose boundary conditions could be easily determined. Surprisingly, due to that the $d\log$-form further simplifies the series expansion of differential equations, we find that power series solutions of the vertex integral family exhibit a simple structure, which allows us to conjecture all order expressions of them directly. These solutions are multivariate hypergeometric functions. We will firstly show two examples in section 3.1 and 3.2: solving the 1-fold and 2-fold Hankel vertex integral families by expanding them around both momentum k_0 of the massless leg equals ∞ and k_n of one massive leg equals ∞, including presenting how to determine their boundary conditions. The provided power series solutions have a region of convergence. Therefore, in section 3.1.4, we discuss how to perform analytic continuation. The numerical efficiency is also presented in this subsection. In one example, we get the numerical result of points 100 points with a relative error of at most $\mathcal{O}(10^{-34})$, and evaluating each point only takes about 0.01s by one core of CPU on a personal computer. Then, in section 3.3, we give the constructed all-order power series solution for arbitrary vertex integral family for both boundary $k_0 \to \infty$ and $k_n \to \infty$, including their boundary coefficient. These are the new results of this paper. Furthermore, in section 4, we compute a 2-vertex example. By this example, we want to show that because of the factorization property of IBP of tree-level cosmological correlators [52], these solutions in section 3.3 directly give all homogeneous solutions of arbitrary tree-level cosmological correlators. Although we have not given all solutions of the non-homogeneous part of arbitrary tree-level cosmological correlators, the calculation of the non-homogeneous solution in section 4 shows using the methods to solve non-homogeneous solutions, including its boundary condition, is also easy and straightforward. Meanwhile, in this section, we also indicate that blow-up is a useful technique for solving

[1] Power series solutions also could be regarded as multivariate hypergeometric functions, and hypergeometric structure of Feynman integrals in flat cases and its evaluation also has been studied in [27] recently.

power series expansion of differential equations around degenerate multivariate singularity. The techniques we develop in this paper could also benefit the evaluation of amplitude in flat space, as we also present related discussions in the section of summary and outlook section 5.

We emphasize that although we have not presented any loop-level example, IBP, differential equations, and generalized power series expansion could be applied to loop-level cosmological correlators straightforwardly as well. Based on our best knowledge, the main possible challenge that could arise at the loop level is determining the boundary conditions. It could be more complicated than tree-level. However, there also are many lessons that can be learned from flat amplitudes for solving boundary conditions. Therefore, it is unlikely to pose a fundamental difficulty.

2 Background

In this section, we review the basic background for later discussions.

2.1 In-in Feynman rules and asymptotic behaviors

The general Feynman rules for cosmological correlators in in-in formalism are displayed as follows (for a modern review, see [61]). The bulk-to-bulk propagators are given by

$$\begin{aligned} G_>(k;\tau_1,\tau_2) &\equiv u(\tau_1,k)u^*(\tau_2,k), \\ G_<(k;\tau_1,\tau_2) &\equiv u^*(\tau_1,k)u(\tau_2,k). \end{aligned} \tag{2.1}$$

$$\begin{aligned} G_{++}(k;\tau_1,\tau_2) &= G_>(k;\tau_1,\tau_2)\theta(\tau_1-\tau_2)+G_<(k;\tau_1,\tau_2)\theta(\tau_2-\tau_1), \\ G_{+-}(k;\tau_1,\tau_2) &= G_<(k;\tau_1,\tau_2), \\ G_{-+}(k;\tau_1,\tau_2) &= G_>(k;\tau_1,\tau_2), \\ G_{--}(k;\tau_1,\tau_2) &= G_<(k;\tau_1,\tau_2)\theta(\tau_1-\tau_2)+G_>(k;\tau_1,\tau_2)\theta(\tau_2-\tau_1), \end{aligned} \tag{2.2}$$

The mode of the field in the time direction is denoted by u, which is

$$u(\tau;k) = -i\frac{\sqrt{\pi}}{2}e^{i\pi(\nu/2+1/4)}H^{(d-1)/2}(-\tau)^{d/2}\mathrm{H}_\nu^{(1)}(-k\tau)\,. \tag{2.3}$$

Here the H is Hubble constant, $\mathrm{H}_\nu^{(1)}(-k\tau)$ is the Hankel function and other parameters are $k=|\boldsymbol{k}|$ where $\boldsymbol{k}$ is the 3-momentum, $\nu=\sqrt{\frac{d^2}{4}-\frac{m^2}{H^2}}$ and $d=3$. From this definition, one can see k typically is real and ν typically is real or imaginary, thus we will usually discuss such cases. The bulk-to-boundary propagators are non-vanishing only when the field is massless

$$\begin{aligned} G_+(k;\tau) &\equiv G_{+\pm}(k;\tau_1,0) = \frac{H^2}{2k^3}(1-ik\tau)e^{ik\tau}, \\ G_-(k;\tau) &\equiv G_{-\pm}(k;\tau_1,0) = \frac{H^2}{2k^3}(1+ik\tau)e^{-ik\tau}. \end{aligned} \tag{2.4}$$

The Hankel functions satisfy[2]

$$\begin{aligned} &\partial_\tau^2\mathrm{H}_\nu^{(1,2)}(-k\tau)+\frac{1}{\tau}\partial_\tau\mathrm{H}_\nu^{(1,2)}(-k\tau)+\Big(k^2-\frac{\nu^2}{H^2\tau^2}\Big)\mathrm{H}_\nu^{(1,2)}(-k\tau)=0\,, \\ &\mathrm{H}_\nu^{(1)}(-k\tau)=\Big(\mathrm{H}_{\nu^\star}^{(2)}(-k^\star\tau^\star)\Big)^\star\,. \end{aligned} \tag{2.5}$$

[2]The definition of $\mathrm{H}_\nu^{(1,2)}$ in textbook is $\mathrm{H}_\nu^{(1)}(z)=\mathrm{J}_\nu(z)+i\mathrm{Y}_\nu(z)$ and $\mathrm{H}_\nu^{(2)}(z)=\mathrm{J}_\nu(z)-i\mathrm{Y}_\nu(z)$. When z,ν are real numbers, it is obviously that $\mathrm{H}_\nu^{(2)}(z)=(\mathrm{H}_\nu^{(1)}(z))^*$. The second line of (2.5) is a generalization of the above result when z,ν are complex numbers.

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The asymptotic behavior of Hankel functions is also important for applying Wick rotation and solving boundary conditions of cosmological correlators. Hence, we list them below. For $\tau \to 0_-$,

$$\begin{aligned}
\mathrm{H}_\nu^{(1)}(-k\tau) &= c_1(-k\tau)^{v}(1+\mathcal{O}(\tau^2)) + c_2(-k\tau)^{-v}(1+\mathcal{O}(\tau^2)) \,, \\
\mathrm{H}_{\nu^\star}^{(2)}(-k\tau) &= c_1^\star(-k\tau)^{v^\star}(1+\mathcal{O}(\tau^2)) + c_2^\star(-k\tau)^{-v^\star}(1+\mathcal{O}(\tau^2)) \,, \\
c_1 = e^{-i\pi\nu}c[\nu] \,, &\qquad c_2 = c[-\nu] \,, \qquad c[\nu] \equiv \frac{2^{-\nu}\Gamma(-\nu)}{i\pi} \,.
\end{aligned} \tag{2.6}$$

For $k\tau \to -\infty$,

$$\begin{aligned}
\mathrm{H}_\nu^{(1)}(-k\tau) &\sim \sqrt{\frac{2}{\pi}}(-k\tau)^{-\frac{1}{2}}e^{-ik\tau - i\pi(\nu/2+1/4)} \,, \\
\mathrm{H}_\nu^{(2)}(-k\tau) &\sim \sqrt{\frac{2}{\pi}}(-k\tau)^{-\frac{1}{2}}e^{ik\tau + i\pi(\nu/2+1/4)} \,.
\end{aligned} \tag{2.7}$$

2.2 Notations for indices

Since we will frequently use the tensor product of 2-component vectors, we also introduce the following notation for convenience:

$$\begin{aligned}
\boldsymbol{a} &\equiv a_1, a_2, \cdots, a_n \,, \qquad a_i = 0, 1 \,, \\
\tilde{\boldsymbol{a}} &\equiv 1 + \sum_{i=1}^{n} a_i 2^{n-i} \,, \\
\mathrm{I}_{\tilde{\boldsymbol{a}}} &\equiv \mathrm{I}_{\{\boldsymbol{a}\}} = \mathrm{I}_{\{a_1, a_2, \cdots\}}
\end{aligned} \tag{2.8}$$

Let us explain the meaning of the above notations. The $\boldsymbol{a}$ is a vector with n components, while each component takes only two values 0 and 1. Thus we can write $\boldsymbol{a}$ as a binary, for example, $\boldsymbol{a} = 10100$. The meaning of $\tilde{\boldsymbol{a}}$ is to transfer the binary number $\boldsymbol{a}$ to a number in decimal system, for example, $\tilde{0} = 1$, $\tilde{1} = 2$ and $\widetilde{1010} = 11$. In other words, we should treat the $\tilde{}$ as an operation acting on $\boldsymbol{a}$. Using this action, we can easily get the location of $\boldsymbol{a}$-th component in the tensor product. For instance, in the 2-fold vertex integral family, there are four master integrals. The 3-th master integrals can be denoted as I_3 or $\mathrm{I}_{\{1,0\}}$. Another example is that $f_{\{a_3=1\}} = f_{\tilde{a}_3=2}$.

2.3 Integral family and differential equations

For in-in Feynman diagrams, we can use the following elements to express integrands of tree diagrams

$$\begin{aligned}
\mathcal{I} &= \int_{-\infty}^{0} \prod_i \mathrm{d}\tau_i \tau_i^{\alpha_i} \prod_j F_j \,, \\
F_j &= e^{ik\tau}, \qquad \mathrm{H}_\nu^{(1,2)}(-k\tau), \qquad \partial_\tau \mathrm{H}_\nu^{(1,2)}(-k\tau), \qquad \theta(\tau_j - \tau_k) \,.
\end{aligned} \tag{2.9}$$

Here, each $\int \mathrm{d}\tau_i$ corresponds to a time-integration of a vertex. In this paper, we will denote the tree-level cosmological correlators with M vertices as "M-vertex correlators", including integrals with respect to M time variables τ_i. We call the integral family with one vertex a "vertex integral family". Since each massive leg contributes a Hankel function in the integrand, we use the n-fold (Hankel) vertex integral family to denote a vertex integral family with n massive legs. Integrals in the family can be written as

$$
\begin{aligned}
&f^{(a_0)}_{\tilde{\boldsymbol{a}},\boldsymbol{s}} = \int_{-\infty}^{0} (-\tau)^{\nu_0+a_0} e^{ik_0\tau} \prod_{i=1}^{n} h^{(s_i)}(\nu_i, a_i; -k_i\tau)\mathrm{d}\tau\,, \\
&a_0 \in \mathbb{Z}\,, \qquad s_{i\geq 1}, a_{i\geq 1} = 0, 1; \qquad \boldsymbol{a} = (a_1, \cdots, a_n); \qquad \boldsymbol{s} = (s_1, \ldots, s_n)
\end{aligned} \tag{2.10}
$$

where h-functions are redefined using the Hankel function and its time derivative for later convenience:

$$
\begin{aligned}
h^{(1 \text{ or } 2)}(\nu, 0; -k\tau) &\equiv (-k\tau)^{-\nu}\mathrm{H}^{(1)}_{\nu}(-k\tau) \text{ (or } \mathrm{H}^{(2)}_{\nu^*}) \propto \tau^{-\frac{3}{2}-\nu} u \text{ (or } u^*), \\
h^{(1 \text{ or } 2)}(\nu, 1; -k\tau) &\equiv -\frac{1}{k}\partial_\tau h^{(1 \text{ or } 2)}(\nu, 0; -k\tau),
\end{aligned} \tag{2.11}
$$

The iterative IBP reduction and d log-form differential equations have been given in [52]. It says n-fold vertex integral family has 2^n master integrals, which we denote as $\mathrm{I}_{\tilde{\boldsymbol{a}}}$. If one selects all $\mathrm{I}_{\tilde{\boldsymbol{a}}} = f^{(0)}_{\tilde{\boldsymbol{a}},\boldsymbol{s}}$,[3] with $a_i = 0, 1$, as master integrals, the differential equations of them are automatically d log-form and given by the uniform formula:

$$
\begin{aligned}
\mathrm{dI} = (\mathrm{d}\Omega)\cdot \mathrm{I} = \sum_{i=0}^{n} \Omega_{k_i}\cdot \mathrm{I} \qquad \mathrm{d}k_i\,, \\
\Omega = \Omega_{ex} - i\mathrm{T}_n^{-1}\cdot \tilde{\Omega}_0 \cdot \mathrm{T}_n \cdot \mathrm{M}_1[\nu_0+1, \boldsymbol{\nu}]\,.
\end{aligned} \tag{2.12}
$$

where $\Omega_{k_i} = \frac{\partial}{\partial k_i}\Omega$ with

$$
\begin{aligned}
\left(\tilde{\Omega}_0\right)_{\boldsymbol{ba}} &\equiv \begin{cases} -i\log\left[k_0 + \sum_i (2a_i-1)k_i\right], & \boldsymbol{b}=\boldsymbol{a} \\ 0, & \boldsymbol{b}\neq\boldsymbol{a} \end{cases}, \\
(\Omega_{ex})_{\boldsymbol{ba}} &\equiv \begin{cases} -\sum_i a_i(2\nu_i+1)\log k_i, & \boldsymbol{b}=\boldsymbol{a} \\ 0, & \boldsymbol{b}\neq\boldsymbol{a} \end{cases},
\end{aligned} \tag{2.13}
$$

and

$$
\begin{aligned}
(\mathrm{M}_1[\nu_0, \boldsymbol{\nu}])_{\boldsymbol{ba}} &= \begin{cases} \nu_0 - \sum_i a_i(2\nu_i+1), & \boldsymbol{b}=\boldsymbol{a} \\ 0, & \boldsymbol{b}\neq\boldsymbol{a} \end{cases}, \\
(\mathrm{T}_n)_{\boldsymbol{ba}} &= \prod_{i=1}^{n} \mathrm{T}_{b_i a_i}\,, \qquad \mathrm{T} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.
\end{aligned} \tag{2.14}
$$

For M-vertex correlators, using 2-vertex correlators as an example for simple, besides the master integrals coming from the product of two 1-vertex master integrals, there will be an extra master integral coming from the remaining term of IBP which appears due to the step functions when applying the IBP method. More discussions can be found in [52]. For

[3] Although the function $f^{(0)}$ depends on the choice of $\boldsymbol{s}$ (so is $\mathrm{I}_{\tilde{\boldsymbol{a}}}$), the matrix $(\mathrm{d}\Omega)$ given in (2.12) does not depend on $\boldsymbol{s}$ [52].

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propagator $G_{\pm\mp}$, the master integrals are just the product of two 1-vertex master integrals. For example, if the propagator is $G_{+-}(\tau_1,\tau_2)$, we have

$$\mathcal{I}_{\tilde{\boldsymbol{r}};+-} \equiv \left(\int \mathrm{d}\tau_1 \hat{\mathrm{I}}^{(0)}_{\boldsymbol{a},\boldsymbol{s}_1;1}\right)\left(\int \mathrm{d}\tau_2 \hat{\mathrm{I}}^{(0)}_{\boldsymbol{b},\boldsymbol{s}_1;2}\right), \qquad \boldsymbol{r}=\boldsymbol{a},\boldsymbol{b} \tag{2.15}$$

with

$$\hat{\mathrm{I}}^{(0)}_{\boldsymbol{a},\boldsymbol{s}_1;1} = (-\tau_1)^{\nu_{0;1}} e^{ik_{0;1}\tau_1}\prod_i h^{(s_i)}(\nu_{i;1},a_i,-k_{i;1}\tau_1). \tag{2.16}$$

For propagator $G_{\pm\pm}$, things are more complicated due to the step function, and the remaining term will appear. In the G_{++} case, the selected master integrals and the d log-form are[4]

$$\begin{aligned}
\mathcal{I}_{\tilde{\boldsymbol{r}};++} &\equiv \int \mathrm{d}\tau_1\mathrm{d}\tau_2 \hat{\mathrm{I}}^{(0)}_{\boldsymbol{a},\boldsymbol{s}_1;1}\theta^{(i,j)}_{1,2}\hat{\mathrm{I}}^{(0)}_{\boldsymbol{b},\boldsymbol{s}_2;2}\,, \qquad \boldsymbol{r}=\boldsymbol{a},\boldsymbol{b}\,,\\
\boldsymbol{R}_{\tilde{\boldsymbol{r}};++} &= -\delta_{a_i,1-b_j}(-1)^{a_i+1}\frac{4i}{\pi}e^{\pi\mathrm{Im}[\nu]}(k_{i;1})^{-2\nu_{i;1}-1}f^{(-2\nu_{i;1})}_{\boldsymbol{a}_{\hat{i}},\boldsymbol{b}_{\hat{j}}}\,, \qquad \boldsymbol{r}=\boldsymbol{a}_{\hat{i}},\boldsymbol{b}_{\hat{j}}\,,\\
\Omega_{\tilde{\boldsymbol{r}}\tilde{\boldsymbol{s}};++} &= -i\left(\mathrm{T}_n^{-1}.\tilde{\Omega}_{0;1}.\mathrm{T}_n\right)_{\boldsymbol{a}(\boldsymbol{c}_{\hat{i}};1-b_j)}\delta_{b_{\hat{j}}d_{\hat{j}}}(-1)^{b_j}\\
&\quad -i\left(\mathrm{T}_n^{-1}.\tilde{\Omega}_{0;2}.\mathrm{T}_n\right)_{\boldsymbol{b}(\boldsymbol{d}_{\hat{j}};1-a_i)}\delta_{a_{\hat{i}}c_{\hat{i}}}(-1)^{a_i}\,, \qquad \boldsymbol{r}=\boldsymbol{a},\boldsymbol{b}; \qquad \boldsymbol{s}=\boldsymbol{c}_{\hat{i}},\boldsymbol{d}_{\hat{j}}\,.
\end{aligned} \tag{2.17}$$

We have used the following notation in the expression above:

$$\begin{aligned}
&h(\nu_{i;1},a_i,-k_{i;1}\tau_1)\theta^{(i,j)}_{1,2}h(\nu_{j;2},b_j,-k_{j;2}\tau_2) \equiv\\
&\quad h^{(1)}(\nu_{i;1},a_i,-k_{i;1}\tau_1)\theta_{12}h^{(2)}(\nu_{j;2},b_j,-k_{j;2}\tau_2) + h^{(2)}(\nu_{i;1},a_i,-k_{i;1}\tau_1)\theta_{21}h^{(1)}(\nu_{j;2},b_j,-k_{j;2}\tau_2)\,,\\
&\nu_{i;1}=\nu_{j;2}\,, \quad k_{i;1}=k_{j;2}\,, \qquad \theta_{ij}=\theta(\tau_i-\tau_j)
\end{aligned}$$

This paper will only discuss the G_{++} case of 2-vertex correlators and hence we will suppress the label $++$ in (2.17) when we consider the 2-vertex case in section 4.

To express the solutions to these differential equations more compactly, we also use the **Pochhammer symbol** $(a)_n \equiv \Gamma(a+n)/\Gamma(a)$ in some expressions.

3 Analytic results of n-fold vertex integral family

In this section, we will derive the analytic expression for a single vertex with n Hankel functions.

3.1 Pedagogical example: 1-fold vertex integral family

3.1.1 Preparation

As a pedagogical example, let us solve the 1-fold Hankel vertex integral family:[5]

$$\int_{-\infty}^{0}\mathrm{d}\tau e^{ik_0\tau}(-\tau)^{\nu_0+a_0}h^{(2)}_{\nu_1}(a,-k_1\tau)\,. \tag{3.1}$$

[4]When we integrate the delta-function, two h will combine to give a simple factor (see eq. (3.11) and eq. (3.12) of [52]), so for remaning part, we have $r=\boldsymbol{a}_{\hat{i}},\boldsymbol{b}_{\hat{j}}$ as given in the second line in (2.17). The matrix $\Omega_{\tilde{\boldsymbol{r}}\tilde{\boldsymbol{s}};++}$ gives the reduction coefficients of the differential of the original sector to the remaining part. More explanation can be found in eq. (3.68) of [52].

[5]For simplicity, we write $h(\nu,a,-k_1\tau)=h_\nu(a,-k_1\tau)$.

To simplify the discussion, we will consider $\nu_i \in \mathbb{R}$ in this section. The methods employed here can be applied directly to the case of imaginary values of ν_i as well. We will also discuss how to extend the results to imaginary ν_i in section 3.1.4.

This function family has 2 master integrals. We use following $\mathrm{I}_{\tilde{a}}$ as master integrals:

$$\begin{aligned}
\mathrm{I}_1 &= \int_{-\infty}^{0} \mathrm{d}\tau e^{ik_0\tau}(-\tau)^{\nu_0} h_{\nu_1}^{(2)}(0, -k_1\tau)\,, \\
\mathrm{I}_2 &= \int_{-\infty}^{0} \mathrm{d}\tau e^{ik_0\tau}(-\tau)^{\nu_0} h_{\nu_1}^{(2)}(1, -k_1\tau)\,.
\end{aligned} \tag{3.2}$$

This choice of master integrals is the same as those constructed in [52], allowing us to directly use the uniform formula of differential equations of them (2.12). The differential equations are as follows:

$$\begin{aligned}
\mathrm{dI}_{\tilde{a}} &= \left(\mathrm{d}\Omega_{\tilde{a}\tilde{b}}\right) \mathrm{I}_{\tilde{b}}\,, \\
\Omega_{11} &= -i\left(\nu_0+1\right)\left(-\frac{1}{2}i\log\left(k_0-k_1\right)-\frac{1}{2}i\log\left(k_0+k_1\right)\right)\,, \\
\Omega_{12} &= -i\left(\nu_0-2\nu_1\right)\left(\frac{1}{2}\log\left(k_0+k_1\right)-\frac{1}{2}\log\left(k_0-k_1\right)\right)\,, \\
\Omega_{21} &= -i\left(\nu_0+1\right)\left(\frac{1}{2}\log\left(k_0-k_1\right)-\frac{1}{2}\log\left(k_0+k_1\right)\right)\,, \\
\Omega_{22} &= -\left(2\nu_1+1\right)\log\left(k_1\right)-i\left(\nu_0-2\nu_1\right)\left(-\frac{1}{2}i\log\left(k_0-k_1\right)-\frac{1}{2}i\log\left(k_0+k_1\right)\right)\,.
\end{aligned} \tag{3.3}$$

This is a system of first-order differential equations. It is well known that, for first-order differential equations, one can determine the solution if the values of the principal integral on the boundary or its asymptotic behavior are provided. Similar to the application of differential equation methods in flat spacetime field theory, a natural approach here is to choose a boundary point that is significantly easier to compute than the original integral. Noting that the Wick rotation of this integral family is

$$\tau \to -i\tau \tag{3.4}$$

and the asymptotic behavior (2.7) of Hankel functions at $k\tau \to -\infty$, taking the limit as $k_i \to \infty$ (since $\tau \leq 0$) simplifies the computation due to the exponential suppression. We will proceed with calculations using this boundary.

In the following part of this section, we will first solve the system of differential equations using the method of power series expansion around $k_0 \to \infty$. We will then find this analytical series solution can be rewritten as hypergeometric functions, thereby obtaining a compact analytical function. Subsequently, we will determine the coefficients of the analytical solution by computing the boundary conditions.

3.1.2 Solutions with the boundary $k_0 \to \infty$

For convenience, we define a new parameter

$$x = \frac{1}{k_0}. \tag{3.5}$$

Then, the matrix of partial differential equations with respect to x is

$$\partial_x \mathrm{I}_{\tilde{a}} = (\Omega_x)_{\tilde{a}\tilde{b}}\, \mathrm{I}_{\tilde{b}}\,, \qquad \Omega_x = \begin{pmatrix} \frac{\nu_0+1}{x-k_1^2x^3} & \frac{ik_1(\nu_0-2\nu_1)}{k_1^2x^2-1} \\ -\frac{ik_1(\nu_0+1)}{k_1^2x^2-1} & \frac{\nu_0-2\nu_1}{x-k_1^2x^3} \end{pmatrix}. \tag{3.6}$$

Due to the differential equation of the chosen master integrals being in d log-form, its series expansion takes a very simple form. The power series expansion of Ω_x around $x = 0$ are

$$\begin{aligned} \Omega_x &= \sum_{i=-1}^{\infty} \Omega_x^{(i)} x^i\,, \\ \Omega_x^{(-1+2j)} &= \begin{pmatrix} (\nu_0+1)\, k_1^{2j} & 0 \\ 0 & (\nu_0-2\nu_1)\, k_1^{2j} \end{pmatrix}, \\ \Omega_x^{(0+2j)} &= \begin{pmatrix} 0 & -i\,(\nu_0-2\nu_1)\, k_1^{1+2j} \\ i\,(\nu_0+1)\, k_1^{1+2j} & 0 \end{pmatrix}, \end{aligned} \tag{3.7}$$

where $j \in \mathbb{N}$. We also denote the ansatz of the power series expansions of the solutions around $x = 0$ as

$$f_i = x^\lambda \sum_{j=0}^{\infty} \mathrm{C}(i,j) x^j \tag{3.8}$$

where λ represents the smallest nonzero exponent among all master integrals of this solution. To determine λ, we consider the indicial equation derived from the leading order of the power series solution of the differential equations:

$$\begin{aligned} \partial_x \mathrm{C}(i,0) x^\lambda &= \left(\Omega_x^{(-1)}\right)_{ij} x^{-1} \mathrm{C}(j,0) x^\lambda \\ \Rightarrow \lambda \begin{pmatrix} \mathrm{C}(1,0) \\ \mathrm{C}(2,0) \end{pmatrix} &= \begin{pmatrix} \nu_0+1 & 0 \\ 0 & \nu_0-2\nu_1 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{C}(1,0) \\ \mathrm{C}(2,0) \end{pmatrix} \end{aligned} \tag{3.9}$$

Solving for $\mathrm{C}(1,0)$ and λ yields two non-trivial solutions. They are

$$\begin{aligned} &\text{solution 1:} \qquad && \lambda = \nu_0 + 1\,, \qquad && \mathrm{C}(2,0) = 0\,, \\ &\text{solution 2:} \qquad && \lambda = \nu_0 - 2\nu_1\,, \qquad && \mathrm{C}(1,0) = 0\,. \end{aligned} \tag{3.10}$$

For each selected solution, one can solve $\mathrm{C}(i,j)$ iteratively. For example, the $x^{\lambda+j_0}$ order of (3.6) gives

$$(\lambda + j_0)\mathrm{C}(i,j_0) = \sum_{j=-1}^{j_0} \Omega_x^{(j)}.\ \mathrm{C}(i, j_0 - j)\,. \tag{3.11}$$

Supposing people have solved $\mathrm{C}(i, j < j_0)$, people can solve $\mathrm{C}(i, j_0)$ from the above equations. As a result, the master integrals could be expressed as

$$\mathrm{I}_{\tilde{\boldsymbol{a}}} = \mathrm{C}^{[1]} f_{\tilde{\boldsymbol{a}}}^{[1]} + \mathrm{C}^{[2]} f_{\tilde{\boldsymbol{a}}}^{[2]}\,, \tag{3.12}$$

where referring to the definition of index $\tilde{\boldsymbol{a}}$ in (2.8), $\mathrm{I}_{\tilde{\boldsymbol{a}}}$ denotes the $\tilde{\boldsymbol{a}}$-th master integral. Here we denote the i-th function in the j-th general solution of the differential equations

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by $f_i^{[j]}$. We denote the boundary coefficients $\mathrm{C}(1,0)$ in solution 1 by $\mathrm{C}^{[1]}$ and the $\mathrm{C}(2,0)$ in solution 2 by $\mathrm{C}^{[2]}$, which will be determined by boundary conditions. $f_i^{[j]}$ and $\mathrm{C}^{[j]}$ together give the particular solutions corresponding to master integrals. The power series solutions $f_i^{[j]}$ of differential equations in (3.12) are

$$\begin{aligned}
f_1^{[1]}(1/x,k_1) &= x^{\nu_0+1}\sum_{m=0}^{\infty}\frac{\left(\frac{\nu_0+1}{2}\right)_m\left(\frac{\nu_0+2}{2}\right)_m}{(\nu_1+1)_m}\frac{(k_1^2x^2)^m}{m!}\,,\\
f_2^{[1]}(1/x,k_1) &= x^{\nu_0+1}ik_1x\sum_{m=0}^{\infty}\frac{\left(\frac{\nu_0+1}{2}\right)_{m+1}\left(\frac{\nu_0+2}{2}\right)_m}{(\nu_1+1)_{m+1}}\frac{(k_1^2x^2)^m}{m!}\,,\\
f_1^{[2]}(1/x,k_1) &= x^{\nu_0-2\nu_1}(-ik_1x)\sum_{m=0}^{\infty}\frac{\left(\frac{\nu_0-2\nu_1}{2}\right)_{m+1}\left(\frac{\nu_0-2\nu_1+1}{2}\right)_m}{(-\nu_1)_{m+1}}\frac{(k_1^2x^2)^m}{m!}\,,\\
f_2^{[2]}(1/x,k_1) &= x^{\nu_0-2\nu_1}\sum_{m=0}^{\infty}\frac{\left(\frac{\nu_0-2\nu_1}{2}\right)_m\left(\frac{\nu_0-2\nu_1+1}{2}\right)_m}{(-\nu_1)_m}\frac{(k_1^2x^2)^m}{m!}\,,
\end{aligned}\tag{3.13}$$

which converge when $|k_1/k_0|<1$. Differential equations being in $\mathrm{d}\log$-form leads to the simple expression of the expansion of differential equations and the power series solution as well. As a result, we can easily see that these solutions can be expressed in terms of well-known hypergeometric functions:

$$\begin{aligned}
f_1^{[1]}(1/x,k_1) &= x^{\nu_0+1}\,{}_2\mathrm{F}_1\left(\frac{\nu_0+1}{2},\frac{\nu_0+2}{2};\nu_1+1;k_1^2x^2\right)\,,\\
f_2^{[1]}(1/x,k_1) &= x^{\nu_0+1}\frac{ik_1\,(\nu_0+1)\,x}{2\,(\nu_1+1)}\,{}_2\mathrm{F}_1\left(\frac{\nu_0+2}{2},\frac{\nu_0+3}{2};\nu_1+2;k_1^2x^2\right)\,,\\
f_1^{[2]}(1/x,k_1) &= x^{\nu_0-2\nu_1}\frac{ik_1\,(\nu_0-2\nu_1)\,x}{2\nu_1}\,{}_2\mathrm{F}_1\left(\frac{\nu_0-2\nu_1+1}{2},\frac{\nu_0-2\nu_1+2}{2};1-\nu_1;k_1^2x^2\right)\,,\\
f_2^{[2]}(1/x,k_1) &= x^{\nu_0-2\nu_1}\,{}_2\mathrm{F}_1\left(\frac{\nu_0-2\nu_1}{2},\frac{\nu_0-2\nu_1+1}{2};-\nu_1;k_1^2x^2\right)\,.
\end{aligned}\tag{3.14}$$

To complete the calculation of the master integrals, we only need to determine the coefficients $\mathrm{C}^{[1]}$ and $\mathrm{C}^{[2]}$ by boundary conditions as follows.

Due to the exponential factor $e^{ik\tau}$ being suppressed when $k_0\to\infty$ after a Wick rotation, only the region where $\tau\to 0$ can contribute non-zero terms. Therefore, we expand the other parts of the integrand around $\tau=0$. Recall (2.6), we have

$$\begin{aligned}
h_\nu^{(1)}(0,-k\tau) &= c_1(1+\mathcal{O}(\tau^2))+c_2(-k\tau)^{-2\nu}(1+\mathcal{O}(\tau^2))\,,\\
h_\nu^{(2)}(0,-k\tau) &= c_1^\star(-k\tau)^{-\nu+\nu^\star}(1+\mathcal{O}(\tau^2))+c_2^\star(-k\tau)^{-\nu-\nu^\star}(1+\mathcal{O}(\tau^2))\,.
\end{aligned}\tag{3.15}$$

We denote the coefficient we need for boundary condition by $\mathrm{C}_{\tilde{a}}^{(k_0)}$. For real ν and $h_\nu^{(2)}(a,-k\tau)$, the case we consider in this section, we have

$$\begin{aligned}
h_\nu^{(2)}(0,-k\tau) &\sim c_1^\star+c_2^\star(-k\tau)^{-2\nu}=\mathrm{C}_1^{(k_0)}(\nu)+\mathcal{O}(\tau^{-2\nu})\,,\\
h_\nu^{(2)}(1,-k\tau) &\sim -2\nu c_2^\star(-k\tau)^{-2\nu-1}=\mathrm{C}_2^{(k_0)}(\nu)(-k\tau)^{-2\nu-1}\,,\\
\mathrm{C}_1^{(k_0)}(\nu) &= c_1^\star=-e^{i\pi\nu}c[\nu]\,,\qquad \mathrm{C}_2^{(k_0)}(\nu)=-2\nu c_2^\star=c[-\nu-1]\,,
\end{aligned}\tag{3.16}$$

where $c[\nu]$ is defined in (2.6). The $\mathcal{O}(\tau^{-2\nu})$ term in $h_\nu^{(2)}(0,-k\tau)$ contributes to next-to-leading-order term of solution two. Hence we do not need its coefficient here. Taking the expansions in the integrands, we have

$$\begin{aligned}
\mathrm{C}^{[1]} &= k_0^{\nu_0+1}\int_{-\infty}^{0} \mathrm{d}\tau e^{ik_0\tau}(-\tau)^{\nu_0}c_1 = (-i)^{\nu_0+1}\Gamma\left(\nu_0+1\right)\mathrm{C}_1^{(k_0)}(\nu_1)\,,\\
\mathrm{C}^{[2]} &= k_0^{\nu_0-2\nu_1}\int_{-\infty}^{0} \mathrm{d}\tau e^{ik_0\tau}(-\tau)^{\nu_0}(-2\nu_1 c_2)(-k_1\tau)^{-2\nu_1-1} = (-i)^{\nu_0-2\nu_1}\Gamma\left(\nu_0-2\nu_1\right)\frac{\mathrm{C}_2^{(k_0)}(\nu_1)}{k_1^{2\nu_1+1}}\,.
\end{aligned} \tag{3.17}$$

After substituting this result into (3.12), the calculation is completed.

3.1.3 Solutions with the boundary $k_1 \to \infty$

In this section, we will present an alternative boundary condition choice $k_1 \to \infty$. The series solution obtained under this choice can cover the region uncovered by the convergence region of the series solution from the previous section. Although (3.14) allows us to directly obtain the power series solutions, this approach is not feasible for more complex cases. Therefore, as an instructional case, again we derive the series solution through a differential equation expansion. We define

$$x = \frac{1}{k_1}. \tag{3.18}$$

Then, the differential equations are

$$\Omega_x = \begin{pmatrix} \frac{\nu_0+1}{x-k_0^2x^3} & \frac{ik_0(\nu_0-2\nu_1)}{k_0^2x^2-1} \\ -\frac{ik_0(\nu_0+1)}{k_0^2x^2-1} & \frac{k_0^2(2\nu_1+1)x^2-\nu_0-1}{x\left(k_0^2x^2-1\right)} \end{pmatrix}. \tag{3.19}$$

The series expansion of Ω_x is

$$\begin{aligned}
\Omega_x &= \sum_{i=-1}^{\infty}\Omega_x^{(i)}x^i\,,\\
\Omega_x^{(-1)} &= \begin{pmatrix} (\nu_0+1) & 0 \\ 0 & (\nu_0+1) \end{pmatrix},\\
\Omega_x^{(0+2j)} &= \begin{pmatrix} 0 & i\left(\nu_0+1\right)k_0^{1+2j} \\ -i\left(\nu_0-2\nu_1\right)k_0^{1+2j} & 0 \end{pmatrix},\\
\Omega_x^{(1+2j)} &= \begin{pmatrix} \left(\nu_0+1\right)k_0^{2j+2} & 0 \\ 0 & \left(\nu_0-2\nu_1\right)k_0^{2j+2} \end{pmatrix},
\end{aligned} \tag{3.20}$$

$\Omega_x^{(-1)}$ gives indicial equations and the solution is

$$\lambda = \nu_0 + 1\,. \tag{3.21}$$

Since $\mathrm{C}(1,0)$ and $\mathrm{C}(2,0)$ are undetermined, the system of differential equations still has two linear independent solutions and corresponding two boundary coefficients $\mathrm{C}^{[1]}$ and $\mathrm{C}^{[2]}$. Then, we solve the equations at each order of x again and find the power series solutions are

$$
\begin{aligned}
f_1^{[1]}(k_0,1/x) &= x^{\nu_0+1}\sum_{m=0}^{\infty}\left(\frac{\nu_0+1}{2}\right)_m\left(\frac{\nu_0-2\nu_1+1}{2}\right)_m\frac{4^m\,(k_0x)^{2m}}{(2m)!}\,,\\
f_2^{[1]}(k_0,1/x) &= x^{\nu_0+1}ik_0x(\nu_0+1)\sum_{m=0}^{\infty}\left(\frac{\nu_0+3}{2}\right)_m\left(\frac{\nu_0-2\nu_1+1}{2}\right)_m\frac{4^m\,(k_0x)^{2m}}{(2m+1)!}\,,\\
f_1^{[2]}(k_0,1/x) &= x^{\nu_0+1}(-ik_0x)(\nu_0-2\nu_1)\sum_{m=0}^{\infty}\left(\frac{\nu_0+2}{2}\right)_m\left(\frac{\nu_0-2\nu_1+2}{2}\right)_m\frac{4^m\,(k_0x)^{2m}}{(2m+1)!}\,,\\
f_2^{[2]}(k_0,1/x) &= x^{\nu_0+1}\sum_{m=0}^{\infty}\left(\frac{\nu_0+2}{2}\right)_m\left(\frac{\nu_0-2\nu_1}{2}\right)_m\frac{4^m\,(k_0x)^{2m}}{(2m)!}\,,
\end{aligned}
\tag{3.22}
$$

which converge when $|k_0/k_1|<1$. They also could be expressed as hypergeometric functions:

$$
\begin{aligned}
f_1^{[1]}(k_0,1/x) &= x^{\nu_0+1}\,{}_2\mathrm{F}_1\left(\frac{\nu_0+1}{2},\frac{\nu_0-2\nu_1+1}{2};\frac{1}{2};k_0^2x^2\right)\,,\\
f_2^{[1]}(k_0,1/x) &= x^{\nu_0+1}ik_0x(\nu_0+1)\,{}_2\mathrm{F}_1\left(\frac{\nu_0+3}{2},\frac{\nu_0-2\nu_1+1}{2};\frac{3}{2};k_0^2x^2\right)\,,\\
f_1^{[2]}(k_0,1/x) &= x^{\nu_0+1}(-ik_0x)(\nu_0+1)\,{}_2\mathrm{F}_1\left(\frac{\nu_0+2}{2},\frac{\nu_0-2\nu_1+2}{2};\frac{3}{2};k_0^2x^2\right)\,,\\
f_2^{[2]}(k_0,1/x) &= x^{\nu_0+1}\,{}_2\mathrm{F}_1\left(\frac{\nu_0+2}{2},\frac{\nu_0-2\nu_1}{2};\frac{1}{2};k_0^2x^2\right)\,.
\end{aligned}
\tag{3.23}
$$

Expanding (3.23) at $k_1\to\infty$, we have

$$
\begin{aligned}
&\int_{-\infty}^{0}\mathrm{d}\tau e^{ik_0\tau}(-\tau)^{\nu_0}h_{\nu_1}^{(2)}(0,-k_1\tau) = \mathrm{C}_1^{(k_1)}x^{\nu_0+1}+\mathcal{O}(x^{\nu_0+2})\,,\\
&\int_{-\infty}^{0}\mathrm{d}\tau e^{ik_0\tau}(-\tau)^{\nu_0}h_{\nu_1}^{(2)}(1,-k_1\tau) = \mathrm{C}_2^{(k_1)}x^{\nu_0+1}+\mathcal{O}(x^{\nu_0+2})\,,\\
&\mathrm{C}_1^{(k_1)}(\nu_0,\nu_1) = \frac{\pi 2^{\nu_0-\nu_1+2}e^{i\pi\nu_0/2}}{\left(1+e^{i\pi\nu_0}\right)\left(1+e^{i\pi(\nu_0-2\nu_1)}\right)\Gamma\left(\frac{1}{2}-\frac{\nu_0}{2}\right)\Gamma\left(-\frac{\nu_0}{2}+\nu_1+\frac{1}{2}\right)}\,,\\
&\mathrm{C}_2^{(k_1)}(\nu_0,\nu_1) = -\frac{i\pi 2^{\nu_0-\nu_1+2}e^{i\pi\nu_0/2}}{\left(-1+e^{i\pi\nu_0}\right)\left(-1+e^{i\pi(\nu_0-2\nu_1)}\right)\Gamma\left(-\frac{\nu_0}{2}\right)\Gamma\left(-\frac{\nu_0}{2}+\nu_1+1\right)}\,,
\end{aligned}
\tag{3.24}
$$

and

$$
\mathrm{C}^{[1]}=\mathrm{C}_1^{(k_1)}(\nu_0,\nu_1)\,,\quad \mathrm{C}^{[2]}=\mathrm{C}_2^{(k_1)}(\nu_0,\nu_1)\,. \tag{3.25}
$$

In this section, obtaining (3.24) seems like circular reasoning. However, (3.24) will assist us in determining the boundary conditions for more complex cases.

3.1.4 Analytic continuation and numerical computation efficiency

Note that the series solutions obtained in previous sections have a finite region of convergence. This happens commonly when using the series expansion method. Although we may be able to solve the series solution in another region like we have done in section 3.1.3, it could not be easy in general cases. Hence, in this section, we will discuss how to extend the series solution to regions outside its convergence domain in the general case. We will continue to use the 1-fold vertex integral family as an example for clarity in some places.

We will outline two methods for analytic continuation. Firstly, in the 1-fold vertex integral family we solved, we expressed the analytical series result in terms of a known hypergeometric function, whose properties are well-studied. One can directly use Gauss inverse relation

$$\begin{aligned}{}_2\mathrm{F}_1\left(a,b;c;z\right)=&\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}{}_2\mathrm{F}_1\left(a,1+a-c;1+a-b;\frac{1}{z}\right)\\&+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}{}_2\mathrm{F}_1\left(b,1+b-c;1+b-a;\frac{1}{z}\right)\end{aligned}\tag{3.26}$$

to get the power series around $z=\infty$. For general cases, we may obtain a series of solutions corresponding to generalized hypergeometric functions with multi-variables, some properties of such functions can be found in [62–66]. However, extending these series of solutions beyond the radius of convergence may not always have a ready-made result. One possible method for achieving such analytic continuation is through the Mellin-Barnes contour [67, 68]. This scenario commonly appears in systems involving IBP and differential equations, and the calculations in flat spacetime QFT have driven related research [69]. Secondly, since there is no fully understood analytic function to represent the integral we need to compute, one might consider defining new "analytic functions" directly through differential equations [70]. With differential equations, we can easily solve a series expansion solution at any point of parameter space and obtain extremely accurate numerical results with remarkable speed [60]. Automatic packages for numerical differential equations [71, 72] are already available and widely used in flat QFT. This means that one can quickly compute function values at any regular point and analyze the asymptotic behavior near any singularities. Then, as long as boundary conditions are given, these functions appear to have not many differences from a so-called "analytic" result.

To illustrate, let us assume that we do not know the properties of the hypergeometric function in (3.23), and only have the series solution given by (3.13) with its domain of convergence to be $|k_1/k_0|<1$. We will demonstrate the second method to obtain function values where $|k_1/k_0|>1$.

The numerical method is straightforward. We begin by obtaining the function value at a point within the convergence domain $|k_1/k_0|<1$. This value is then used as a new boundary condition to expand. Solving the linear system like (3.11) again provides the function value at another point. By selecting a series of points to form a path that bypasses the singularities, we can extend the solution to region $|k_1/k_0|>1$. For example, consider $v_0=54/5$ and $v_1=11/7$. We first use (3.13) to compute the sum up to $m=50$, and give the function values at $(k_0,k_1)=(5,2)$:

$$\mathrm{I}_1(5,2)=(2.81\ldots+i\;1.68\ldots)\times10^{-4}\,,\quad \mathrm{I}_2(5,2)=(3.38\ldots-i\;5.65\ldots)\times10^{-4}\tag{3.27}$$

We could choose the path with four steps:

$$(k_0,k_1)=(5,2)\to\left(\frac{7}{2}-i,2\right)\to(2-i,2)\to\left(\frac{3}{2}-\frac{i}{2},2\right)\to\left(\frac{3}{2},2\right)\,,\tag{3.28}$$

where the final point satisfies $k_0 < k_1$ and is outside the convergence domain of (3.13). We compute them using (3.11) up to the 90th order at each point. Each step takes about 0.2 seconds. We obtained the final results with relative errors of $\mathcal{O}(10^{-34})$:

$$\mathrm{I}_1(3/2,2) = 0.201\ldots + i\ 0.120\ldots\,, \quad \mathrm{I}_2(3/2,2) = 0.176\ldots - i\ 0.295\ldots \tag{3.29}$$

Subsequently, one can use point $(3/2, 2)$ as a new boundary and solve for the function values in the region $k_0 < k_1$ along the real axis. Since the distance for each expansion is relatively short and the series solution converges quickly, we can obtain the values more efficiently. For instance, we tested to evaluate the master integrals at $(k_0, k_1) = (3/2 - j/100, 2)$, for $j = 1, \cdots, 100$ and with 20th order expansion. We find that each point took approximately 0.01 seconds to compute while maintaining a relative error of at most $\mathcal{O}(10^{-34})$. All these calculations were performed using Mathematica on a single-core CPU of a personal computer. Additionally, one can use the function values at these regular points to match the expansion near the singularity $k_1 \to \infty$ and determine the boundary condition coefficients for this expansion, as also has been discussed in the [60].

3.2 Example: 2-fold vertex integral family

3.2.1 Preparation

In this section, we use a 2-fold vertex integral family as an example to solve its series solution by the expansions of $k_0 \to \infty$ and $k_2 \to \infty$ using the d log-form differential equations. This function family has 4 master integrals. Again, We use $\mathrm{I}_{\tilde{a}}$ as master integrals:

$$\begin{aligned}
\mathrm{I}_1 &= \int_{-\infty}^{0} \mathrm{d}\tau e^{ik_0\tau}(-\tau)^{\nu_0} h^{(2)}_{\nu_1}(0, -k_1\tau) h^{(2)}_{\nu_2}(0, -k_2\tau)\,, \\
\mathrm{I}_2 &= \int_{-\infty}^{0} \mathrm{d}\tau e^{ik_0\tau}(-\tau)^{\nu_0} h^{(2)}_{\nu_1}(0, -k_1\tau) h^{(2)}_{\nu_2}(1, -k_2\tau) \\
\mathrm{I}_3 &= \int_{-\infty}^{0} \mathrm{d}\tau e^{ik_0\tau}(-\tau)^{\nu_0} h^{(2)}_{\nu_1}(1, -k_1\tau) h^{(2)}_{\nu_2}(0, -k_2\tau)\,, \\
\mathrm{I}_4 &= \int_{-\infty}^{0} \mathrm{d}\tau e^{ik_0\tau}(-\tau)^{\nu_0} h^{(2)}_{\nu_1}(1, -k_1\tau) h^{(2)}_{\nu_2}(1, -k_2\tau)\,.
\end{aligned} \tag{3.30}$$

The matrices of differential equations are

$$\begin{aligned}
\Omega = {} & A_1 \log\left(k_{--}\right) + A_2 \log\left(k_{-+}\right) + A_3 \log\left(k_{+-}\right) + A_4 \log\left(k_{++}\right) \\
& + A_5 \log\left(k_2\right) + A_6 \log\left(k_1\right)\,,
\end{aligned} \tag{3.31}$$

where

$$\begin{aligned}
& k_{\pm\pm} \equiv k_0 \pm k_1 \pm k_2\,, \\
& A_1 = \frac{1}{4}\begin{pmatrix} -\nu_0 - 1 & i\left(\nu_0 - 2\nu_2\right) & i\left(\nu_0 - 2\nu_1\right) & \nu_0 - 2\nu_1 - 2\nu_2 - 1 \\ -i\left(\nu_0 + 1\right) & 2\nu_2 - \nu_0 & 2\nu_1 - \nu_0 & i\left(\nu_0 - 2\nu_1 - 2\nu_2 - 1\right) \\ -i\left(\nu_0 + 1\right) & 2\nu_2 - \nu_0 & 2\nu_1 - \nu_0 & i\left(\nu_0 - 2\nu_1 - 2\nu_2 - 1\right) \\ \nu_0 + 1 & -i\left(\nu_0 - 2\nu_2\right) & -i\left(\nu_0 - 2\nu_1\right) & -\nu_0 + 2\nu_1 + 2\nu_2 + 1 \end{pmatrix},
\end{aligned} \tag{3.32}$$

$$A_2 = \frac{1}{4}\begin{pmatrix} -\nu_0-1 & -i(\nu_0-2\nu_2) & i(\nu_0-2\nu_1) & -\nu_0+2\nu_1+2\nu_2+1 \\ i(\nu_0+1) & 2\nu_2-\nu_0 & \nu_0-2\nu_1 & i(\nu_0-2\nu_1-2\nu_2-1) \\ -i(\nu_0+1) & \nu_0-2\nu_2 & 2\nu_1-\nu_0 & -i(\nu_0-2\nu_1-2\nu_2-1) \\ -\nu_0-1 & -i(\nu_0-2\nu_2) & i(\nu_0-2\nu_1) & -\nu_0+2\nu_1+2\nu_2+1 \end{pmatrix},$$

$$A_3 = \frac{1}{4}\begin{pmatrix} -\nu_0-1 & i(\nu_0-2\nu_2) & -i(\nu_0-2\nu_1) & -\nu_0+2\nu_1+2\nu_2+1 \\ -i(\nu_0+1) & 2\nu_2-\nu_0 & \nu_0-2\nu_1 & -i(\nu_0-2\nu_1-2\nu_2-1) \\ i(\nu_0+1) & \nu_0-2\nu_2 & 2\nu_1-\nu_0 & i(\nu_0-2\nu_1-2\nu_2-1) \\ -\nu_0-1 & i(\nu_0-2\nu_2) & -i(\nu_0-2\nu_1) & -\nu_0+2\nu_1+2\nu_2+1 \end{pmatrix},$$

$$A_4 = \frac{1}{4}\begin{pmatrix} -\nu_0-1 & -i(\nu_0-2\nu_2) & -i(\nu_0-2\nu_1) & \nu_0-2\nu_1-2\nu_2-1 \\ i(\nu_0+1) & 2\nu_2-\nu_0 & 2\nu_1-\nu_0 & -i(\nu_0-2\nu_1-2\nu_2-1) \\ i(\nu_0+1) & 2\nu_2-\nu_0 & 2\nu_1-\nu_0 & -i(\nu_0-2\nu_1-2\nu_2-1) \\ \nu_0+1 & i(\nu_0-2\nu_2) & i(\nu_0-2\nu_1) & -\nu_0+2\nu_1+2\nu_2+1 \end{pmatrix},$$

$$A_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2\nu_1-1 & 0 \\ 0 & 0 & 0 & -2\nu_1-1 \end{pmatrix}, \quad A_6 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\nu_2-1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\nu_2-1 \end{pmatrix}. \tag{3.33}$$

3.2.2 Solutions with the boundary $k_0 \to \infty$

Defining $x = 1/k_0$ and following the same steps, indicial equations give

$$\begin{aligned} &\text{Solution 1: } \mathrm{C}(i \neq 1, 0) = 0, \quad \lambda = \nu_0 + 1\,, \\ &\text{Solution 2: } \mathrm{C}(i \neq 2, 0) = 0, \quad \lambda = \nu_0 - 2\nu_2\,, \\ &\text{Solution 3: } \mathrm{C}(i \neq 3, 0) = 0, \quad \lambda = \nu_0 - 2\nu_1\,, \\ &\text{Solution 4: } \mathrm{C}(i \neq 4, 0) = 0, \quad \lambda = \nu_0 - 2\nu_1 - 2\nu_2 - 1\,. \end{aligned} \tag{3.34}$$

We have four solutions:

$$\begin{aligned}
\mathrm{I}_{\tilde{\boldsymbol{a}}} &= \sum_{\tilde{\boldsymbol{b}}=1}^{4} \mathrm{C}^{[\tilde{\boldsymbol{b}}]} f_{\tilde{\boldsymbol{a}}}^{[\tilde{\boldsymbol{b}}]}\,, \\
f_1^{[1]} &= x^{\nu_0+1} \mathrm{F}_4\left(\frac{\nu_0+1}{2}, \frac{\nu_0+2}{2}; \nu_1+1, \nu_2+1; k_1^2x^2, k_2^2x^2\right), \\
f_2^{[1]} &= x^{\nu_0+2} \frac{ik_2(\nu_0+1)}{2(\nu_2+1)} \mathrm{F}_4\left(\frac{\nu_0+2}{2}, \frac{\nu_0+3}{2}; \nu_1+1, \nu_2+2; k_1^2x^2, k_2^2x^2\right), \\
f_3^{[1]} &= x^{\nu_0+2} \frac{ik_1(\nu_0+1)}{2(\nu_1+1)} \mathrm{F}_4\left(\frac{\nu_0+2}{2}, \frac{\nu_0+3}{2}; \nu_1+2, \nu_2+1; k_1^2x^2, k_2^2x^2\right), \\
f_4^{[1]} &= x^{\nu_0+3} \frac{-k_1k_2(\nu_0+1)(\nu_0+2)}{4(\nu_1+1)(\nu_2+1)} \mathrm{F}_4\left(\frac{\nu_0+3}{2}, \frac{\nu_0+4}{2}; \nu_1+2, \nu_2+2; k_1^2x^2, k_2^2x^2\right), \\
f_1^{[2]} &= x^{\nu_0-2\nu_2+1} \frac{ik_2(\nu_0-2\nu_2)}{2\nu_2} \mathrm{F}_4\left(\frac{\nu_0-2\nu_2+1}{2}, \frac{\nu_0-2\nu_2+2}{2}; \nu_1+1, 1-\nu_2; k_1^2x^2, k_2^2x^2\right), \\
f_2^{[2]} &= x^{\nu_0-2\nu_2} \mathrm{F}_4\left(\frac{\nu_0-2\nu_2}{2}, \frac{\nu_0-2\nu_2+1}{2}; \nu_1+1, -\nu_2; k_1^2x^2, k_2^2x^2\right), \\
f_3^{[2]} &= x^{\nu_0-2\nu_2+2} \frac{-k_1k_2(\nu_0-2\nu_2)(\nu_0-2\nu_2+1)}{4\nu_2(\nu_1+1)} \\
&\quad \times \mathrm{F}_4\left(\frac{\nu_0-2\nu_2+2}{2}, \frac{\nu_0-2\nu_2+3}{2}; \nu_1+2, 1-\nu_2; k_1^2x^2, k_2^2x^2\right),
\end{aligned}$$

$$
\begin{aligned}
f_4^{[2]} &= x^{\nu_0-2\nu_2+1}\frac{ik_1\left(\nu_0-2\nu_2\right)}{2\left(\nu_1+1\right)}\mathrm{F}_4\left(\frac{\nu_0-2\nu_2+1}{2},\frac{\nu_0-2\nu_2+2}{2};\nu_1+2,-\nu_2;k_1^2x^2,k_2^2x^2\right)\,,\\
f_1^{[3]} &= x^{\nu_0-2\nu_1+1}\frac{ik_1\left(\nu_0-2\nu_1\right)}{2\nu_1}\mathrm{F}_4\left(\frac{\nu_0-2\nu_1+1}{2},\frac{\nu_0-2\nu_1+2}{2};1-\nu_1,\nu_2+1;k_1^2x^2,k_2^2x^2\right)\,,\\
f_2^{[3]} &= x^{\nu_0-2\nu_1+2}\frac{-k_1k_2\left(\nu_0-2\nu_1\right)\left(\nu_0-2\nu_1+1\right)}{4\nu_1\left(\nu_2+1\right)}\\
&\quad\times\mathrm{F}_4\left(\frac{\nu_0-2\nu_1+2}{2},\frac{\nu_0-2\nu_1+3}{2};1-\nu_1,\nu_2+2;k_1^2x^2,k_2^2x^2\right)\,,\\
f_3^{[3]} &= x^{\nu_0-2\nu_1}\mathrm{F}_4\left(\frac{\nu_0-2\nu_1}{2},\frac{\nu_0-2\nu_1+1}{2};-\nu_1,\nu_2+1;k_1^2x^2,k_2^2x^2\right)\,,\\
f_4^{[3]} &= x^{\nu_0-2\nu_1+1}\frac{ik_2\left(\nu_0-2\nu_1\right)}{2\left(\nu_2+1\right)}\mathrm{F}_4\left(\frac{\nu_0-2\nu_1+1}{2},\frac{\nu_0-2\nu_1+2}{2};-\nu_1,\nu_2+2;k_1^2x^2,k_2^2x^2\right)\,,\\
f_1^{[4]} &= x^{\nu_0-2\nu_1-2\nu_2+1}\frac{-k_1k_2\left(\nu_0-2\nu_1-2\nu_2-1\right)\left(\nu_0-2\nu_1-2\nu_2\right)}{4\nu_1\nu_2}\\
&\quad\times\mathrm{F}_4\left(\frac{1}{2}\left(\nu_0-2\nu_1-2\nu_2+1\right),\frac{1}{2}\left(\nu_0-2\nu_1-2\nu_2+2\right);1-\nu_1,1-\nu_2;k_1^2x^2,k_2^2x^2\right)\,,\\
f_2^{[4]} &= x^{\nu_0-2\nu_1-2\nu_2}\frac{ik_1\left(\nu_0-2\nu_1-2\nu_2-1\right)}{2\nu_1}\\
&\quad\times\mathrm{F}_4\left(\frac{1}{2}\left(\nu_0-2\nu_1-2\nu_2\right),\frac{1}{2}\left(\nu_0-2\nu_1-2\nu_2+1\right);1-\nu_1,-\nu_2;k_1^2x^2,k_2^2x^2\right)\,,\\
f_3^{[4]} &= x^{\nu_0-2\nu_1-2\nu_2}\frac{ik_2\left(\nu_0-2\nu_1-2\nu_2-1\right)}{2\nu_2}\\
&\quad\times\mathrm{F}_4\left(\frac{1}{2}\left(\nu_0-2\nu_1-2\nu_2\right),\frac{1}{2}\left(\nu_0-2\nu_1-2\nu_2+1\right);-\nu_1,1-\nu_2;k_1^2x^2,k_2^2x^2\right)\,,\\
f_4^{[4]} &= x^{\nu_0-2\nu_1-2\nu_2-1}\\
&\quad\times\mathrm{F}_4\left(\frac{1}{2}\left(\nu_0-2\nu_1-2\nu_2-1\right),\frac{1}{2}\left(\nu_0-2\nu_1-2\nu_2\right);-\nu_1,-\nu_2;k_1^2x^2,k_2^2x^2\right)\,. \qquad (3.35)
\end{aligned}
$$

Here the F_4 are [73]

$$
\mathrm{F}_4(a,b;c_1,c_2;x,y)\equiv\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(c_1)_m(c_2)_n}\frac{x^my^n}{m!n!}\,. \qquad (3.36)
$$

To determine $\mathrm{C}^{[\tilde{\boldsymbol{b}}]}$, we only need one term in the expansion of integrand again. For example, referring to (3.16), we have

$$
\mathrm{I}_1\sim\int_{-\infty}^{0}\mathrm{d}\tau e^{ik_0\tau}(-\tau)^{\nu_0}\left(\mathrm{C}_1^{(k_0)}(\nu_1)+\mathcal{O}(t^{-2\nu_1})\right)\left(\mathrm{C}_1^{(k_0)}(\nu_2)+\mathcal{O}(\tau^{-2\nu_2})\right)\,. \qquad (3.37)
$$

Here only the $\mathrm{C}_1^{(k_0)}(\nu_1)\mathrm{C}_1^{(k_0)}(\nu_2)$ contributes to leading-order of solution one, and thus it contributes to $\mathrm{C}^{[1]}$. Meanwhile, $\mathrm{C}_1^{(k_0)}(\nu_1)\mathcal{O}(\tau^{-2\nu_2})$ contributes to the second and high order of solution two, $\mathrm{C}_1^{(k_0)}(\nu_2)\mathcal{O}(\tau^{-2\nu_1})$ contributes to the second and high order of solution three, $\mathcal{O}(\tau^{-2\nu_2})\mathcal{O}(\tau^{-2\nu_1})$ contributes to the second and high order of solution four. Through similar analysis, we focus only on the following terms in the expansion (recalling the definition (2.8) that $\mathrm{I}_{\{a_1,a_2\}}\equiv\mathrm{I}_{\tilde{\boldsymbol{a}}}$):

$$
\mathrm{I}_{\{0,0\}}\sim\mathrm{C}^{[\{0,0\}]}x^{\nu_0+1}=\int_{-\infty}^{0}\mathrm{d}\tau\;\mathrm{C}_{\tilde{0}}^{(k_0)}(\nu_1)\;\mathrm{C}_{\tilde{0}}^{(k_0)}(\nu_2)\;e^{ik_0\tau}(-\tau)^{\nu_0}\,,
$$

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$$
\begin{aligned}
\mathrm{I}_{\{0,1\}} &\sim \mathrm{C}^{[\{0,1\}]} x^{\nu_0-2\nu_2} = \int_{-\infty}^{0} \mathrm{d}\tau\ \mathrm{C}^{(k_0)}_{\tilde{0}}(\nu_1)\ \mathrm{C}^{(k_0)}_{\tilde{1}}(\nu_2)\ e^{ik_0\tau}(-\tau)^{\nu_0}(-k_2\tau)^{-2\nu_2-1}\,,\\
\mathrm{I}_{\{1,0\}} &\sim \mathrm{C}^{[\{1,0\}]} x^{\nu_0-2\nu_2} = \int_{-\infty}^{0} \mathrm{d}\tau\ \mathrm{C}^{(k_0)}_{\tilde{1}}(\nu_1)\ \mathrm{C}^{(k_0)}_{\tilde{0}}(\nu_2)\ e^{ik_0\tau}(-\tau)^{\nu_0}(-k_1\tau)^{-2\nu_1-1}\,,\\
\mathrm{I}_{\{1,1\}} &\sim \mathrm{C}^{[\{1,1\}]} x^{\nu_0-2\nu_2-2\nu_1-1}\\
&= \int_{-\infty}^{0} \mathrm{d}\tau\ \mathrm{C}^{(k_0)}_{\tilde{1}}(\nu_1)\ \mathrm{C}^{(k_0)}_{\tilde{1}}(\nu_2)\ e^{ik_0\tau}(-\tau)^{\nu_0}(-k_1\tau)^{-2\nu_1-1}(-k_2\tau)^{-2\nu_2-1}\,.
\end{aligned}
\tag{3.38}
$$

Hence,

$$
\begin{aligned}
\mathrm{C}^{[\{0,0\}]} &= (-i)^{\nu_0+1}\Gamma\left(\nu_0+1\right)\ \mathrm{C}^{(k_0)}_{\tilde{0}}(\nu_1)\ \mathrm{C}^{(k_0)}_{\tilde{0}}(\nu_2)\,,\\
\mathrm{C}^{[\{0,1\}]} &= (-i)^{\nu_0-2\nu_2} k_2^{-2\nu_2-1}\Gamma\left(\nu_0-2\nu_2\right)\ \mathrm{C}^{(k_0)}_{\tilde{0}}(\nu_1)\ \mathrm{C}^{(k_0)}_{\tilde{1}}(\nu_2)\,,\\
\mathrm{C}^{[\{0,1\}]} &= (-i)^{\nu_0-2\nu_1} k_1^{-2\nu_1-1}\Gamma\left(\nu_0-2\nu_1\right)\ \mathrm{C}^{(k_0)}_{\tilde{1}}(\nu_1)\ \mathrm{C}^{(k_0)}_{\tilde{0}}(\nu_2)\,,\\
\mathrm{C}^{[\{1,1\}]} &= (-i)^{\nu_0-2(\nu_1+\nu_2)-1} k_1^{-2\nu_1-1} k_2^{-2\nu_2-1}\Gamma\left(\nu_0-2\nu_1-2\nu_2-1\right)\ \mathrm{C}^{(k_0)}_{\tilde{1}}(\nu_1)\ \mathrm{C}^{(k_0)}_{\tilde{1}}(\nu_2)\,.
\end{aligned}
\tag{3.39}
$$

3.2.3 Solutions with the boundary $k_2 \to \infty$

Without loss of generality, we consider the expansion near the boundary as $k_2 \to \infty$. Defining $x = 1/k_2$ and following the same steps, indicial equations give

$$
\begin{aligned}
&\text{Solution 1 \& 2:} \qquad \lambda = \nu_0+1\,, \qquad \mathrm{C}(3,0)=\mathrm{C}(4,0)=0\,,\\
&\text{Solution 3 \& 4:} \qquad \lambda = \nu_0-2\nu_1\,, \qquad \mathrm{C}(1,0)=\mathrm{C}(2,0)=0\,.
\end{aligned}
\tag{3.40}
$$

We denote $\mathrm{C}(1,0)$ and $\mathrm{C}(2,0)$ in solution 1&2 by $\mathrm{C}^{[1]}$ and $\mathrm{C}^{[2]}$, denote $\mathrm{C}(3,0)$ and $\mathrm{C}(4,0)$ in solution 3&4 by $\mathrm{C}^{[3]}$ and $\mathrm{C}^{[4]}$, and have four solutions:

$$
\begin{aligned}
\mathrm{I}_{\tilde{a}} &= \sum_{\tilde{b}=1}^{4} \mathrm{C}^{[\tilde{b}]} f^{[\tilde{b}]}_{\tilde{a}}\,,\\
f^{[1]}_1 &= x^{\nu_0+1}\mathrm{F}_4\left(\frac{1}{2}(\nu_0+1),\frac{1}{2}(\nu_0-2\nu_2+1);\frac{1}{2},\nu_1+1;k_0^2x^2,k_1^2x^2\right),\\
f^{[1]}_2 &= x^{\nu_0+2} ik_0(\nu_0+1)\,\mathrm{F}_4\left(\frac{1}{2}(\nu_0+3),\frac{1}{2}(\nu_0-2\nu_2+1);\frac{3}{2},\nu_1+1;k_0^2x^2,k_1^2x^2\right),\\
f^{[1]}_3 &= x^{\nu_0+3}\frac{-ik_0k_1(\nu_0+1)(\nu_0-2\nu_2+1)}{2(\nu_1+1)}\\
&\quad\times\mathrm{F}_4\left(\frac{1}{2}(\nu_0+3),\frac{1}{2}(\nu_0-2\nu_2+3);\frac{3}{2},\nu_1+2;k_0^2x^2,k_1^2x^2\right),\\
f^{[1]}_4 &= x^{\nu_0+2}\frac{k_1(\nu_0+1)}{2(\nu_1+1)}\mathrm{F}_4\left(\frac{1}{2}(\nu_0+3),\frac{1}{2}(\nu_0-2\nu_2+1);\frac{1}{2},\nu_1+2;k_0^2x^2,k_1^2x^2\right),\\
f^{[2]}_1 &= x^{\nu_0+2}(-ik_0)(\nu_0-2\nu_2)\,\mathrm{F}_4\left(\frac{1}{2}(\nu_0+2),\frac{1}{2}(\nu_0-2\nu_2+2);\frac{3}{2},\nu_1+1;k_0^2x^2,k_1^2x^2\right),\\
f^{[2]}_2 &= x^{\nu_0+1}\mathrm{F}_4\left(\frac{1}{2}(\nu_0+2),\frac{1}{2}(\nu_0-2\nu_2);\frac{1}{2},\nu_1+1;k_0^2x^2,k_1^2x^2\right),\\
f^{[2]}_3 &= x^{\nu_0+2}\frac{-k_1(\nu_0-2\nu_2)}{2(\nu_1+1)}\mathrm{F}_4\left(\frac{1}{2}(\nu_0+2),\frac{1}{2}(\nu_0-2\nu_2+2);\frac{1}{2},\nu_1+2;k_0^2x^2,k_1^2x^2\right),
\end{aligned}
$$

$$
\begin{aligned}
f_4^{[2]} &= x^{\nu_0+3}\frac{-ik_0k_1(\nu_0+2)(\nu_0-2\nu_2)}{2(\nu_1+1)}\\
&\quad\times \mathrm{F}_4\left(\frac{1}{2}(\nu_0+4),\frac{1}{2}(\nu_0-2\nu_2+2);\frac{3}{2},\nu_1+2;k_0^2x^2,k_1^2x^2\right),\\
f_1^{[3]} &= x^{\nu_0-2\nu_1+2}\frac{-ik_0k_1(\nu_0-2\nu_1)(\nu_0-2(\nu_1+\nu_2))}{2\nu_1}\\
&\quad\times \mathrm{F}_4\left(\frac{1}{2}(\nu_0-2(\nu_1+\nu_2-1)),\frac{1}{2}(\nu_0-2\nu_1+2);\frac{3}{2},1-\nu_1;k_0^2x^2,k_1^2x^2\right),\\
f_2^{[3]} &= x^{\nu_0-2\nu_1+1}\frac{k_1(\nu_0-2\nu_1)}{2\nu_1}\mathrm{F}_4\left(\frac{1}{2}(\nu_0-2(\nu_1+\nu_2)),\frac{1}{2}(\nu_0-2\nu_1+2);\frac{1}{2},1-\nu_1;k_0^2x^2,k_1^2x^2\right),\\
f_3^{[3]} &= x^{\nu_0-2\nu_1}\mathrm{F}_4\left(\frac{1}{2}(\nu_0-2(\nu_1+\nu_2)),\frac{1}{2}(\nu_0-2\nu_1);\frac{1}{2},-\nu_1;k_0^2x^2,k_1^2x^2\right),\\
f_4^{[3]} &= x^{\nu_0-2\nu_1+1}ik_0(\nu_0-2\nu_1)\,\mathrm{F}_4\left(\frac{1}{2}(\nu_0-2(\nu_1+\nu_2)),\frac{1}{2}(\nu_0-2\nu_1+2);\frac{3}{2},-\nu_1;k_0^2x^2,k_1^2x^2\right),\\
f_1^{[4]} &= x^{\nu_0-2\nu_1+1}\frac{k_1(-\nu_0+2\nu_1+2\nu_2+1)}{2\nu_1}\\
&\quad\times \mathrm{F}_4\left(\frac{1}{2}(\nu_0-2\nu_1-2\nu_2+1),\frac{1}{2}(\nu_0-2\nu_1+1);\frac{1}{2},1-\nu_1;k_0^2x^2,k_1^2x^2\right),\\
f_2^{[4]} &= x^{\nu_0-2\nu_1+2}\frac{-ik_0k_1(\nu_0-2\nu_1+1)(\nu_0-2\nu_1-2\nu_2-1)}{2\nu_1}\\
&\quad\times \mathrm{F}_4\left(\frac{1}{2}(\nu_0-2\nu_1-2\nu_2+1),\frac{1}{2}(\nu_0-2\nu_1+3);\frac{3}{2},1-\nu_1;k_0^2x^2,k_1^2x^2\right),\\
f_3^{[4]} &= x^{\nu_0-2\nu_1+1}(-ik_0)(\nu_0-2\nu_1-2\nu_2-1)\\
&\quad\times \mathrm{F}_4\left(\frac{1}{2}(\nu_0-2\nu_1-2\nu_2+1),\frac{1}{2}(\nu_0-2\nu_1+1);\frac{3}{2},-\nu_1;k_0^2x^2,k_1^2x^2\right),\\
f_4^{[4]} &= x^{\nu_0-2\nu_1}\mathrm{F}_4\left(\frac{1}{2}(\nu_0-2\nu_1-2\nu_2-1),\frac{1}{2}(\nu_0-2\nu_1+1);\frac{1}{2},-\nu_1;k_0^2x^2,k_1^2x^2\right).
\end{aligned}
\tag{3.41}
$$

Recalling (2.7), all terms in the integrand, except for the Hankel function corresponding to k_2, are exponentially suppressed and thus can be expanded around $\tau=0$. Then, recalling (2.6), all boundary coefficients could be determined by integrals taking the form just like the two in (3.24). Let us consider I_3 as an example. In the $k_2\to\infty$ limitation,

$$
\begin{aligned}
\mathrm{I}_3 &= \int_{-\infty}^{0}\mathrm{d}\tau e^{ik_0\tau}(-\tau)^{\nu_0}h_{\nu_1}^{(2)}(1,-k_1\tau)h_{\nu_2}^{(2)}(0,-k_2\tau)\\
&\sim \int_{-\infty}^{0}\mathrm{d}\tau e^{ik_0\tau}(-\tau)^{\nu_0}(-k_1\tau)\mathrm{C}_2^{(k_0)}(\nu_1)h_{\nu_2}^{(2)}(1,-k_2\tau)\\
&= \mathrm{C}_2^{(k_0)}(\nu_1)\mathrm{C}_1^{(k_n)}(\nu_0-2\nu_1-1,\nu_2)\;k_1^{-2\nu_1-1}x^{\nu_0-2\nu_1}\\
&= \mathrm{C}^{[3]}x^{\nu_0-2\nu_1}\,.
\end{aligned}
\tag{3.42}
$$

As a result, we have

$$
\begin{aligned}
\mathrm{C}^{[1]} &= \mathrm{C}_1^{(k_0)}(\nu_1)\,\mathrm{C}_1^{(k_n)}(\nu_0,\nu_2)\;,\\
\mathrm{C}^{[2]} &= \mathrm{C}_1^{(k_0)}(\nu_1)\,\mathrm{C}_2^{(k_n)}(\nu_0,\nu_2)\;,\\
\mathrm{C}^{[3]} &= \mathrm{C}_2^{(k_0)}(\nu_1)\,\mathrm{C}_1^{(k_n)}(\nu_0-2\nu_1-1,\nu_2)\,k_1^{-2\nu_1-1}\,,\\
\mathrm{C}^{[4]} &= \mathrm{C}_2^{(k_0)}(\nu_1)\,\mathrm{C}_2^{(k_n)}(\nu_0-2\nu_1-1,\nu_2)\,k_1^{-2\nu_1-1}\,.
\end{aligned}
\tag{3.43}
$$

3.3 Multivariate hypergeometric solutions of arbitrary vertex integral family

3.3.1 Solutions with the boundary $k_0 \to \infty$

For the master integrals of an arbitrary n-fold vertex function family with $h_\nu^{(2)}(a, -k\tau)$ and all ν_i being real (we will generalize the results to general cases later in section 3.3.3), we could easily derive $\mathrm{C}^{[\tilde{\boldsymbol{b}}]}$ by computation similar to previous subsections, and by observation, we conjecture solutions $f_{\tilde{\boldsymbol{a}}}^{[\tilde{\boldsymbol{b}}]}$ of expansion $k_0 \to \infty$ as follows:

$$
\begin{aligned}
\mathrm{I}_{\tilde{\boldsymbol{a}}} &= \sum_{\tilde{\boldsymbol{b}}=1}^{2^n} \mathrm{C}^{[\tilde{\boldsymbol{b}}]} f_{\tilde{\boldsymbol{a}}}^{[\tilde{\boldsymbol{b}}]} \,, \\
f_{\tilde{\boldsymbol{a}}}^{[\tilde{\boldsymbol{b}}]} &= x^{\tilde{\mathrm{A}}} \frac{(\tilde{\mathrm{A}})_{|\boldsymbol{a}-\boldsymbol{b}|\cdot\mathbf{1}}}{2^{|\boldsymbol{a}-\boldsymbol{b}|\cdot\mathbf{1}}} \prod_{j=1}^{n} \left(\frac{(-1)^{b_j} i k_j x}{\tilde{\mathrm{B}}_j} \right)^{|a_j-b_j|} \times \tilde{\mathrm{F}}_4 \left(\mathrm{A}_1, \mathrm{A}_2;\ \mathrm{B}_1, \cdots, \mathrm{B}_n;\ k_1^2 x^2, \cdots, k_n^2 x^2 \right) , \\
\tilde{\mathrm{A}} &= \nu_0 + 1 - \boldsymbol{b} \cdot (2\boldsymbol{\nu} + \mathbf{1}) \,, \qquad \mathrm{A}_j = \frac{1}{2} \left(\tilde{\mathrm{A}} + |\boldsymbol{a} - \boldsymbol{b}| \cdot \mathbf{1} - 1 + j \right) , \\
\tilde{\mathrm{B}}_j &= \nu_j + 1 - b_j (2\nu_j + 1) \,, \qquad \mathrm{B}_j = \tilde{\mathrm{B}}_j + |a_j - b_j| \,,
\end{aligned} \tag{3.44}
$$

$$
\mathrm{C}^{[\tilde{\boldsymbol{b}}]} = (-i)^{\nu_0+1} \Gamma(\tilde{\mathrm{A}}) \prod_{j=1}^{n} (-ik_i)^{-b_j(2\nu_j+1)} \mathrm{C}_{\tilde{b}_j}^{(k_0)}(\nu_j) \,, \tag{3.45}
$$

where

$$
\begin{aligned}
&\tilde{\mathrm{F}}_4 \left(\mathrm{A}_1, \mathrm{A}_2; \mathbf{B}; \boldsymbol{z} \right) \equiv \sum_{m_1, \cdots, m_n = 0}^{\infty} \frac{(\mathrm{A}_1)_{\boldsymbol{m}\cdot\mathbf{1}} (\mathrm{A}_2)_{\boldsymbol{m}\cdot\mathbf{1}}}{\prod_{i=1}^{n} (\mathrm{B}_i)_{m_i}} \prod_{i=1}^{n} \frac{z_i^{m_i}}{m_i!} \,, \\
&\boldsymbol{a} = a_1, a_2, \cdots, a_n \ , \qquad \boldsymbol{b} = b_1, b_2, \cdots, b_n \ , \\
&\boldsymbol{\nu} = \nu_1, \nu_2, \cdots, \nu_n \ , \qquad \mathbf{1} = 1, 1, \cdots, 1 \ , \\
&|\boldsymbol{a} - \boldsymbol{b}| = |a_1 - b_1|, |a_2 - b_2|, \cdots, |a_n - b_n| \ .
\end{aligned} \tag{3.46}
$$

We have verified the result to be right by comparing it with both the power series solution from differential equations and results of direct numerical integration, up to $n = 4$.

3.3.2 Solutions with the boundary $k_n \to \infty$

Similarly, the multivariate hypergeometric solutions around $k_n \to \infty$ also could be given by

$$
\begin{aligned}
\mathrm{I}_{\tilde{\boldsymbol{a}}} &= \sum_{\tilde{\boldsymbol{b}}=1}^{2^n} \mathrm{C}^{[\tilde{\boldsymbol{b}}]} f_{\tilde{\boldsymbol{a}}}^{[\tilde{\boldsymbol{b}}]} \,, \\
f_{\tilde{\boldsymbol{a}}}^{[\tilde{\boldsymbol{b}}]} &= x^{\nu_0 + 1 - \hat{\boldsymbol{b}}\cdot(2\hat{\boldsymbol{\nu}}+\mathbf{1})} (-1)^{\lfloor \left(\mathrm{mod}[|\boldsymbol{a}-\boldsymbol{b}|\cdot\mathbf{1}, 2] + \hat{\boldsymbol{b}}\cdot\mathbf{1} + \mathrm{mod}[\tilde{\boldsymbol{b}}-1, 2] \right)/2 \rfloor} \\
&\quad \times (2ik_0 x)^{\mathrm{mod}[|\boldsymbol{a}-\boldsymbol{b}|\cdot\mathbf{1}, 2]} \prod_{j=1}^{n-1} \left(\frac{(-1)^{b_j} k_j x}{\nu_j + 1 - b_j(2\nu_j + 1)} \right)^{|a_j - b_j|} \\
&\quad \times \prod_{j=0}^{\mathrm{mod}[|\boldsymbol{a}-\boldsymbol{b}|\cdot\mathbf{1}, 2] + \hat{\boldsymbol{b}}\cdot\mathbf{1} - 1} \left(\frac{\nu_j + 1 - b_j(2\nu_j + 1)}{2} - \mathrm{mod}[b_n + j, 2] \frac{2\nu_n + 1}{2} \right) \\
&\quad \times \tilde{\mathrm{F}}_4 \left(\mathrm{A}_1, \mathrm{A}_2;\ \mathrm{B}_0, \mathrm{B}_1, \cdots, \mathrm{B}_{n-1};\ k_0^2 x^2, \cdots, k_{n-1}^2 x^2 \right) ,
\end{aligned} \tag{3.47}
$$

where

$$
\begin{aligned}
\mathrm{A}_i &= \left(\frac{\nu_j+1-b_j(2\nu_j+1)}{2} - \mathrm{mod}[b_n+j,2]\frac{2\nu_n+1}{2}\right)\Big|_{j=\mathrm{mod}[|\boldsymbol{a}-\boldsymbol{b}|\cdot\boldsymbol{1},2]+\hat{\boldsymbol{b}}\cdot\boldsymbol{1}-1+i}\,,\\
\mathrm{B}_0 &= \frac{1}{2}+\mathrm{mod}[|\boldsymbol{a}-\boldsymbol{b}|\cdot\boldsymbol{1},2]\,,\\
\mathrm{B}_{i>0} &= \nu_i+1-b_i(2\nu_i+1)+|a_i-b_i|\,, \qquad (3.48)\\
\mathrm{C}^{[\tilde{\boldsymbol{b}}]} &= \mathrm{C}^{(k_n)}_{\tilde{b}_n}(\nu_0-\hat{\boldsymbol{b}}\cdot(2\hat{\boldsymbol{\nu}}+\boldsymbol{1}),\nu_n)\prod_{j=1}^{n-1}k_j^{-b_j(2\nu_j+1)}\mathrm{C}^{(k_0)}_{\tilde{b}_j}(\nu_j)\,, \qquad (3.49)
\end{aligned}
$$

$\mathrm{mod}[a,b]$ represents the remainder when a is divided by b, the $\lfloor\ \rfloor$ indicates rounding up, and

$$
\hat{\boldsymbol{a}} = a_1,a_2,\cdots,a_{n-1}\ ,\qquad \hat{\boldsymbol{b}} = b_1,b_2,\cdots,b_{n-1}\ ,\qquad \hat{\boldsymbol{\nu}} = \nu_1,\nu_2,\cdots,\nu_{n-1}\,. \qquad (3.50)
$$

We have verified the result to be right by comparing it with power series solutions, which are directly solved from differential equations, up to $n=5$ and $\mathcal{O}[x^{\lambda+20}]$.

3.3.3 Results for $h^{(1,2)w}$ and real/imaginary ν

Now, let us discuss more general cases: each h-function in n-fold Hankel vertex integral family could be $h^{(1)}_\nu(a,-k\tau)$ or $h^{(2)}_\nu(a,-k\tau)$, and each ν_i could be real or imaginary. Since for all these cases, the differential equations are the same [52], we have the same solutions $f^{[\tilde{\boldsymbol{b}}]}_{\tilde{\boldsymbol{a}}}$. However, the $\mathrm{C}^{(k_0)}_{\tilde{a}}(\nu)$ and $\mathrm{C}^{(k_n)}_{\tilde{a}_n}(\nu,\nu_n)$ in the boundary coefficients $\mathrm{C}^{[\tilde{\boldsymbol{b}}]}$ could be changed and should be re-determined. $\mathrm{C}^{(k_0)}_{\tilde{a}}(\nu)$ should be re-determined by (3.15) like (3.16). For ease of reading, we recall (2.6), (3.15) and (3.16) here:

$$
\begin{aligned}
&h^{(1)}_\nu(0,-k\tau) = c_1(1+\mathcal{O}(\tau^2))+c_2(-k\tau)^{-2\nu}(1+\mathcal{O}(\tau^2))\,,\\
&h^{(2)}_\nu(0,-k\tau) = c_1^\star(-k\tau)^{-\nu+\nu^\star}(1+\mathcal{O}(\tau^2))+c_2^\star(-k\tau)^{-\nu-\nu^\star}(1+\mathcal{O}(\tau^2))\,,\\
&c_1 = e^{-i\pi\nu}c[\nu]\,,\qquad c_2=c[-\nu]\,,\qquad c[\nu]\equiv\frac{2^{-\nu}\Gamma(-\nu)}{i\pi}\,. \qquad (3.51)
\end{aligned}
$$

For $h^{(2)}_\nu(a,-k\tau)$ with real ν, the corresponding $\mathrm{C}^{(k_0)}_1(\nu)$ in the boundary coefficients have been determined as follow:

$$
\begin{aligned}
&h^{(2)}_\nu(0,-k\tau) \sim c_1^\star + c_2^\star(-k\tau)^{-2\nu} = \mathrm{C}^{(k_0)}_1(\nu)+\mathcal{O}(\tau^{-2\nu})\,,\\
&h^{(2)}_\nu(1,-k\tau) \sim -2\nu c_2^\star(-k\tau)^{-2\nu-1} = \mathrm{C}^{(k_0)}_2(\nu)(-k\tau)^{-2\nu-1}\,,\\
&\mathrm{C}^{(k_0)}_1(\nu) = c_1^\star = -e^{i\pi\nu}c[\nu]\,,\qquad \mathrm{C}^{(k_0)}_2(\nu) = -2\nu c_2^\star = c[-\nu-1]\,; \qquad (3.52)
\end{aligned}
$$

For a $h^{(2)}_\nu(a,-k\tau)$ with imaginary ν, the corresponding $\mathrm{C}^{(k_0)}_1(\nu)$ in the boundary coefficients are determined as follow:

$$
\begin{aligned}
&h^{(2)}_\nu(0,-k\tau) \sim c_2^\star + c_1^\star(-k\tau)^{-2\nu} = \mathrm{C}^{(k_0)}_1(\nu)+\mathcal{O}(\tau^{-2\nu})\,,\\
&h^{(2)}_\nu(1,-k\tau) \sim -2\nu c_1^\star(-k\tau)^{-2\nu-1} = \mathrm{C}^{(k_0)}_2(\nu)(-k\tau)^{-2\nu-1}\,,\\
&\mathrm{C}^{(k_0)}_1(\nu) = c_2^\star = -c[\nu]\,,\qquad \mathrm{C}^{(k_0)}_2(\nu) = -2\nu c_1^\star = e^{-i\pi\nu}c[-\nu-1]\,; \qquad (3.53)
\end{aligned}
$$

Since for both cases that ν is real or imaginary in $h_\nu^{(1)}(a,-k\tau)$, they do not involve $\nu^\star$ at the beginning, these two cases have the same corresponding $\mathrm{C}_{\tilde{a}}^{(k_0)}(\nu)$:

$$
\begin{aligned}
&h_\nu^{(1)}(0,-k\tau) \sim c_1 + c_2(-k\tau)^{-2\nu} = \mathrm{C}_1^{(k_0)}(\nu) + \mathcal{O}(\tau^{-2\nu})\,,\\
&h_\nu^{(1)}(1,-k\tau) \sim -2\nu c_2(-k\tau)^{-2\nu-1} = \mathrm{C}_2^{(k_0)}(\nu)(-k\tau)^{-2\nu-1}\,,\\
&\mathrm{C}_1^{(k_0)}(\nu) = c_1 = e^{-i\pi\nu}c[\nu]\,, \qquad \mathrm{C}_2^{(k_0)}(\nu) = -2\nu c_2 = -c[-\nu-1]\,,
\end{aligned}
\tag{3.54}
$$

Let's turn to consider $\mathrm{C}_{\tilde{a}_n}^{(k_n)}(\nu,\nu_n)$ in boundary coefficients. It is defined by:

$$
\int_{-\infty}^{0} \mathrm{d}\tau e^{ik_0\tau}(-\tau)^\nu h_{\nu_n}^{(1,2)}(a_n,-k_n\tau) \sim \mathrm{C}_{\tilde{a}_n}^{(k_n)} x^{\nu+1}\,,
\tag{3.55}
$$

while the left-hand side could be given by expanding the results of the 1-fold vertex integral family which we just obtained.

For $h_\nu^{(2)}(a,-k\tau)$ with real ν_n, recall (3.24):

$$
\begin{aligned}
\mathrm{C}_1^{(k_n)}(\nu,\nu_n) &= \frac{\pi 2^{\nu-\nu_n+2}e^{i\pi\nu/2}}{\left(1+e^{i\pi\nu}\right)\left(1+e^{i\pi(\nu-2\nu_n)}\right)\Gamma\left(\frac{1}{2}-\frac{\nu}{2}\right)\Gamma\left(-\frac{\nu}{2}+\nu_n+\frac{1}{2}\right)}\,,\\
\mathrm{C}_2^{(k_n)}(\nu,\nu_n) &= -\frac{i\pi 2^{\nu-\nu_n+2}e^{i\pi\nu/2}}{\left(1-e^{i\pi\nu}\right)\left(1-e^{i\pi(\nu-2\nu_n)}\right)\Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(-\frac{\nu}{2}+\nu_n+1\right)}\,.
\end{aligned}
\tag{3.56}
$$

For $h_\nu^{(2)}(a,-k\tau)$ with imaginary ν, we have

$$
\begin{aligned}
\mathrm{C}_1^{(k_n)}(\nu,\nu_n) &= \frac{\pi 2^{\nu-\nu_n+2}e^{\frac{1}{2}i\pi(\nu-2\nu_n)}}{\left(1+e^{i\pi\nu}\right)\left(1+e^{i\pi(\nu-2\nu_n)}\right)\Gamma\left(\frac{1}{2}-\frac{\nu}{2}\right)\Gamma\left(-\frac{\nu}{2}+\nu_n+\frac{1}{2}\right)}\,,\\
\mathrm{C}_2^{(k_n)}(\nu,\nu_n) &= -\frac{i\pi 2^{\nu-\nu_n+2}e^{\frac{1}{2}i\pi(\nu-2\nu_n)}}{\left(1-e^{i\pi\nu}\right)\left(1-e^{i\pi(\nu-2\nu_n)}\right)\Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(-\frac{\nu}{2}+\nu_n+1\right)}\,.
\end{aligned}
\tag{3.57}
$$

For $h_\nu^{(1)}(a,-k\tau)$ with real or imaginary ν, the $\mathrm{C}_{\tilde{a}_n}^{(k_n)}(\nu,\nu_n)$ should be the complex conjugate of (3.56) when ν and ν_n are real:

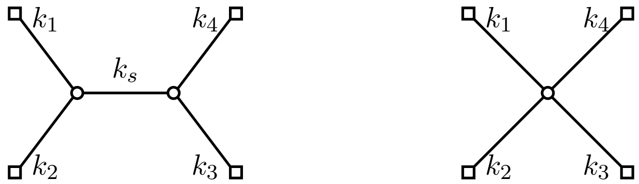
$$
\begin{aligned}
\mathrm{C}_1^{(k_n)}(\nu,\nu_n) &= \frac{\pi 2^{\nu-\nu_n+2}e^{-i\pi\nu/2}}{\left(1+e^{-i\pi\nu}\right)\left(1+e^{-i\pi(\nu-2\nu_n)}\right)\Gamma\left(\frac{1}{2}-\frac{\nu}{2}\right)\Gamma\left(-\frac{\nu}{2}+\nu_n+\frac{1}{2}\right)}\,,\\
\mathrm{C}_2^{(k_n)}(\nu,\nu_n) &= \frac{i\pi 2^{\nu-\nu_n+2}e^{-i\pi\nu/2}}{\left(1-e^{-i\pi\nu}\right)\left(1-e^{-i\pi(\nu-2\nu_n)}\right)\Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(-\frac{\nu}{2}+\nu_n+1\right)}\,,
\end{aligned}
\tag{3.58}
$$

and the result could be directly extended to the imaginary case.

With these results, we have completed the solving of the arbitrary n-fold vertex integral family of cosmological correlators.

4 Properties of time-order n-vertex cosmological correlators

While the two vertices linked by $G_{\pm\mp}$ are directly factorized as two integral, time-order propagators $G_{\pm\pm}$ combine two integrations of τ_i together and is likely to be much more complicated to solve. However, time-order propagators have elegant factorization properties,



(a) 2-vertex sector $\qquad$ (b) 1-vertex sub-sector

Figure 1. (a) is a diagram of the 2-vertex correlators, with the vertices could be scalar or time derivative interactions. $\mathrm{I}_{1,2,3,4}$ belong to this sector. The IBP relation of integrals in (a) will automatically involve (b), which is given by pinching the propagator of (a) [52]. I_5 belong to this sub-sector.

simplifying the IBP relation and differential equations [52]. This property further leads to the simplification of solutions as well. Thus, in this section, we will show the properties of time-order n-vertex cosmological correlators by solving the integral family of tree-level 4-pt 2-vertex correlators as an example. The master integrals of this integral family for the s-channel with G_{++} can be written as follows (with $k_{1;1} = k_{1;2} = k_s$ and $\nu_{1;1} = \nu_{1;2} = \nu_1$):

$$\mathrm{I}_{\{a,b\}} \equiv \int \mathrm{d}\tau_1 \mathrm{d}\tau_2 (-\tau_1)^{\nu_0} e^{ik_{12}\tau_1} (-\tau_2)^{\nu_0} e^{ik_{34}\tau_2} h(\nu_1, a, -k_s\tau_1) \theta^{(1,1)}_{1,2} h(\nu_1, b, -k_s\tau_2) \tag{4.1}$$

and the remaining term

$$\mathrm{I}_R = -\frac{4i}{\pi} e^{\pi \mathrm{Im}\nu_1} (k_s)^{-2\nu_1-1} \int \mathrm{d}\tau (-\tau)^{2\nu_0-2\nu_1} e^{i(k_{12}+k_{34})\tau} \,. \tag{4.2}$$

Here we have used the notation $k_{ij} = k_i + k_j$. In the following discussion, we will use $\boldsymbol{f} = \{f_i\}$ to denote the master integrals with $\mathrm{I}_1 = f_{\{0,0\}}$, $\mathrm{I}_2 = \mathrm{I}_{\{0,1\}}$, $\mathrm{I}_3 = \mathrm{I}_{\{1,0\}}$, $\mathrm{I}_4 = \mathrm{I}_{\{1,1\}}$, and $\mathrm{I}_5 = \mathrm{I}_R$. Notice that arbitrary cases of scalar or time derivative interaction are automatically included in the IBP system [52].

4.1 Differential equations

The differential equations of the time-order 2-vertex integral family can be written as follows

$$\mathrm{d}\mathbf{I} = \mathrm{d}\Omega \mathbf{I} \tag{4.3}$$

with $5{\times}5$ $\mathrm{d}\log$ matrix

$$\Omega = \begin{pmatrix} \mathbf{A} & \mathbf{R} \\ \mathbf{0} & \mathbf{C} \end{pmatrix}. \tag{4.4}$$

and

$$\begin{aligned}
\mathbf{A} &= \Omega_1(k_{12}, k_s) \otimes \mathbf{1}_{2\times 2} + \mathbf{1}_{2\times 2} \otimes \Omega_1(k_{34}, k_s), \\
\mathbf{C} &= (-2\nu_0 + 2\nu_1 - 1)\log(k_{12} + k_{34}) + (-2\nu_1 - 1)\log(k_s) \\
\mathbf{R} &= \begin{pmatrix}
\frac{1}{2} i(\log(k_{12} - k_s) - \log(k_{12} + k_s) + \log(k_{34} - k_s) - \log(k_{34} + k_s)) \\
\frac{1}{2}(\log(k_{12} - k_s) + \log(k_{12} + k_s) - \log(k_{34} - k_s) - \log(k_{34} + k_s)) \\
\frac{1}{2}(-\log(k_{12} - k_s) - \log(k_{12} + k_s) + \log(k_{34} - k_s) + \log(k_{34} + k_s)) \\
\frac{1}{2} i(\log(k_{12} - k_s) - \log(k_{12} + k_s) + \log(k_{34} - k_s) - \log(k_{34} + k_s))
\end{pmatrix}
\end{aligned}$$

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where $\Omega_1(k_0, k_1)$ equals the matrix Ω defined in (3.3). Following the last section, we find that taking one of k_i to ∞ of each vertex integral is likely to be a good choice for easily determining boundary conditions. However, note that there is a factor $1/(k_{12} + k_{34})$ in the differential equation. This factor leads to that $(k_{12}, k_{34}) = (\infty, \infty)$ a degenerate multivariate pole. To see that, consider $t_1 = 1/k_{12}, t_2 = 1/k_{34}$, then there are denominators t_1, t_2 and $t_1 + t_2$ in the differential equations with respect to t_1 and t_2. The denominators equal to zero provide three hyper-surfaces across $(0, 0)$ in this two-variable problem of t_1 and t_2. Thus, $(0, 0)$ is a degenerate pole of (t_1, t_2). By other words, (∞, ∞) is a degenerate pole of (k_{12}, k_{34}). Hence, we need to apply blow-up to them when using the power series method around $(k_{12}, k_{34}) = (\infty, \infty)$. Otherwise, the expansion will be illness and depend on expanding which variables first. As an alternative choice, we choose the transformation including blow-up to be $x = k_{34}/k_{12}$ and $y = 1/k_{34}$. By this choice, $(x, y) = (0, 0)$ is an non-degenerate pole. Now assuming all master integrals have a general form

$$x^\lambda y^\mu \sum_{j,k=0}^{\infty} \mathrm{C}(i, j, k) x^j y^k$$

with lowest weights λ, μ for x and y, there are five non-trivial solution sets from the equation for $\mathrm{C}(i,0,0)$:

$$\begin{aligned}
&\{\mathrm{C}(i \neq 1, 0, 0) = 0, \lambda = \nu_0 + 1, \mu = 2\nu_0 + 2\},\\
&\{\mathrm{C}(i \neq 2, 0, 0) = 0, \lambda = \nu_0 + 1, \mu = 1 + 2\nu_0 - 2\nu_1\},\\
&\{\mathrm{C}(i \neq 3, 0, 0) = 0, \lambda = \nu_0 - 2\nu_1, \mu = 1 + 2\nu_0 - 2\nu_1\},\\
&\{\mathrm{C}(i \neq 4, 0, 0) = 0, \lambda = \nu_0 - 2\nu_1, \mu = 2\nu_0 - 4\nu_1\},\\
&\Big\{\mathrm{C}(2, 0, 0) = \frac{\mathrm{C}(5, 0, 0)}{2\nu_1 - \nu_0}, \mathrm{C}(3, 0, 0) = \frac{\mathrm{C}(5, 0, 0)}{\nu_0 + 1},\\
&\mathrm{C}(1, 0, 0) = \mathrm{C}(4, 0, 0) = 0, \lambda = \mu = 2\nu_0 - 2\nu_1 + 1\Big\}
\end{aligned} \tag{4.5}$$

where the $\mathrm{C}(i,0,0)$s can be determined by boundary conditions. To avoid ambiguity, we will denote the non-zero coefficient in the i-th solution by $\mathrm{C}^{[i]}$ in the following discussion such that

$$\mathrm{I} = \sum_{i=1}^{5} \mathrm{C}^{[i]} \boldsymbol{f}^{[i]} \tag{4.6}$$

The first four general solutions are just direct products of solutions of the 1-fold general solutions. This is due to the structure of $\mathbf{A}$ in (4.1) as we will present more discussion in section 4.3. To avoid confusion with symbols, let's re-denote the results of 1-vertex by

$$V_j^{[i]}(x) = f_j^{[i]}(1/x, k_s)\,, \tag{4.7}$$

where the $f_j^{[i]}$ are the ones given in (3.13). We can write the first 4 solutions in a compact form:

$$f_{\tilde{\boldsymbol{b}}}^{[\tilde{a}]} = V_{\tilde{b}_1}^{[\tilde{a}_1]}(xy) V_{\tilde{b}_2}^{[\tilde{a}_2]}(y)\,, \quad f_5^{[\tilde{a}]} = \mathbf{0}\,, \tag{4.8}$$

with $\boldsymbol{a} = a_1, a_2$, $\boldsymbol{b} = b_1, b_2$, $a_i = 0, 1$, $b_i = 0, 1$. For example, solution $\boldsymbol{f}^{[1]}$ is

$$\begin{aligned}
\boldsymbol{f}^{[\{0,0\}]} &= \begin{pmatrix} \begin{pmatrix} V_{\tilde{0}}^{[\tilde{0}]}(xy) \\ V_{\tilde{1}}^{[\tilde{0}]}(xy) \end{pmatrix} \otimes \begin{pmatrix} V_{\tilde{0}}^{[\tilde{0}]}(y) \\ V_{\tilde{1}}^{[\tilde{0}]}(y) \end{pmatrix} \\ 0 \end{pmatrix} = \begin{pmatrix} V_{\tilde{0}}^{[\tilde{0}]}(xy) \ V_{\tilde{0}}^{[\tilde{0}]}(y) \\ V_{\tilde{0}}^{[\tilde{0}]}(xy) \ V_{\tilde{1}}^{[\tilde{0}]}(y) \\ V_{\tilde{1}}^{[\tilde{0}]}(xy) \ V_{\tilde{0}}^{[\tilde{0}]}(y) \\ V_{\tilde{1}}^{[\tilde{0}]}(xy) \ V_{\tilde{1}}^{[\tilde{0}]}(y) \\ 0 \end{pmatrix} \\
&= x^{\nu_0+1} y^{2\nu_0+2} \begin{pmatrix} {}_2\mathrm{F}_1\left(\frac{\nu_0+1}{2}, \frac{\nu_0+2}{2}; \nu_1+1; k_s^2 y^2\right) {}_2\mathrm{F}_1\left(\frac{\nu_0+1}{2}, \frac{\nu_0+2}{2}; \nu_1+1; k_s^2 x^2 y^2\right) \\ \frac{ik_s(\nu_0+1)y \, {}_2\mathrm{F}_1\left(\frac{\nu_0+2}{2}, \frac{\nu_0+3}{2}; \nu_1+2; k_s^2 y^2\right) {}_2\mathrm{F}_1\left(\frac{\nu_0+1}{2}, \frac{\nu_0+2}{2}; \nu_1+1; k_s^2 x^2 y^2\right)}{2(\nu_1+1)} \\ \frac{ik_s(\nu_0+1)xy \, {}_2\mathrm{F}_1\left(\frac{\nu_0+1}{2}, \frac{\nu_0+2}{2}; \nu_1+1; k_s^2 y^2\right) {}_2\mathrm{F}_1\left(\frac{\nu_0+2}{2}, \frac{\nu_0+3}{2}; \nu_1+2; k_s^2 x^2 y^2\right)}{2(\nu_1+1)} \\ \frac{-k_s^2(\nu_0+1)^2 xy^2 \, {}_2\mathrm{F}_1\left(\frac{\nu_0+2}{2}, \frac{\nu_0+3}{2}; \nu_1+2; k_s^2 y^2\right) {}_2\mathrm{F}_1\left(\frac{\nu_0+2}{2}, \frac{\nu_0+3}{2}; \nu_1+2; k_s^2 x^2 y^2\right)}{4(\nu_1+1)^2} \\ 0 \end{pmatrix}
\end{aligned} \tag{4.9}$$

The first 4 solutions have no contribution from the remaining integral f_5. Thus, they are homogeneous parts of solutions. The non-homogeneous solution related to non-zero f_5 also can be solved via power series expansion straightforwardly. It is given as follows:

$$\begin{aligned}
\boldsymbol{f}^{[5]} &= x^{2\nu_0-2\nu_1+1} y^{2\nu_0-2\nu_1+1} \\
&\times \begin{pmatrix} \sum_{m,n=0}^{\infty} \frac{-(-x)^m \left(\frac{1}{4}k_s^2 xy\right)^{n+1} (2\nu_0-2\nu_1+1)_{m+2n+1} (4ix^n y^n)}{4m! k_s \left(\frac{\nu_0+1}{2}+\frac{m}{2}\right)_{n+1} \left(\frac{1}{2}(m+\nu_0-2\nu_1+1)\right)_{n+1}} \\ \sum_{m,n=0}^{\infty} \frac{-(-x)^m \left(\frac{1}{4}k_s^2 x^2 y^2\right)^n (2\nu_0-2\nu_1+1)_{m+2n}}{2m! \left(\frac{\nu_0+2}{2}+\frac{m}{2}\right)_n \left(\frac{1}{2}(m+\nu_0-2\nu_1)\right)_{n+1}} \\ \sum_{m,n=0}^{\infty} \frac{(-x)^m (2\nu_0-2\nu_1+1)_{m+2n} \left(\frac{1}{4}k_s^2 x^2 y^2\right)^n}{2m! \left(\frac{\nu_0+1}{2}+\frac{m}{2}\right)_{n+1} \left(\frac{1}{2}(m+\nu_0-2\nu_1+1)\right)_n} \\ \sum_{m,n=0}^{\infty} \frac{-(-x)^m \left(\frac{1}{4}k_s^2 xy\right)^{n+1} (2\nu_0-2\nu_1+1)_{m+2n+1} (4ix^n y^n)}{4m! k_s \left(\frac{\nu_0+2}{2}+\frac{m}{2}\right)_{n+1} \left(\frac{1}{2}(m+\nu_0-2\nu_1)\right)_{n+1}} \\ (1+x)^{-1-2\nu_0+2\nu_1} \end{pmatrix} \\
&= x^{2\nu_0-2\nu_1+1} y^{2\nu_0-2\nu_1+1} \\
&\times \begin{pmatrix} \sum_{m=0}^{\infty} \frac{-ik_s xy(-x)^m (2\nu_0-2\nu_1+1)_{m+1}}{m!(\nu_0+m+1)(\nu_0-2\nu_1+m+1)} {}_3\mathrm{F}_2\left[\left.\begin{matrix} \nu_0-\nu_1+\frac{m+2}{2}, \nu_0-\nu_1+\frac{m+3}{2}, 1 \\ \frac{\nu_0+m+3}{2}, \frac{\nu_0-2\nu_1+m+3}{2} \end{matrix}\right| k_s^2 x^2 y^2\right] \\ \sum_{m=0}^{\infty} \frac{-(-x)^m (2\nu_0-2\nu_1+1)_m}{m!(\nu_0-2\nu_1+m)} {}_3\mathrm{F}_2\left[\left.\begin{matrix} \nu_0-\nu_1+\frac{m+1}{2}, \nu_0-\nu_1+\frac{m+2}{2}, 1 \\ \frac{\nu_0+m+2}{2}, \frac{\nu_0-2\nu_1+m+2}{2} \end{matrix}\right| k_s^2 x^2 y^2\right] \\ \sum_{m=0}^{\infty} \frac{(-x)^m (2\nu_0-2\nu_1+1)_m}{m!(\nu_0+m+1)} {}_3\mathrm{F}_2\left[\left.\begin{matrix} \nu_0-\nu_1+\frac{m+1}{2}, \nu_0-\nu_1+\frac{m+2}{2}, 1 \\ \frac{\nu_0+m+3}{2}, \frac{\nu_0-2\nu_1+m+1}{2} \end{matrix}\right| k_s^2 x^2 y^2\right] \\ \sum_{m=0}^{\infty} \frac{-ik_s xy(-x)^m (2\nu_0-2\nu_1+1)_{m+1}}{m!(\nu_0+m+2)(\nu_0-2\nu_1+m)} {}_3\mathrm{F}_2\left[\left.\begin{matrix} \nu_0-\nu_1+\frac{m+2}{2}, \nu_0-\nu_1+\frac{m+3}{2}, 1 \\ \frac{\nu_0+m+4}{2}, \frac{\nu_0-2\nu_1+m+2}{2} \end{matrix}\right| k_s^2 x^2 y^2\right] \\ (1+x)^{-1-2\nu_0+2\nu_1} \end{pmatrix}
\end{aligned} \tag{4.10}$$

In the next subsection, we will show how to obtain the relative coefficients $\mathrm{C}^{[i]}$.

4.2 Boundary conditions

When $x, y \to 0$, only the lowest power terms dominate. One can expand the Hankel function around $t = 0$ under this limit since the integrand will be localized at that point due to the exponential terms, and then integrate it to determine the relative coefficients C(i,0,0) of

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these solution sets. We can start with the remaining term first since it can be integrated analytically. We will get

$$\mathrm{C}^{[5]} = -\frac{4ie^{\pi \mathrm{Im}[\nu_1]}k_s^{-2\nu_1-1}e^{i\pi(\nu_1-\nu_0)}\Gamma(2\nu_0-2\nu_1+1)}{\pi}. \tag{4.11}$$

Notice that in solution $\boldsymbol{f}^{[5]}$, $f_2^{[5]}$ and $f_3^{[5]}$ are also non-zero at the leading order. An alternative way to determine $\mathrm{C}^{[5]}$ is expanding $\theta(\tau_i-\tau_j)$ in I_2 or I_3 as $0-\tau_j\delta(\tau_i)$ or $1+\tau_i\delta(\tau_j)$ for the selected blow-up and computing the contribution of the δ function part.

For the other 4 coefficients, we need to expand the Hankel functions in integrands of $\mathbf{I}$ to the leading order and then integrate. For example, the master integral I_1 has the following expression after Wick rotation:

$$\mathrm{I}_1 = -\int \mathrm{d}\tau_1 \mathrm{d}\tau_2 (i\tau_1)^{\nu_0} e^{\tau_1/(xy)} (i\tau_2)^{\nu_0} e^{\tau_2/y} h(\nu_1, 0, ik_s\tau_1)\theta_{1,2}^{(1,2)} h(\nu_1, 0, ik_s\tau_2) \tag{4.12}$$

Note that after taking $x, y \to 0$ we must have $-1 \ll \tau_2 < \tau_1 < 0$ due to the blow-up process. The theta function at the leading order will consequently be taken to 0 or 1. Using the expansion of the Hankel function (3.16), for real ν_1 we have

$$\begin{aligned}
\mathrm{C}^{[1]} &= -e^{-i\pi\nu_0}\Gamma(\nu_0+1)^2 \mathrm{C}_1^{*(k_0)}(\nu_1)\mathrm{C}_1^{(k_0)}(\nu_1) \\
\mathrm{C}^{[2]} &= -ie^{-i\pi(\nu_0-\nu_1)}k_s^{-2\nu_1-1}\Gamma(\nu_0+1)\Gamma(\nu_0-2\nu_1)\mathrm{C}_1^{*(k_0)}(\nu_1)\mathrm{C}_2^{(k_0)}(\nu_1) \\
\mathrm{C}^{[3]} &= -ie^{-i\pi(\nu_0-\nu_1)}k_s^{-2\nu_1-1}\Gamma(\nu_0+1)\Gamma(\nu_0-2\nu_1)\mathrm{C}_2^{*(k_0)}(\nu_1)\mathrm{C}_1^{(k_0)}(\nu_1) \\
\mathrm{C}^{[4]} &= \left(k_s^2\right)^{-2\nu_1-1} e^{-i\pi(\nu_0-2\nu_1)}\Gamma(\nu_0-2\nu_1)^2 \mathrm{C}_2^{*(k_0)}(\nu_1)\mathrm{C}_2^{(k_0)}(\nu_1).
\end{aligned} \tag{4.13}$$

Here, we use the $\mathrm{C}_{\tilde{a}}^{(k_0)}$ in (3.52), and the $\mathrm{C}_{\tilde{a}}^{*(k_0)}(\nu_1)$ here is just the $\mathrm{C}_{\tilde{a}}^{(k_0)}$ in (3.54). Thus, one can find that these boundary coefficients are exactly the product of the corresponding 1-vertex ones.

4.3 Factorization of homogeneous solutions

For this first-order linear differential equation system with 5 master integrals, there must be 5 arbitrary constants (coefficients) that need to be fixed by boundary conditions. This means that there are always 5 independent solution sets. One can set $f_5 = f_R = 0$ and then the differential system will become 4 master integrals that satisfy the $\mathrm{d}\log$ form differential equations $\mathbf{A}$ in (4.1). By factorization of IBP and differential equations [52], $\mathbf{A}$ consists of the 1-vertex 1-fold $\mathrm{d}\log$ form differential equations. Obviously, they can be written as the product of the two general solutions of 1-vertex 1-fold, as we have shown in (4.9). These solutions are also called "homogeneous parts". In addition, the left one with $f_5 \neq 0$ will correspond to the "non-homogeneous part".

The coefficients for the homogeneous part can also be written as the product of coefficients of 1-vertex 1-fold solutions. When we consider the boundary conditions, by choosing blow-up, the theta function will be expanded as 0 or 1 at the leading order. For example, since there are $e^{ik_{12}\tau_1}$ and $e^{ik_{34}\tau_2}$ in the integrand, the limitation $k_{12} \gg 1$ and $k_{34} \gg 1$ together with Wick rotation lead to that only the region $|\tau_1| \ll 1$ and $|\tau_2| \ll 1$ could contribute. The blow-up further choose that $k_{12} \gg k_{34}$, thus only the region $|\tau_1| \ll |\tau_2| \ll 1$ could contribute. Obviously, $\theta(\tau_1-\tau_2)=1$ in the region contribute. This will cause the integrals involving two times to be factorized and exactly equal to the product of 1-vertex 1-fold integrals at leading order:

$$\begin{aligned}&\int \mathrm{d}\tau_1\mathrm{d}\tau_2(-\tau_1)^{\nu_0}e^{ik_{12}\tau_1}(-\tau_2)^{\nu_0}e^{ik_{34}\tau_2}h(\nu_1,a,-k_s\tau_1)\theta^{(1,2)}_{1,2}h(\nu_1,b,-k_s\tau_2)\\ \longrightarrow&\left[\int \mathrm{d}\tau_1(-\tau_1)^{\nu_0}e^{ik_{12}\tau_1}h^{(1)}(\nu_1,a,-k_s\tau_1)\right]\left[\int \mathrm{d}\tau_2(-\tau_2)^{\nu_0}e^{ik_{34}\tau_2}h^{(2)}(\nu_1,b,-k_s\tau_2)\right].\end{aligned}\tag{4.14}$$

The coefficients determined by them will consequently be the product 1-vertex 1-fold boundary coefficients.

Furthermore, this argument can also be generalized to general tree-level cases. One can set all or a part of sub-sectors to be zero first. Then one can choose a blow-up transformation. This leads to step functions becoming one or zero correspondingly. As a result, boundary coefficients determined by leading order expansion are factorized. These factorized general solutions together with factorized boundary coefficients give the factorized particular solutions that are exactly the product of several particular solutions of the vertices which come from dividing the diagram by cutting several propagators. The dividing is not necessary to be cutting all the propagators and it is not necessary to be 1-vertex at each part. This is the factorization property of solutions of tree-level cosmological correlators. If one sets all integrals in sub-sectors to be zero, it corresponds to cutting all the propagators in the dividing. Then, the remaining solutions are homogeneous solutions at the top-sector level. They are the product of particular solutions of the vertex integral family we have given in 3.3.

5 Summary and outlook

In this work, we use power series expansion to solve the d log-form differential equations of cosmological correlators. It gives multivariate hypergeometric solutions. We also analyze the properties of integrands of the cosmological correlator and find the boundary conditions easy to solve for arbitrary vertex integral families. The solutions are given in 3.3. Analytic continuation of the series expansion solution is also discussed. We show the recommended numerical analytic continuation by differential equations is efficient, straightforward, and convenient. We indicate that blow-up transformation could be applied to solve differential equations by the power series expansion around degenerate poles. With this technique, we also find the boundary conditions easy to determine for 2-vertex correlators, which is likely to work for arbitrary n-vertex correlators cases. Then, we give the particular solutions to this 2-vertex example. By this example, we also discuss the factorization property of homogeneous part solutions of n-vertex correlators, with which, one can easily give them as the product of the solutions of vertex integral family.

We want to remind the readers, that our results not only elucidate the mathematical structure of cosmological correlator, benefit their evaluation and related applications in cosmological phenomenology, but also offer new insights into computational techniques for evaluating integrals of perturbative QFT including flat cases, mainly in two aspects. Firstly, we indicate that using blow-up one could easily handle the degenerate pole of differential equations in multivariate limitation. Secondly, our results show a potential way toward analytic evaluation of field theory integrals beyond MPL: d log-form differential equations. Let us call some background of analytic evaluation perturbative QFT here. [74] carry out the canonical differential equations method, the most powerful analytic method currently. It is the IBP-based differential equation that takes a special form called canonical differential equations. which means the differential equations is proportional to dimension regulator ε, or says ε-form for short, and also a d log-form at the same time. This method works for those integrals that are multiple polylogarithms (MPL) in the expansion of ε. Even though, its scope includes all one-loop Feynman integrals and a large part of common multi-loop integrals in flat QFT. However, the developing phenomenology of particle physics still calls for analytic methods that can go beyond MPL. The most frequently considered method of generalizing canonical differential equations to these cases currently is keeping the differential equations in ε-form rather than d log-form (see [75–78] and their references for examples and discussions). In this case, the elliptic symbol or beyond appears in the elements of differential equations. The emerging new functions and series solutions are studied. Unlike d log-form, coefficients in differential equations and master integrals are non-algebraic for these cases, thus they are more complicated. Finding such ε-form usually is also more difficult than d log-form cases. However, our paper results imply that keeping d log-form rather than ε-form is another way worth to be considered. We find it easy to get all order series solutions to d log-form differential equations in our example. This is partially due to the expansion of d log-forms are simple, avoiding non-algebraic expressions. It is simple, especially for the case that the function in log is rational, as shown in (3.7), since in such cases the log can always be regarded as $\mathrm{d}\log(z-c)$ for a selected parameter z.

Our work leads to many topics that could be explored in the future. We merely list a part of them. Firstly, one can apply our methods to the correlators important to phenomenology that have not been evaluated. Even for the loop level, the framework of IBP already has been discussed in [52]. Although d log-form differential equations may not be easy to get in this case, we want to remind people, the (generalized) power series expansion method does not rely on d log-form differential equations. The efficiency of this method for differential equations in general (usually rational) form has been confirmed in [60] at the beginning, and a subsequent series of works following it have also validated this. Our section 3.1.4 confirm this as well. Automatic tools of this method, which have been widely applied in flat QFT, could also help, for example, DiffEXP [71].

Secondly, although in this work, we show all order expressions of multivariate hypergeometric solutions, the attainment of this result is partially dependent on the special structure of the dlog differential equations we have computed. For general d log-form differential equations, could we find a formula to determine its all-order power series solution or express it as multivariate hypergeometric functions like in this paper? Alternatively, one can also explore that is there exists an algorithm, that is more efficient than the naive (generalized) power series expansion, to evaluate master integrals in the special case of d log-form differential equations. Then, since one can quickly get numerical results at any point of phase space and analyze asymptotic behavior around the arbitrary singularity, this algorithm together with d log-form matrix defines a series of new "analytic functions".

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Notes on selection rules of canonical differential equations and relative cohomology

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Abstract: We give an explanation of the $\mathrm{d}\log$-form of the coefficient matrix of canonical differential equations using the projection of $(n+1)$-$\mathrm{d}\log$ forms onto n-$\mathrm{d}\log$ forms. This projection is done using the leading-order formula for intersection numbers. This formula gives a simple way to compute the coefficient matrix. When combined with the relative twisted cohomology, redundancy in computation using the regulator method can be avoided.

Keywords: Scattering Amplitudes, Higher-Order Perturbative Calculations, Differential and Algebraic Geometry

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Contents

1 Introduction

For the study of theories and phenomenology in high energy physics, perturbative quantum field theory plays a crucial role. One of its central tasks is to compute Feynman integrals. For Feynman integrals, linear relationships can be established through Integration-By-Parts (IBP) [1], thereby expressing integrals within the same function family as linear combinations of a chosen finite set of integrals. These selected integrals are referred to as master integrals, and this process is known as the IBP reduction. Subsequently, the computation in perturbative field theory is transformed into completing the reduction and calculating the master integrals. Based on IBP, one can take partial derivatives of master integrals and then use IBP reduction to express the resulting integrals in terms of master integrals. This process leads to first-order differential equations satisfied by the master integrals [2–5]. To simplify the computations, the **canonical differential equations (CDE)** method [6] are developed. It says that for many cases, when the master integrals are appropriately chosen, the differential equations will be transformed into $\mathrm{d}\log$-form proportional to ϵ. In such a case, we call it a canonical form of the differential equations system. i.e.,

$$\mathrm{d}f_I = (\mathrm{d}\Omega)_{IK} f_K\,, \quad (\mathrm{d}\Omega)_{IK} = \epsilon \sum_i \mathrm{C}^{(i)}_{IK}\, \mathrm{d}\log \mathrm{W}^{(i)}(\boldsymbol{s})\,. \tag{1.1}$$

Then each order of ϵ of the master integrals can be iteratively resolved as iterative integration of a series of $\mathrm{d}\log \mathrm{W}^{(i)}(\boldsymbol{s})$ integrals, which leads to multi-polylogarithm [7, 8]. The $\mathrm{W}^{(i)}(\boldsymbol{s})$'s are called symbol letters and they contain the information on the analytic structure of Feynman integrals. The complete set of letters is defined as a symbol alphabet. The symbol has been studied in various researches [9–27] and could be used for bootstrap [13, 19, 28–50]. Information of symbol is encoded in the coefficient matrix $(\mathrm{d}\Omega)_{IK}$ of the CDE.

In the past decade, the CDE method has been the most crucial technique for analytically computing Feynman integrals. People have developed many methods to "appropriately choose the master integral" to get the CDE, such as d log-form and the closely related leading singularity analysis [6, 51–56], which are also inspired by previous work such as [57, 58]. There are also some automatic packages with other algorithms such as [59–64]. People found that if one can construct d log-from integrands as the integrands of master integrals, the differential equations of this system will become canonical form automatically in practice. Baikov representation [65] shows advantage in such construction [51–53]. However, why and how such a structure could emerge from d log-form integrand of master integral, is not fully understood. In the last several years, a mathematical tool called "intersection theory" was introduced to Feynman integral and developed in [51, 52, 66–86]. It could be used as a reduction method equivalent to the IBP method. Recently, the polynomial division method is introduced [84] and applied [85] to avoid the algebraic extension in the computation of intersection numbers. Based on these methods, people [86] successfully applied intersection theory, together with companion tensor algebra, to reduce the 2-loop 5-point Feynman integral family. In one of the previous works [83], how the CDE emerges from d log-from integrand has been partially understood by using intersection theory, especially the method of computing intersection numbers by higher-order partial differential equations [70]. Furthermore, the selection rules of the coefficient matrix of the CDE could be given, including which element of the matrix is non-zero and which letter could appear. For these computations, all people need are two universal formulas, i.e., formulas for the leading-order (LO) contribution and next-to-leading-order (NLO) contribution of the intersection number.

In this paper, we are going to improve the selection rules of the coefficient matrix of the CDE, mainly in two aspects. Firstly one could notice that the differential action on n-d log-form basis could be easily transformed to a $(n+1)$-d log-form. Such rewriting has been mentioned in some previous works such as [81]. This rewriting is very useful since the usage of the formula of NLO contribution of intersection number could be avoided when computing the coefficient matrix of the CDE. Furthermore, it is easy to see that the coefficient matrix is nothing but the d log-form coefficient when projecting $(n+1)$-d log-form to n-d log-form. Secondly, in the previous work [83], factors with integer powers such as propagators are handled by adding regulators that ultimately need to be taken to zero. This makes both the computational process as well as the selection rules more complicated. In contrast, relative twisted cohomology of the dual form [78–81] in intersection theory could be a natural language to deal with these cases, allowing to avoid regulator from the beginning, and thus obtaining the simpler selection rules.

The organization of this paper is as follows. In section 2, we recall intersection theory with regulator and the previous version of the selection rules of the CDE, but from a new perspective, i.e., the d log projecting and with only LO formula, instead of LO and NLO in [83]. An important technical point is given in the subsection 2.4, where we have carefully discussed the factorization of poles, including the understanding of relations between them and integration contours and regions. Then, with the preparation of section 2 we can smoothly go to the computation of intersection number with dual form in relative cohomology in section 3. We will introduce this mathematical tool in an easier practice way. Then the improved selection rules of the CDE are presented. In section 4 and 5, we show two examples and compare the computing processes of two methods, i.e., with regulator and using relative cohomology. From the comparison, one can see the simplification of the latter. Finally, a summary is given in section 6.

2 $\hat{\mathrm{d}}\log$-form differential equations from projecting $\mathrm{D}\log$ to $\mathrm{d}\log$

2.1 Intersection theory

Feynman integrals in Baikov representation are in the form

$$I[u,\varphi] \equiv \int u\,\varphi\,, \tag{2.1}$$

where

$$\begin{aligned} \varphi &\equiv \hat{\varphi}(\boldsymbol{z}) \bigwedge_j \mathrm{d}z_j = \frac{Q(\boldsymbol{z})}{\left(\prod_k D_k^{a_k}\right)\left(\prod_i P_i^{b_i}\right)} \bigwedge_j \mathrm{d}z_j\,, \\ u &= \prod_i \left[P_i(\boldsymbol{z})\right]^{\beta_i}\,, \qquad a_k, b_j \in \mathbb{N}\,. \end{aligned} \tag{2.2}$$

The propagators are denoted by $\boldsymbol{z} = (z_1, \ldots, z_n)$. The polynomials/monomials $D_k(\boldsymbol{z})$ denotes the denominators with integer power a_k. They usually are the propagators. The $P_i(\boldsymbol{z})$ denotes the denominators with complex powers $\beta_i - b_i$ in the complete integrand $u\varphi$. They typically are Gram determinants $\mathrm{G}(\boldsymbol{q}) \equiv \det(q_i \cdot q_j)$ of loop and external momenta. The numerator $Q(\boldsymbol{z})$ is an arbitrary polynomial of $\boldsymbol{z}$. In this section, we are supposed to use a regulator to deal with integer power denominators. Then

$$u = \prod_i \left[P_i(\boldsymbol{z})\right]^{\beta_i} \prod_k D_k^{\delta_k} \tag{2.3}$$

where regulators δ_k are kept in the computation, which will be taken to be zero in the final result. In the next section, we will discuss how to avoid regulators from the beginning of the computation.

Despite traditional IBP reduction, one can also reduce integrals in such an integral family to master integrals via intersection theory. To do so, one needs to define IBP-equivalence classes of cocycles [66–68, 70] first:

$$\langle \varphi_L | \equiv \varphi_L \sim \varphi_L + \sum_i \nabla_i \xi_i\,, \quad \nabla_i = dz_i \wedge (\partial_{z_i} + \hat{\omega}_i)\,, \quad \hat{\omega}_i \equiv \partial_{z_i} \log(u). \tag{2.4}$$

The dual form in dual space is important in this paper and we will discuss it more carefully later. At this moment, let us roughly regard the dual space as being consisting of equivalence classes $|\varphi\rangle$ (φ is a part of integrand) which corresponds to $I[u^{-1},\varphi]$. Then, the intersection number is given by

$$\langle\varphi_L|\varphi_R\rangle = \sum_{\boldsymbol{p}} \mathrm{Res}_{\boldsymbol{z}=\boldsymbol{p}}\left(\psi_L\varphi_R\right) , \quad \nabla_1\cdots\nabla_n\psi_L = \varphi_L , \tag{2.5}$$

where $\boldsymbol{p}$ are isolated intersection points of n hypersurfaces belonging to $\mathcal{B} = \{P_1 = 0,\infty,\cdots, D_1 = 0,\infty,\cdots\}$ ($P_1 = 0,\infty$ means $P_1 = 0, P_1 = \infty$).

With the algorithm of intersection number as the IBP-invariant inner product, the integral reduction becomes entirely a projection in vector space. For example, to reduce $f_0 = \int u\varphi_0$ to master integrals $f_I = \int u\varphi_I$

$$f_0 = \sum_{I=1}^{n} c_I f_I \tag{2.6}$$

c_I can be calculated via

$$c_I = \sum_J \langle\varphi_0|\varphi'_J\rangle\left(\eta^{-1}\right)_{JI} , \quad \eta_{IJ} \equiv \langle\varphi_I|\varphi'_J\rangle . \tag{2.7}$$

(Dual basis φ'_I are not necessarily equal to φ_I.)

2.2 d log projection of the CDE

For the CDE considered in this paper, β_i in (2.2) are proportional to ϵ, and integrands φ_I of master integrals f_I are n-d log-forms

$$\varphi_I = \bigwedge_j \mathrm{d}\log \mathrm{W}_j^{(I)}(\boldsymbol{z}), \tag{2.8}$$

which typically has two types of building blocks:

$$\begin{aligned}
\mathrm{d}\log(z-c) &= \frac{dz}{z-c} ,\\
\mathrm{d}\log(\tau[z,c;c_\pm]) &= \frac{\sqrt{(c-c_+)(c-c_-)}dz}{(z-c)\sqrt{(z-c_+)(z-c_-)}} ,\\
\tau[z,c;c_\pm] &\equiv \frac{\sqrt{c-c_+}\sqrt{z-c_-}+\sqrt{c-c_-}\sqrt{z-c_+}}{\sqrt{c-c_+}\sqrt{z-c_-}-\sqrt{c-c_-}\sqrt{z-c_+}} ,
\end{aligned} \tag{2.9}$$

where we denote the first type as “rational-type” and the second as “sqrt-type”. For convenience and distinction, we denote the differentiation over integration variables $\boldsymbol{z}$ as d, the differentiation over arbitrarily selected one parameter such as a kinetic parameter or mass as $\hat{\mathrm{d}}$, and

$$\mathrm{D} = \mathrm{d} + \hat{\mathrm{d}} \tag{2.10}$$

Differential equations are given by the IBP reduction of $\partial_s \boldsymbol{f}$ (s could be any selected parameter whose corresponding total derivative is denoted as $\hat{\mathrm{d}}$ as we mentioned)

$$\hat{\mathrm{d}}\boldsymbol{f} = \Omega_s.\boldsymbol{f}\ \hat{\mathrm{d}}s. \tag{2.11}$$

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Carrying out the computation we have

$$\begin{aligned}\hat{\mathrm{d}} f_I &= \int \hat{\mathrm{d}}\left(u \bigwedge_j \mathrm{d}\log \mathrm{W}_j^{(I)}(\boldsymbol{z})\right) \\ &= \int u\ \hat{\mathrm{d}}\log u \bigwedge_j \mathrm{d}\log \mathrm{W}_j^{(I)} + \int u \sum_k \left[(-1)^k \left(\mathrm{d}\wedge \hat{\mathrm{d}}\log \mathrm{W}_k^{(I)}\right) \bigwedge_{j\neq k} \mathrm{d}\log \mathrm{W}_j^{(I)}\right] \\ &= \int u\ \hat{\mathrm{d}}\log u \bigwedge_j \mathrm{d}\log \mathrm{W}_j^{(I)} + \int u\ \mathrm{d}\log u \wedge \sum_k \left[(-1)^{k+1} \left(\hat{\mathrm{d}}\log \mathrm{W}_k^{(I)}\right) \bigwedge_{j\neq k} \mathrm{d}\log \mathrm{W}_j^{(I)}\right] \\ &= \int u\ \mathrm{D}\log u \bigwedge_j \mathrm{D}\log \mathrm{W}_j^{(I)}\,. \end{aligned} \tag{2.12}$$

In this equation, from line 2 to line 3, we apply the IBP of the d in $\mathrm{d}\wedge\hat{\mathrm{d}}\log \mathrm{W}_k^{(I)}$ and act it to u, which gives $\mathrm{d}u = u\ \mathrm{d}\log u$. Denote $\hat{\mathrm{d}} f_I \equiv \int u \dot{\varphi}_I$, we have

$$\dot{\varphi}_I = \mathrm{D}\log u \bigwedge_j \mathrm{D}\log \mathrm{W}_j^{(I)} \tag{2.13}$$

Obviously, they are n-dlog-form with $\hat{\mathrm{d}}\log$ coefficient. Result (2.13) is important as we will show that in the calculation of the matrix $\hat{\mathrm{d}}\Omega$ (here dual basis are selected to be the same as original one)

$$\begin{aligned}\langle\dot{\varphi}_I| &= \left(\hat{\mathrm{d}}\Omega\right)_{IJ}\langle\varphi_J|\,, \\ \left(\hat{\mathrm{d}}\Omega\right)_{IK} &= \langle\dot{\varphi}_I|\varphi_J\rangle\left(\eta^{-1}\right)_{JK}\,,\end{aligned} \tag{2.14}$$

(Note that not to forget the summation over the repeated index J here.) people only need to compute the so-called leading order (LO) contribution to the intersection number. This will lead to a great simplification. In the next two subsections, details of related mathematical techniques of such computations will be presented.

2.3 LO contribution of intersection number

The computation of intersection numbers involves multivariate residues. For fractional polynomials, multivariate residues can be computed by transformation law, and global residue can be calculated via the Bezoutian method [87, 88]. Since ψ_L in the computation of intersection number usually is not polynomial, for keeping more information explicitly, we choose to compute multivariate residue (2.5) directly by solving ψ in multivariate Laurent expansion [70].

However, multivariate residue and Laurent expansion are highly non-trivial and usually cannot be computed variable-by-variable. For example, the Laurent expansion of $1/(z_1(z_1+z_2))$ depends on the order of the expansion variables. To overcome this, one can factorize the poles first, then the Laurent expansion is legal and the high order differential equation of ψ_L can be solved using the expansion. This method is developed in [70, 83] and we are going to give more discussions about it here. Factorization also transforms the n-variable residue problem into n one-variable residue problem, whose computations become trivial.

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A further technical difficulty is when the pole $\boldsymbol{p}$ is degenerate and thus also non-factorized. To deal with it, one needs to involve several regions. Each region has different factorization transformations and different residue contributions. As indicated in [70], this transformation is like the one applied in sector decomposition [89–92].

We denote all factorization transformations as $\mathrm{T}^{(\alpha)}: z_i \to f_i^{(\alpha)}(\boldsymbol{x}^{(\alpha)})$, and the corresponding pole after transformation as $\boldsymbol{\rho}^{(\alpha)} = \{\rho_1^{(\alpha)}, \rho_1^{(\alpha)}, \cdots, \rho_n^{(\alpha)}\}$. Around a factorized pole, an n-form φ can be Laurent-expanded safely as

$$\varphi = \sum_{\boldsymbol{b}} \varphi^{(\boldsymbol{b})}, \quad \varphi^{(\boldsymbol{b})} = C^{(\boldsymbol{b})} \bigwedge_i \left[x_i^{(\alpha)} - \rho_i^{(\alpha)}\right]^{b_i} \mathrm{d}x_i^{(\alpha)}, \tag{2.15}$$

where the powers $\boldsymbol{b} = (b_1, \ldots, b_n)$. The u could be written as

$$\mathrm{T}^{(\alpha)}[u] \equiv u(\mathrm{T}^{(\alpha)}[\boldsymbol{z}]) = \bar{u}_\alpha(\boldsymbol{x}^{(\alpha)}) \prod_i \left[x_i^{(\alpha)} - \rho_i^{(\alpha)}\right]^{\gamma_i^{(\alpha)}}. \tag{2.16}$$

These remaining hypersurfaces in $\bar{u}_\alpha(\boldsymbol{x}^{(\alpha)})$ will not intersect at the point $\boldsymbol{\rho}^{(\alpha)}$, so $\bar{u}_\alpha(\boldsymbol{\rho}^{(\alpha)}) \neq 0$. Thus the leading term of u around the pole is

$$u(\mathrm{T}^{(\alpha)}[\boldsymbol{z}])\big|_{\boldsymbol{x}^{(\alpha)} \to \boldsymbol{\rho}^{(\alpha)}} = \bar{u}_\alpha(\boldsymbol{\rho}^{(\alpha)}) \prod_i \left[x_i^{(\alpha)} - \rho_i^{(\alpha)}\right]^{\gamma_i^{(\alpha)}}. \tag{2.17}$$

We define $\gamma_i^{(\alpha)}$ as the **hypersurface-power** for each variable, where α corresponds to the transformation $\mathrm{T}^{(\alpha)}$.

After the above transformation, the intersection number becomes

$$\begin{aligned}\langle \varphi_L | \varphi_R \rangle &= \sum_\alpha \mathrm{Res}_{\boldsymbol{\rho}^{(\alpha)}} \mathrm{T}^{(\alpha)}[\psi_L \varphi_R] \\ &= \sum_\alpha \mathrm{Res}_{\boldsymbol{\rho}^{(\alpha)}} \left[\left(\sum_{\boldsymbol{b}_L} \nabla_1^{-1} \cdots \nabla_n^{-1} \varphi_L^{(\boldsymbol{b}_L)}\right) \sum_{\boldsymbol{b}_R} \varphi_R^{(\boldsymbol{b}_R)}\right],\end{aligned} \tag{2.18}$$

where any $g = \nabla_i^{-1} f$ means the g satisfies $\nabla_i g = f$. As has been discussed in [83], only when there exist non-zero terms $\varphi_L^{(\boldsymbol{b}_L)}$ and $\varphi_R^{(\boldsymbol{b}_R)}$ in their expansion which satisfy $b_{L,i} + b_{R,i} \leq -2$ for all i, intersection number gets non-zero contribution from such pairs. When $b_{L,i} + b_{R,i} = -2$, we say it gives a **LO contribution** to the intersection number. A special case is the intersection number of d log-form. Since the d log-form has only multivariate simple poles, all terms in its expansion (after factorization) have all $b_i \geq -1$. Hence, all non-zero contributions come from terms with $\boldsymbol{b}_L = \boldsymbol{b}_R = \boldsymbol{-1} = \{-1, -1, \cdots, -1\}$. Thus the formula of LO contribution can be easily read as

$$\begin{aligned}\mathrm{Res}_{\boldsymbol{\rho}^{(\alpha)}} \mathrm{T}^{(\alpha)}\left[\left(\nabla_1^{-1} \cdots \nabla_n^{-1} \varphi_L\right) \varphi_R\right] &= \frac{\mathrm{Res}_{\boldsymbol{\rho}^{(\alpha)}} \mathrm{T}^{(\alpha)}[\varphi_L] \times \mathrm{Res}_{\boldsymbol{\rho}^{(\alpha)}} \mathrm{T}^{(\alpha)}[\varphi_R]}{\prod_i \mathrm{Res}_{\rho_i^{(\alpha)}} \partial_{x_i^{(\alpha)}} \log\left(\mathrm{T}^{(\alpha)}[u]\right) \mathrm{d}x_i^{(\alpha)}} \\ &= \frac{C_L^{(\boldsymbol{b}_L)} C_R^{(\boldsymbol{b}_R)}}{\boldsymbol{\gamma}^{(\alpha)}}, \qquad \boldsymbol{\gamma}^{(\alpha)} = \prod_i \gamma_i^{(\alpha)}\end{aligned} \tag{2.19}$$

Let us give some explanations of the result (2.19). First the term $\partial_{x_i^{(\alpha)}} \log\left(\mathrm{T}^{(\alpha)}[u]\right) \mathrm{d}x_i^{(\alpha)}$ comes from

$$\mathrm{T}^{(\alpha)}[\nabla_i] \equiv \mathrm{d}x_i^{(\alpha)} \wedge \left(\partial_{x_i^{(\alpha)}} + \partial_{x_i^{(\alpha)}} \log\left(\mathrm{T}^{(\alpha)}[u]\right)\right) \tag{2.20}$$

which keeps the structure of the commutator

$$\left[\mathrm{T}^{(\alpha)}[\nabla_i], \mathrm{T}^{(\alpha)}[\nabla_j]\right] = [\nabla_i, \nabla_j] = 0\,. \tag{2.21}$$

Secondly after solving high-order differential equation

$$\prod_i \mathrm{T}^{(\alpha)}[\nabla_i]\mathrm{T}^{(\alpha)}[\psi_L] = \mathrm{T}^{(\alpha)}[\varphi_L] \tag{2.22}$$

in Laurent series expansion, we pick out the coefficient of the order that will contribute and get the part $\frac{\mathrm{Res}_{\rho^{(\alpha)}} \mathrm{T}^{(\alpha)}[\varphi_L]}{\prod_i \mathrm{Res}_{\rho_i^{(\alpha)}} \partial_{z_i} \log\left(\mathrm{T}^{(\alpha)}[u]\right)\mathrm{d}z_i}$

For LO contributions of other cases, they can be transformed to the case of $\boldsymbol{b}_L = \boldsymbol{b}_R = \boldsymbol{-1}$ by some rescaling transformations

$$\tilde{u} = uP^{\beta}\,,\ \tilde{\varphi}_L = \varphi_L/P^{\beta}\,,\ \tilde{\varphi}_R = \varphi_R P^{\beta}\,. \tag{2.23}$$

Obviously, $\tilde{u}\tilde{\varphi}_L = u\varphi_L$ and $\tilde{u}^{-1}\tilde{\varphi}_R = u^{-1}\varphi_R$, so this transformation do not change integrals. Then, one could apply (2.19) and get the formula for the general LO contribution of intersection number

$$\begin{aligned} &\mathrm{Res}_{\boldsymbol{\rho}^{(\alpha)}} \mathrm{T}^{(\alpha)}\left[\left(\nabla_1^{-1}\cdots\nabla_n^{-1}\varphi_L^{(\boldsymbol{b}_L)}\right)\varphi_R^{(\boldsymbol{b}_R)}\right] = \frac{C_L^{(\boldsymbol{b}_L)}C_R^{(\boldsymbol{b}_R)}}{\tilde{\boldsymbol{\gamma}}^{(\alpha)}}\,, \\ &\tilde{\boldsymbol{\gamma}} \equiv \prod_i \tilde{\gamma}_i\,, \quad \tilde{\gamma}_i^{(\alpha)} = \gamma_i^{(\alpha)} - b_{R,i} - 1\,, \quad \boldsymbol{b}_L + \boldsymbol{b}_R = \boldsymbol{-2}\,, \end{aligned} \tag{2.24}$$

where the shifting of $\tilde{\boldsymbol{\gamma}}$ comes from the factor P^{β} making $\tilde{\varphi}_R$ having $\tilde{\boldsymbol{b}}_R = \boldsymbol{-1}$.

2.4 Factorization of poles

For a degenerate pole, there are more than one independent integration cycle. We will show its properties and factorization transformations correspond to these independent cycles.

2.4.1 Factorization and contour

For multivariate residue (see [88, 93]), usually the integration circle is defined by $|P_i(\vec{z})| = r_i$ instead of $|z - z_0| = r$ for the univariate case. However, for the degenerate case, for example, the pole $(0, 0)$ coming from the intersection of three surfaces (of denominators) $\{P_1 = x_1 = 0, P_2 = x_2 = 0, P_3 = f(x_1, x_2) = 0\}$, the circle defined by taking two of three P_i's is ill-defined. To see it, let us consider the integration cycle

$$\oint_{|P_1|=|x_1|=r_1} \mathrm{d}x_1 \oint_{|P_2|=|x_2|=r_2} \mathrm{d}x_2 \times \cdots\,. \tag{2.25}$$

Obviously, $x_1 = 0$ is inside the circle of $x_1 = r_1$, and $x_2 = 0$ is inside the circle of $x_2 = r_2$. From the implicit function P_3, we can solve the variable x_1 by the variable x_2, which we denote as $\bar{f}$, i.e.,

$$x_1 = \bar{f}(x_2) = x_2^{a_1} + \mathcal{O}(x_2^{a_2}) \tag{2.26}$$

with $0 < a_1 < a_2$ for P_3 passing through the point $(0,0)$. The inverse of $\bar{f}$ is given by

$$x_2 = \bar{f}^{-1}(x_1) = x_1^{1/a_1} + \mathcal{O}(x_1^{\frac{1}{a_1}+\frac{a_2-a_1}{a_1}}) \tag{2.27}$$

If $\bar{f}(r_2) < r_1$, $P_3 = 0$ is in the circle $x_1 = r_1$, but meanwhile this implies $\bar{f}^{-1}(r_1) > r_2$, which means $P_3 = 0$ is not in the circle $x_2 = r_2$. We denote this case as $(\{P_2\}, \{P_3, P_1\})$. On the contrary, If $\bar{f}(r_2) > r_1$, which implies $\bar{f}^{-1}(r_1) < r_2$, $P_3 = 0$ is in the cycle of $x_2 = r_2$ and not in the circle of $x_1 = r_1$. We denote this case as $(\{P_1\}, \{P_2, P_3\})$. By selecting different $|P_i| = r_i$ as did in (2.25), one can also find another contour corresponds to the combination $(\{P_3\}, \{P_1, P_2\})$. The analysis tells us that multivariate residues depend not only on the location of the pole but also the shape of the cycle enclosing the pole as shown in [88, 93].

As pointed out in [70], to deal with degenerated cases one can use the method of resolution of singularities, which closely relates to the sector decomposition method. To understand the procedure, let's show a simple example. Consider $u = z_1^{\beta_1} z_2^{\beta_2} (z_1 + z_2)^{\beta_3} \prod_{i=4}^{n} (\mathrm{C}_i + \mathcal{O}(\boldsymbol{z}))^{\beta_i}$. The pole $\boldsymbol{p} = (0,0)$ is non-factorized and degenerate, since there are three hypersurfaces

$$P_1 = z_1 = 0\,, \quad P_2 = z_2 = 0\,, \quad P_3 = z_1 + z_2 = 0 \tag{2.28}$$

that meet at $(0,0)$ but it is only a two-dimension (or say 2-variable) problem.

One could find three transformations:

$$\begin{aligned} &\mathrm{T}^{(1)} : z_1 \to x_1^{(1)} x_2^{(1)}\,,\ z_2 \to x_2^{(1)}\,,\\ &\mathrm{T}^{(2)} : z_1 \to x_2^{(2)}\,,\ z_2 \to x_1^{(2)} x_2^{(2)}\,,\\ &\mathrm{T}^{(3)} : z_1 \to x_1^{(3)} x_2^{(3)} - x_2^{(3)}\,,\ z_2 \to x_2^{(3)}\,. \end{aligned} \tag{2.29}$$

They are built according to the following logic. For the n-variable problem, firstly we choose n P_i as n new variables x_i. Then we turn all the remaining factors P_j which lead to degeneration into the form $x_t^a(\mathrm{C} + \mathcal{O}(\boldsymbol{z}))$ by a series of blow-up transformations

$$x_i \to x_i x_j^b \tag{2.30}$$

with a chosen pair of (x_i, x_j). The choice of b is important for the intersection number and we will discuss it later. For example, we select P_2 as x_2, P_3 as x_1. With this choice, we have the shift transformation

$$\mathrm{t}_1 : z_1 \to x_1 - x_2\,,\ z_2 \to x_2\,, \Longrightarrow P_1 = x_1 - x_2\,,\ P_2 = x_2\,,\ P_3 = x_1\,. \tag{2.31}$$

Now we need to turn P_1 to the form $x_i^a(\mathrm{C} + \mathcal{O}(\boldsymbol{z}))$. There are two different choices. Let us factorize x_2 with the second transformation[1]

$$\mathrm{t}_2 : x_1 \to x_1^{(3)} x_2^{(3)}\,,\ x_2 \to x_2^{(3)}\,, \tag{2.32}$$

where the $b = 1$. Putting these two together, we have $\mathrm{T}^{(3)} = \mathrm{t}_2 \circ \mathrm{t}_1$.

[1]Another choice is to factorize x_1 with the second transformation $x_2 \to x_1^{(3)} x_2^{(3)}$, $x_1 \to x_1^{(3)}$, which is just $\mathrm{T}^{(3)}$ with the relabeling of $x_1 \leftrightarrow x_2$.

Since x_2 has been factorized from P_1, we combine P_1 with $P_2 = x_2$ to write this case as $(\{P_3\}, \{P_1 P_2\})$.[2] As we will show shortly, it just corresponds to the contour $(\{P_3\}, \{P_1, P_2\})$. Then, we have

$$\mathrm{T}^{(1)} : (\{P_1\}, \{P_2, P_3\}), \quad \mathrm{T}^{(2)} : (\{P_2\}, \{P_1, P_3\}), \quad \mathrm{T}^{(3)} : (\{P_3\}, \{P_1, P_2\}). \tag{2.33}$$

From it we can read the hypersurface-powers from $\mathrm{T}^{(\alpha)}[u]$:

$$\begin{aligned}
\mathrm{T}^{(1)}[u] &= \left(x_1^{(1)}\right)^{\beta_1} \left(x_2^{(1)}\right)^{\beta_1+\beta_2+\beta_3} \left(x_1^{(1)} + 1\right)^{\beta_3} \times \cdots, \qquad \gamma_1^{(1)} = \beta_1, \ \gamma_2^{(1)} = \beta_1 + \beta_2 + \beta_3, \\
\mathrm{T}^{(2)}[u] &= \left(x_1^{(2)}\right)^{\beta_2} \left(x_2^{(2)}\right)^{\beta_1+\beta_2+\beta_3} \left(x_1^{(2)} + 1\right)^{\beta_3} \times \cdots, \qquad \gamma_1^{(2)} = \beta_2, \ \gamma_2^{(2)} = \beta_1 + \beta_2 + \beta_3, \\
\mathrm{T}^{(3)}[u] &= \left(x_1^{(3)} - 1\right)^{\beta_1} \left(x_2^{(3)}\right)^{\beta_1+\beta_2+\beta_3} \left(x_1^{(3)}\right)^{\beta_3} \times \cdots, \qquad \gamma_1^{(3)} = \beta_3, \ \gamma_2^{(3)} = \beta_1 + \beta_2 + \beta_3.
\end{aligned} \tag{2.34}$$

Under our regularization scheme, all factors in φ are also shown in u, so the factorization of u will factorize φ automatically.

Since we are going to take residue around $x_i^{(\alpha)} = 0$, let us consider the limit to $(0, 0)$ for each transformation. Through the transformation we have

$$\begin{aligned}
\mathrm{T}^{(1)} : &\quad P_1 = x_1^{(1)} x_2^{(1)}, \quad P_2 = x_2^{(1)}, \quad P_3 = x_2^{(1)}(x_1^{(1)} + 1); \\
\mathrm{T}^{(2)} : &\quad P_1 = x_2^{(2)}, \quad P_2 = x_1^{(2)} x_2^{(2)}, \quad P_3 = x_2^{(2)}(1 + x_1^{(2)}); \\
\mathrm{T}^{(3)} : &\quad P_1 = x_2^{(3)}(x_1^{(3)} - 1), \quad P_2 = x_2^{(3)}, \quad P_3 = x_1^{(3)} x_2^{(3)}.
\end{aligned} \tag{2.35}$$

The limit going to $(0, 0)$ can be described as $x_1 \to \lambda^{\beta_1}, x_1 \to \lambda^{\beta_2}$ with $\beta_1 > 0, \beta_2 > 0$. If we describe the limit of P_i as $P_i \to \lambda^{a_i}$, we will have

$$\begin{aligned}
\mathrm{T}^{(1)} : &\quad a_1 = \beta_1 + \beta_2, \quad a_2 = \beta_2, \quad a_3 = \beta_2; \\
\mathrm{T}^{(2)} : &\quad a_1 = \beta_2, \quad a_2 = \beta_1 + \beta_2, \quad a_3 = \beta_2; \\
\mathrm{T}^{(3)} : &\quad a_1 = \beta_2, \quad a_2 = \beta_2, \quad a_3 = \beta_1 + \beta_2.
\end{aligned} \tag{2.36}$$

From them, we can read out the character of each transformation to be

$$\begin{aligned}
\mathcal{R}^{(1)} &: \{P_2 \to 0, P_3 \to 0, \frac{P_1}{P_2} \to 0, \frac{P_1}{P_3} \to 0, \frac{P_3}{P_2} \to 1\}, \\
\mathcal{R}^{(2)} &: \{P_1 \to 0, P_3 \to 0, \frac{P_2}{P_1} \to 0, \frac{P_2}{P_3} \to 0, \frac{P_3}{P_1} \to 1\}, \\
\mathcal{R}^{(3)} &: \{P_1 \to 0, P_2 \to 0, \frac{P_3}{P_1} \to 0, \frac{P_3}{P_2} \to 0, \frac{P_2}{P_1} \to -1\}.
\end{aligned} \tag{2.37}$$

For a better understanding, let us consider the coordinate system of a_1, a_3. From (2.36) one can see that each transformation defines a "region" in this coordinate system as

$$\begin{aligned}
\mathcal{R}_{1,3}^{(1)} &: \ \{a_1 > 0, \ a_3 > 0, \ a_1 > a_3\}, \\
\mathcal{R}_{1,3}^{(2)} &: \ \{a_1 > 0, \ a_3 > 0, \ a_1 = a_3\}, \\
\mathcal{R}_{1,3}^{(3)} &: \ \{a_1 > 0, \ a_3 > 0, \ a_1 < a_3\}.
\end{aligned} \tag{2.38}$$

[2] Another understanding is to use the "region" concept in (2.35).

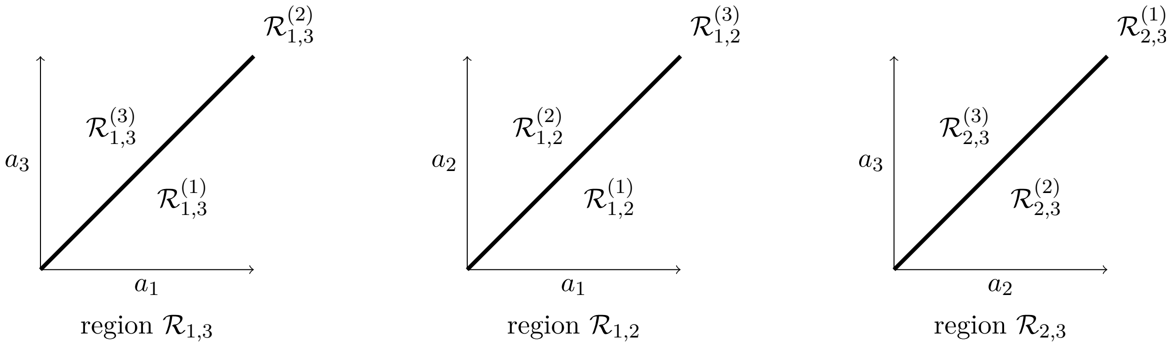

Figure 1. At the left it is the region of P_1, P_3, the middle the region of P_1, P_2 and the right the region of P_2, P_3.

The combination of these three regions fills the first quadrant as shown in the left picture in figure 1, although the measure of $\mathcal{R}_{1,3}^{(2)}$ is zero compared to the other two regions. The same phenomenon also occurs for $\mathcal{R}_{1,2}$ and $\mathcal{R}_{2,3}$ as shown in figure 1

$$
\begin{aligned}
\mathcal{R}_{1,2}^{(1)}:&\ \{a_1 > 0\,,\ a_2 > 0\,,\ a_1 > a_2\}\,,\\
\mathcal{R}_{1,2}^{(2)}:&\ \{a_1 > 0\,,\ a_2 > 0\,,\ a_1 < a_2\}\,,\\
\mathcal{R}_{1,2}^{(3)}:&\ \{a_1 > 0\,,\ a_2 > 0\,,\ a_1 = a_2\}\,,\\
\mathcal{R}_{2,3}^{(1)}:&\ \{a_2 > 0\,,\ a_3 > 0\,,\ a_2 = a_3\}\,,\\
\mathcal{R}_{2,3}^{(2)}:&\ \{a_2 > 0\,,\ a_3 > 0\,,\ a_2 > a_3\}\,,\\
\mathcal{R}_{2,3}^{(3)}:&\ \{a_2 > 0\,,\ a_3 > 0\,,\ a_2 < a_3\}\,.
\end{aligned}
\tag{2.39}
$$

For $\mathrm{T}^{(1)}$, P_1 is much smaller than P_2 and P_3. It means for

$$
\oint_{|P_1|=r_1} \oint_{|P_2|=r_2} \mathrm{d}P_1 \wedge \mathrm{d}P_2 \times \cdots\,, \tag{2.40}
$$

we have $r_1 < |P_2|$. Hence $P_2 = 0$ and $P_3 = 0$ is not in the circle of $|P_1| = r_1$, but in the circle of $|P_2| = r_2$. That's why it does correspond to the contour $(\{P_1\}, \{P_2, P_3\})$. One can also consider this problem starting from

$$
\oint_{|P_1|=r_1} \oint_{|P_3|=r_2} \mathrm{d}P_1 \wedge \mathrm{d}P_3 \times \cdots\,, \tag{2.41}
$$

and get the same conclusion. Similarly, one could know the contour of $\mathrm{T}^{(2)}$ is $(\{P_2\}, \{P_1, P_3\})$ and the contour of $\mathrm{T}^{(3)}$ is $(\{P_3\}, \{P_1, P_2\})$.

Now, as an exercise let's compute a multivariate residue at $(0, 0)$ of

$$
\varphi = \frac{(z_1 + z_2)\, R_{1,2} + z_2 R_{1,3} - z_1 R_{2,3}}{z_1 z_2\, (z_1 + z_2)} \mathrm{d}z_1 \mathrm{d}z_2\,. \tag{2.42}
$$

For this simple case, the denominator of the rational function can be directly separated to

$$
\varphi = \frac{R_{1,2}}{P_1 P_2} \mathrm{d}P_1 \wedge \mathrm{d}P_2 + \frac{R_{1,3}}{P_1 P_3} \mathrm{d}P_1 \wedge \mathrm{d}P_3 + \frac{R_{2,3}}{P_2 P_3} \mathrm{d}P_2 \wedge \mathrm{d}P_3\,. \tag{2.43}
$$

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Using the above analysis for the degenerated case, for the transformation $\mathrm{T}^{(1)}$ we have

$$\varphi|_{\mathrm{T}^{(1)}} = \frac{(x_1^{(1)}+1)R_{1,2} + R_{1,3} - x_1^{(1)}R_{2,3}}{x_1^{(1)}x_2^{(1)}(x_1^{(1)}+1)} dx_1^{(1)} \wedge dx_2^{(1)} \,, \tag{2.44}$$

thus

$$\mathrm{Res}_{(0,0)}\, \mathrm{T}^{(1)}[\varphi] = R_{1,2} + R_{1,3} \,. \tag{2.45}$$

On the other side, from the analysis of contour, since the transformation $\mathrm{T}^{(1)}$ corresponds to the contour $(\{P_1\},\{P_2P_3\})$, $1/(P_1P_2)$ and $1/(P_1P_3)$ could contribute. Thus, the result is $R_{1,2}+R_{1,3}$. This result is the same as the one we get from transformation $\mathrm{T}^{(1)}$. Similarly

$$\begin{aligned} R^{(2)} &= \mathrm{Res}_{(0,0)}\, \mathrm{T}^{(2)}[\varphi] = -R_{1,2} + R_{2,3} \,, \\ R^{(3)} &= \mathrm{Res}_{(0,0)}\, \mathrm{T}^{(3)}[\varphi] = -R_{1,3} - R_{2,3} \,, \end{aligned} \tag{2.46}$$

where the minus comes from $\mathrm{d}P_i \wedge \mathrm{d}P_j = -\mathrm{d}P_j \wedge \mathrm{d}P_i$. $R^{(1)} + R^{(2)} + R^{(3)} = 0$ is due to the three contours are not independent [88].

Before ending this part, let us mention that there are other transformations. For example, one could select P_3 as x_1 and P_1 as x_2, then, factorize a x_2 in the remaining degenerate factor P_2. This gives a factorization transformation

$$\mathrm{T}^{(4)} : z_1 \to x_2^{(4)} \,, \;\; z_2 \to x_1^{(4)}x_2^{(4)} - x_2^{(4)} \,, \tag{2.47}$$

which leads to

$$P_1 = x_2^{(4)}, \quad P_2 = x_2^{(4)}(x_1^{(4)} - 1), \quad P_3 = x_1^{(4)}x_2^{(4)} \,, \tag{2.48}$$

and

$$\begin{aligned} \mathrm{T}^{(4)}[u] &= \left(x_2^{(4)}\right)^{\beta_1+\beta_2+\beta_3} \left(x_1^{(4)} - 1\right)^{\beta_2} \left(x_1^{(4)}\right)^{\beta_3} \times \cdots \,, \\ \gamma_1^{(4)} &= \beta_3 \,, \;\; \gamma_2^{(4)} = \beta_1 + \beta_2 + \beta_3 \,. \end{aligned} \tag{2.49}$$

Comparing (2.48) with $\mathrm{T}^{(3)}$ in (2.35), we see they are same after noting $P_1|_{\mathrm{T}^{(3)}} \sim x_2^{(3)}$ and $P_2|_{\mathrm{T}^{(4)}} \sim x_2^{(4)}$. Similarly, we have $\gamma_1^{(4)} = \gamma_1^{(3)}$, $\gamma_2^{(4)} = \gamma_2^{(3)}$ and $\mathcal{R}^{(4)} = \mathcal{R}^{(3)}$ by the analysis of region. The contour corresponds to $\mathrm{T}^{(4)}[u]$ is also $(\{P_3\},\{P_1,P_2\})$, the same as $\mathrm{T}^{(3)}$. These arguments show it is equivalent to the transformation $\mathrm{T}^{(3)}$, thus we can neglect it.

2.4.2 Region of factorization

Now let's turn to a subtler issue about the power b in the blow-up transformation (2.30) for factorizing the degenerated poles. The choice of b relates also to the concept of "region" defined by the coordinate system of a_i, which describes the limit behavior $P_i \sim \lambda^{a_i}$. For the example presented in the previous subsubsection, the three factorization transformations $\mathrm{T}^{(\alpha)}$ in (2.29) are chosen such that their sum of regions will fill the whole region defined by $\mathcal{R}_{i,j} = \{a_i > 0, a_j > 0\}$ for all pairs of $P_i = 0, P_j = 0$ as demonstrated in figure 1. We claim that only when the factorization transformation satisfies this criterion, we will get the right LO contribution using the formula (2.19). Unfortunately, due to our limited mathematical knowledge, the rigorous mathematical reasons for this phenomenon are unclear to us. To reconfirm our calculations of the intersection numbers (such as the computations in section 5) using the region method, we compute them again by the fibration algorithm [67] of multivariate intersection number. It is another algorithm that can compute multivariate intersection numbers but does not suffer from the complication of the multivariate complex function.

Now let us illustrate some features of the subtler issue about blow-up transformation. This problem arises from that we are not just computing the multivariate residue of a rational function, but also the residue of factorization transformation as the denominator in (2.19). For a rational function, once it is factorized, further rescaling transformations will not change the result. For example

$$\mathrm{Res}_{(0,0)} \frac{\mathrm{d}z_1 \wedge \mathrm{d}z_2}{z_1 z_2} = 1 \tag{2.50}$$

A further rescaling transformation[3]

$$\mathrm{T}^{(1)} : z_1 \to x_1^{(1)} \left(x_2^{(1)}\right)^2 , \; z_2 \to x_2^{(1)} \tag{2.51}$$

will not change the result

$$\mathrm{Res}_{(0,0)}\, \mathrm{T}^{(1)} \left[\frac{\mathrm{d}z_1 \wedge \mathrm{d}z_2}{z_1 z_2}\right] = \mathrm{Res}_{(0,0)} \frac{\mathrm{d}x_1^{(1)} \wedge \mathrm{d}x_2^{(1)}}{x_1^{(1)} x_2^{(1)}} = 1\,. \tag{2.52}$$

However, the formula of LO contribution of intersection number (2.19) contains the inverse of a residue in the denominator

$$\frac{1}{\prod_i \mathrm{Res}_{\rho_i^{(\alpha)}} \partial_{z_i} \log\left(\mathrm{T}^{(\alpha)}\left[u\right]\right) \mathrm{d}z_i} \tag{2.53}$$

which will change under the rescaling transformations. For example, for $u = z_1^{\beta_1} z_2^{\beta_2} \prod_{i=3}^n (\mathrm{C}_i + \mathcal{O}(\boldsymbol{z}))^{\beta_i}$ which has been completely factorized, formula (2.19) gives

$$\frac{1}{\gamma} \equiv \frac{1}{\prod_i \mathrm{Res}_{z_i=0}\, \partial_{z_i} \log u \; \mathrm{d}z_i} = \frac{1}{\beta_1 \beta_2}\,. \tag{2.54}$$

Now we consider a rescaling transformation

$$\mathrm{T}^{(1)} : z_1 \to x_1^{(1)} \left(x_2^{(1)}\right)^b , \; z_2 \to x_2^{(1)} \tag{2.55}$$

such that

$$\begin{aligned} \mathrm{T}^{(1)}[u] &= \left(x_1^{(1)}\right)^{\beta_1} \left(x_2^{(1)}\right)^{b\beta_1+\beta_2} \prod_{i=3}^n (\mathrm{C}_i + \mathcal{O}(\boldsymbol{z}))^{\beta_i} \\ \frac{1}{\gamma^{(1)}} &= \frac{1}{\prod_i \mathrm{Res}_{\rho_i^{(\alpha)}} \partial_{z_i} \log\left(\mathrm{T}^{(1)}\left[u\right]\right) \mathrm{d}z_i} = \frac{1}{\beta_1(b\beta_1 + \beta_2)} \end{aligned} \tag{2.56}$$

[3]More generally, under the rescaling transformation $z_1 \to t_1^a t_2^b$, $z_2 \to t_1^c t_2^d$, the residue of (2.50) is invariant when $ad - bc = 1$. The blow-up transformations (2.30) belong to the set of rescaling transformations.

Clearly, $1/\gamma \neq 1/\gamma^{(1)}$, the further rescaling transformation will change the result of the intersection number. The reason is that before the rescaling transformation, the region is the complete first quadrant $\mathcal{R}_{1,2} = \{a_1 > 0, a_2 > 0\}$. After the rescaling transformation, we have

$$P_1 = z_1 = x_1^{(1)} \left(x_2^{(1)}\right)^b, \qquad P_2 = z_2 = x_2^{(1)} \tag{2.57}$$

thus the region is given by $\mathcal{R}_{1,2}^{(1)} = \{a_1 > ba_2 > 0\}$, which is only part of first quadrant. To get the right result, we must sum up another rescaling transformation, which gives the region $\mathcal{R}_{1,2}^{(1)} = \{ba_2 > a_1 > 0\}$. One can see that such a rescaling transformation is given by

$$\mathrm{T}^{(2)} : z_1 \to x_1^{(2)}\,,\ z_2 \to x_2^{(2)} \left(x_1^{(2)}\right)^{\frac{1}{b}} \tag{2.58}$$

then

$$\begin{aligned} \mathrm{T}^{(2)}[u] &= \left(x_1^{(2)}\right)^{\beta_1+\beta_2/b} \left(x_2^{(2)}\right)^{\beta_2} \prod_{i=3}^{n} (\mathrm{C}_i + \mathcal{O}(\boldsymbol{z}))^{\beta_i} \\ \frac{1}{\gamma^{(2)}} &= \frac{1}{\prod_i \mathrm{Res}_{\rho_i^{(\alpha)}} \partial_{z_i} \log\left(\mathrm{T}^{(2)}\,[u]\right) \mathrm{d}z_i} = \frac{1}{(\beta_1 + \beta_2/b)\beta_2}\,. \end{aligned} \tag{2.59}$$

It is easy to check that[4]

$$\mathcal{R}_{1,2} = \mathcal{R}_{1,2}^{(1)} + \mathcal{R}_{1,2}^{(2)}\,, \quad \frac{1}{\gamma} = \frac{1}{\gamma^{(1)}} + \frac{1}{\gamma^{(2)}}\,. \tag{2.60}$$

Having the above example, we can come back to the example in the previous subsubsection for the degenerated pole. Let us keep factorization transformation of $\mathrm{T}^{(1)}, \mathrm{T}^{(2)}$ untouched, but change the factorizaton transformation of $\mathrm{T}^{(3)}$. Again, in the first step, we do

$$\mathrm{t}_1 : z_1 \to x_1 - x_2\,,\ z_2 \to x_2\,, \Longrightarrow P_1 = x_1 - x_2\,,\ P_2 = x_2\,,\ P_3 = x_1\,. \tag{2.61}$$

But in the second step, we do

$$\mathrm{t}_2 : x_1 \to x_1^{(3a)}(x_2^{(3a)})^b\,, \qquad x_2 \to x_2^{(3a)}\,, \quad b > 1 \tag{2.62}$$

Overall we have

$$P_1 = x_2^{(3a)}(x_1^{(3a)}(x_2^{(3a)})^{b-1} - 1), \quad P_2 = x_2^{(3a)}, \quad P_3 = x_1^{(3a)}(x_2^{(3a)})^b \tag{2.63}$$

and

$$\begin{aligned} \mathrm{T}^{(3a)}[u] &= \left(x_1^{(3a)}(x_2^{(3a)})^{b-1} - 1\right)^{\beta_1} \left(x_2^{(3a)}\right)^{\beta_1+\beta_2+b\beta_3} \left(x_1^{(3a)}\right)^{b\beta_3} \times \cdots \\ \gamma_1^{(3a)} &= \beta_3\,, \qquad \gamma_2^{(3a)} = \beta_1 + \beta_2 + b\beta_3 \end{aligned} \tag{2.64}$$

It is easy to read out the region by $\mathrm{T}^{(3a)}$ as

$$\mathcal{R}_{1,2}^{(3a)} = \{a_1 = a_2 > 0\}, \quad \mathcal{R}_{1,3}^{(3a)} = \{a_3 > ba_1 > 0\}, \quad \mathcal{R}_{2,3}^{(3a)} = \{a_3 > ba_2 > 0\} \tag{2.65}$$

[4] More rigirously, there should be a region $\mathcal{R}_{1,2}^{(1)} = \{ba_2 = a_1 > 0\}$ to completely fill the complete first quadrant. However, the measure of this region is zero, so we can neglect it.

If we check the figure 1 again, we see that the region $\mathcal{R}_{1,3}$ and $\mathcal{R}_{2,3}$ is not complete. To find the missing region, we need to consider another transformation

$$\widetilde{\mathrm{t}}_2: \quad x_1 \to x_1^{(3b)}(x_2^{(3b)})^{1+\frac{1}{b-1}}, \qquad x_2 \to x_1^{(3b)}(x_2^{(3b)})^{\frac{1}{b-1}} \tag{2.66}$$

Thus we have

$$P_1 = x_1^{(3b)}(x_2^{(3b)})^{\frac{1}{b-1}}(x_2^{(3b)}-1), \qquad P_2 = x_1^{(3b)}(x_2^{(3b)})^{\frac{1}{b-1}}, \qquad P_3 = x_1^{(3b)}(x_2^{(3b)})^{1+\frac{1}{b-1}}, \tag{2.67}$$

and

$$\begin{aligned} \mathrm{T}^{(3b)}[u] &= \left((x_2^{(3b)})-1\right)^{\beta_1}\left(x_1^{(3b)}\right)^{\beta_1+\beta_2+\beta_3}\left(x_2^{(3a)}\right)^{\frac{\beta_1+\beta_2+b\beta_3}{b-1}} \times \cdots, \\ \gamma_1^{(3b)} &= \beta_1+\beta_2+\beta_3, \qquad \gamma_2^{(3b)} = \frac{\beta_1+\beta_2+b\beta_3}{b-1}. \end{aligned} \tag{2.68}$$

It is important to notice that both (2.63) and (2.67) gives the contour $\{\{P_1, P_2\}, P_3\}$. Furthermore, from (2.67) we find the region

$$\mathcal{R}_{1,2}^{(3b)} = \{a_1 = a_2 > 0\}, \ \mathcal{R}_{1,3}^{(3b)} = \{ba_1 > a_3 > a_1 > 0\}, \ \mathcal{R}_{2,3}^{(3b)} = \{ba_2 > a_3 > a_2 > 0\}. \tag{2.69}$$

In fact, one can check that

$$\begin{aligned} \frac{1}{\gamma^{(3a)}} + \frac{1}{\gamma^{(3b)}} &= \frac{1}{\beta_3(\beta_1+\beta_2+b\beta_3)} + \frac{1}{(\beta_1+\beta_2+\beta_3)\left(\frac{\beta_1+\beta_2+b\beta_3}{b-1}\right)} \\ &= \frac{1}{\beta_3(\beta_1+\beta_2+\beta_3)} = \frac{1}{\gamma^{(3)}} \end{aligned} \tag{2.70}$$

when compared with (2.34).

We emphasize that the region rule is only observations in our practice, not proof. It suggests that if one does not want to lose contributions in this method, one should consider the regions of factorization transformations carefully.

2.5 The selection rules of the CDE — I

With the above mathematical preparations, we move to the discussion of the matrix $(\hat{\mathrm{d}}\Omega)$ defined in (2.14). We will focus on the selection rules for non-zero components of $(\hat{\mathrm{d}}\Omega)$ by two different methods. In this section, we will discuss the computation of $(\hat{\mathrm{d}}\Omega)$ without using relative cohomology. Thus for denominators D_i with integer power, we need to add a regulator $D_i^{\delta_i}$ to u and take the zero limit for the final result. As a consequence, the selection rules of the CDE we get in this section will have some redundant terms, which will vanish together with the regulator power δ_i. This problem will be avoided by using relative cohomology in the next section. The discussion in this section will be the expansion of [83]. From (2.14) we see that the coefficient matrix of the CDE (or "CDE matrix" for short) has two factors:

$$(\hat{\mathrm{d}}\Omega)_{IK} = \langle \dot{\varphi}_I | \varphi_J \rangle \left(\eta^{-1}\right)_{JK}. \tag{2.71}$$

We will address these two factors one by one.

2.5.1 Condition of $\langle \dot{\varphi}_I | \varphi_J \rangle \neq 0$

Let's consider $\langle \dot{\varphi}_I | \varphi_J \rangle$ first. By general arguments, to have a non-zero contribution to intersection number, one must have $b_{\dot{I},i} + b_{J,i} \leq -2$ for the series expansion around the pole p_i. For φ_J as a d log-form, $b_{J,i} \geq -1$, thus only those terms in $\dot{\varphi}_I$ that satisfy $b_{\dot{I},i} \leq -1$ lead to non-zero contributions to this intersection number. Furthermore, around each pole's region (α), the action of $\hat{\mathrm{d}}$ operator decreases only one index i of $b_{I,i}$ in the expansion of φ_I by 1. Combining the fact that φ_I is also a d log-form, we have the condition: only when one index j satisfies $b_{I,j} + b_{J,j} \leq -1$ and other indices satisfy $b_{I,i} + b_{J,i} = -2, i \neq j$, $\langle \dot{\varphi}_I | \varphi_J \rangle$ could be non-zero.

With the above explanations, now we define two notations. For a pole's region (α), if they have a pair in the expansion which satisfy $b_{I,j} + b_{J,j} = -1$ and $b_{I,i} + b_{J,i} = -2$ for other $i \neq j$, we say they share a $(n-1)$**-variable Simple Pole ($(n-1)$-SP)**; if $b_{I,i} + b_{J,i} = -2$ for all i, we say they share a n**-SP**. Using these notations, the above discussions can be summarized as follows:

$$\begin{cases} \mathrm{d}\log\text{-form} & : \ b_{I,i} \geq -1 \,, \ b_{J,i} \geq -1 \,, \\ \dot{\varphi}_I & : \ b_{\dot{I},j} = b_{I,j} - 1 \,, \ \text{for one } j \,, \\ \langle \dot{\varphi}_I | \varphi_J \rangle \neq 0 & : \ b_{\dot{I},i} + b_{J,i} \leq -2, \ \forall i \end{cases}$$
$$\Rightarrow \langle \dot{\varphi}_I | \varphi_J \rangle \neq 0 : \ (-2 \leq b_{I,j} + b_{J,j} \leq -1) \ \& \ (b_{I,i} = b_{J,i} = -1, i \neq j) \tag{2.72}$$

and

$$\begin{aligned} (n-1)\text{-SP} : & \ (b_{I,j} + b_{J,j} = -1) \ \& \ (b_{I,i} = b_{J,i} = -1, i \neq j) \,, \\ n\text{-SP} : & \ b_{I,i} = b_{J,i} = -1 \,. \end{aligned} \tag{2.73}$$

Before the computations of the above two cases, let us rewrite $\dot{\varphi}_I = \mathrm{D}\log u \bigwedge_j \mathrm{D}\log \mathrm{W}_j^{(I)}$ given in (2.13) to more explicit form using the expansion (2.17) for factorization in the region (α) as

$$\begin{aligned} & \mathrm{D}\log\left(\mathrm{T}^{(\alpha)}[u]\right) \bigwedge_j \mathrm{D}\log \mathrm{T}^{(\alpha)}\left[\mathrm{W}_j^{(I)}\right] \,, \\ & \mathrm{D}\log\left(\mathrm{T}^{(\alpha)}[u]\right) = \mathrm{D}\log \bar{u}_\alpha(\boldsymbol{x}^{(\alpha)}) + \mathrm{D}\log\left(\prod_i \left[x_i^{(\alpha)} - \rho_i^{(\alpha)}\right]^{\gamma_i^{(\alpha)}}\right) \,. \end{aligned} \tag{2.74}$$

2.5.2 $(n-1)$-SP contribution for $\langle \dot{\varphi}_I | \varphi_J \rangle$

For φ_I and φ_J share a $(n-1)$-SP, $\mathrm{D}\log\left(\mathrm{T}^{(\alpha)}[u]\right)$ provides the remaining "one pole" via $b_{\dot{I},j} = b_{I,j} - 1$. The result is

$$-\frac{\gamma_j^{(\alpha)}}{\gamma^{(\alpha)}} \hat{\mathrm{d}} \int C_I^{(\boldsymbol{b}_I)} C_J^{(\boldsymbol{b}_J)} \, d\rho_j^{(\alpha)} \,, \tag{2.75}$$

which has been given in [83] (include why is it a $\hat{\mathrm{d}}\log$), so we are not going to explain these details again, but merely show one typical example: the $C_I^{(\boldsymbol{b}_I)}$ and $C_J^{(\boldsymbol{b}_J)}$ in the formula could be

$$\begin{aligned} C_I^{(b_j=0)} &= \left(\partial_z \log\tau[z, c_2; c_\pm]\right)\Big|_{z=c_1} = \partial_{c_1} \log\tau[c_1, c_2; c_\pm] \\ C_J^{(b_j=-1)} &= \mathrm{Res}_{z=c_1} \, \mathrm{d}\log\tau[z, c_1; c_\pm] = 1 \,, \end{aligned} \tag{2.76}$$

then

$$\hat{\mathrm{d}} \int C_I^{(b_j=0)} C_J^{(b_j=-1)}\, dc_1 = \hat{\mathrm{d}} \log \tau[c_1, c_2; c_\pm] \tag{2.77}$$

Let us remind the reader that there are three cases for the j in $b_{\dot{I},j} = b_{I,j} - 1$:

$$\begin{aligned}
&b_{I,j} = 0\,, && b_{J,j} = -1\,,\\
&b_{I,j} = -1\,, && b_{J,j} = 0\,,\\
&b_{I,j} = -\frac{1}{2}\,, && b_{J,j} = -\frac{1}{2}\,,
\end{aligned} \tag{2.78}$$

where the third case could emerge from sqrt-type d log. According to the formula of LO contribution (2.24), $\tilde{\gamma}_j^{(\alpha)}$'s in the denominator rely on $b_{I,j}$ and $b_{J,j}$, and are different for these three cases. However, the $\hat{\mathrm{d}}$ act on $\int u\varphi_I$ also leads to a external factor $\tilde{\gamma}_j^{(\alpha)}$ at numerator. For example, to use (2.19), we need to apply the rescaling transformation

$$\tilde{u} = u(z_j - \rho_j^{(\alpha)})^{-b_{I,j}}\,, \tag{2.79}$$

and we have $\tilde{\gamma}_i^{(\alpha)} = \gamma_i^{(\alpha)}$ for $i \neq j$. Then, $\dot{\varphi}$ take the form as follow:

$$\begin{aligned}
&\hat{\mathrm{d}}(z_j - \rho_j^{(\alpha)})^{\tilde{\gamma}_j^{(\alpha)}} \times \prod_{i\neq j}(z_i - \rho_i^{(\alpha)})^{\gamma_i^{(\alpha)}-1} f\\
&= -\tilde{\gamma}_j^{(\alpha)} \hat{\mathrm{d}}\rho_j^{(\alpha)} \times \prod_i (z_i - \rho_i^{(\alpha)})^{\tilde{\gamma}_i^{(\alpha)}-1} f + \cdots\,,
\end{aligned} \tag{2.80}$$

where the $\cdots$ part does not contribute to the intersection number. The $\tilde{\gamma}_j^{(\alpha)}$ in the above equation will cancel the $\tilde{\gamma}_j^{(\alpha)}$ in $\tilde{\boldsymbol{\gamma}}^{(\alpha)}$ in the equation (2.24):

$$\frac{\tilde{\gamma}_j^{(\alpha)}}{\tilde{\boldsymbol{\gamma}}^{(\alpha)}} = \frac{1}{\prod_{i\neq j} \gamma_i^{(\alpha)}} = \frac{\gamma_j^{(\alpha)}}{\boldsymbol{\gamma}^{(\alpha)}}\,. \tag{2.81}$$

Hence, we have the coefficient in (2.75).

2.5.3 n-SP contribution for $\langle \dot{\varphi}_I | \varphi_J \rangle$

For φ_I and φ_J sharing a n-SP, at first glance, the second term $\mathrm{D}\log\left(\prod_i \left[x_i^{(\alpha)} - \rho_i^{(\alpha)}\right]^{\gamma_i^{(\alpha)}}\right)$ in $\mathrm{D}\log\left(\mathrm{T}^{(\alpha)}[u]\right)$ (see (2.74)) seems to lead to a pair of expanded terms that satisfy $b_{L,j} + b_{R,j} = -3$ for one j, and $b_{L,i} + b_{R,i} = -2$ for other $i \neq j$. This could give an NLO contribution (as defined in [83]) to the intersection number. However, as we will show shortly, all these potential contributions are canceled here. For rational-type d log-form,

$$\mathrm{D}\log \mathrm{T}^{(\alpha)}\left[\mathrm{W}_j^{(I)}\right] = \mathrm{D}\log\left(x_j^{(\alpha)} - \rho_j^{(\alpha)}\right) \tag{2.82}$$

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expanding $D = \hat{\mathrm{d}} + \mathrm{d}$ where $\hat{\mathrm{d}}$ is differentiation with respect to $\rho_j^{(\alpha)}$ and d_j differentiation with respect to $x_j^{(\alpha)}$, we have

$$\begin{aligned}
&\left[\mathrm{d}_j \log\left(\left(x_j^{(\alpha)}-\rho_j^{(\alpha)}\right)^{\gamma_j^{(\alpha)}}\right) \wedge \hat{\mathrm{d}} \log\left(x_j^{(\alpha)}-\rho_j^{(\alpha)}\right)\right.\\
&\quad\left.+\hat{\mathrm{d}} \log\left(\left(x_j^{(\alpha)}-\rho_j^{(\alpha)}\right)^{\gamma_j^{(\alpha)}}\right) \wedge \mathrm{d}_j \log\left(x_j^{(\alpha)}-\rho_j^{(\alpha)}\right)\right] \bigwedge_{i\neq j} \mathrm{D}\log \mathrm{T}^{(\alpha)}\left[\mathrm{W}_i^{(I)}\right]\\
&=\gamma_j^{(\alpha)}\left[\mathrm{d}_j \log\left(x_j^{(\alpha)}-\rho_j^{(\alpha)}\right) \wedge \hat{\mathrm{d}} \log\left(x_j^{(\alpha)}-\rho_j^{(\alpha)}\right)\right.\\
&\qquad\left.+\hat{\mathrm{d}} \log\left(x_j^{(\alpha)}-\rho_j^{(\alpha)}\right) \wedge \mathrm{d}_j \log\left(x_j^{(\alpha)}-\rho_j^{(\alpha)}\right)\right] \bigwedge_{i\neq j} \mathrm{D}\log \mathrm{T}^{(\alpha)}\left[\mathrm{W}_i^{(I)}\right]\\
&=0\,.
\end{aligned} \tag{2.83}$$

Similarly, for sqrt-type d log-form

$$\mathrm{D}\log \mathrm{T}^{(\alpha)}\left[\mathrm{W}_j^{(I)}\right] = \mathrm{D}\log \tau\left[x_j^{(\alpha)}, \rho_j^{(\alpha)}; c_\pm\right] \tag{2.84}$$

we have

$$\begin{aligned}
&\left[\mathrm{d}_j \log\left(\left(x_j^{(\alpha)}-\rho_j^{(\alpha)}\right)^{\gamma_j^{(\alpha)}}\right) \wedge \hat{\mathrm{d}} \log\tau\left[x_j^{(\alpha)}, \rho_j^{(\alpha)}; c_\pm\right]\right.\\
&\quad\left.+\hat{\mathrm{d}} \log\left(\left(x_j^{(\alpha)}-\rho_j^{(\alpha)}\right)^{\gamma_j^{(\alpha)}}\right) \wedge \mathrm{d}_j \log\tau\left[x_j^{(\alpha)}, \rho_j^{(\alpha)}; c_\pm\right]\right] \bigwedge_{i\neq j} \mathrm{D}\log \mathrm{T}^{(\alpha)}\left[\mathrm{W}_i^{(I)}\right]\\
&=\left[\mathrm{d}_j \log\left(x_j^{(\alpha)}-\rho_j^{(\alpha)}\right) \wedge \hat{\mathrm{d}} \log\left(x_j^{(\alpha)}-\rho_j^{(\alpha)}\right)+\hat{\mathrm{d}} \log\left(x_j^{(\alpha)}-\rho_j^{(\alpha)}\right) \wedge \mathrm{d}_j \log\left(x_j^{(\alpha)}-\rho_j^{(\alpha)}\right)\right]\\
&\quad\times\frac{\gamma_j^{(\alpha)}\sqrt{\left(\rho_j^{(\alpha)}-c_+\right)\left(\rho_j^{(\alpha)}-c_-\right)}}{\sqrt{\left(x_j^{(\alpha)}-c_+\right)\left(x_j^{(\alpha)}-c_-\right)}} \bigwedge_{i\neq j} \mathrm{D}\log \mathrm{T}^{(\alpha)}\left[\mathrm{W}_i^{(I)}\right]\\
&=0
\end{aligned} \tag{2.85}$$

After showing the second term giving no contribution, using the result (2.24) the first term gives the result

$$\frac{C_I^{(-\mathbf{1})} C_J^{(-\mathbf{1})}}{\boldsymbol{\gamma}^{(\alpha)}}\, \hat{\mathrm{d}} \log\left(\bar{u}_\alpha(\boldsymbol{\rho}^{(\alpha)})\right). \tag{2.86}$$

It is worth pointing out that in [83], one needs a formula of the NLO contribution of the intersection number for this result. Here we avoid using the NLO formula because we have transformed the related term to the LO case via IBP in (2.12) (from the second line to the third line) at the beginning.

2.5.4 Condition of $(\boldsymbol{\eta}^{-1})_{JK} \neq 0$

For $(\eta^{-1})_{JK}$ in the CDE, since $\eta^{-1} = \frac{1}{|\eta|}\eta^*$ where η^* is the adjugate matrix, the element could be written as

$$(\eta^{-1})_{JK} = (-1)^{J+K}\frac{\eta^{(KJ)}}{|\eta|} \tag{2.87}$$

where $\eta^{(KJ)}$ is the minor of the element η_{JK} in the matrix η. The minor in the Laplace expansion of the determinant is the sum of the form

$$(-1)^a \eta_{Ji_1}\eta_{i_1 i_2}\cdots\eta_{i_{v-1}i_v}\eta_{i_v K}\prod_k \eta_{j_k j_k}\,. \tag{2.88}$$

For $\eta^{(KJ)} \neq 0$, at least one term of the form (2.88) should be non-zero, which is equivalent to the statement that every element η_{ij} in (2.88) should be non-zero. For the d log-form $\eta_{JK} = \langle\varphi_J|\varphi_K\rangle$ could be non-zero only when φ_J and φ_K share at least one n-SP. Since φ_{j_k} must exhibit n-SP with itself, all $\eta_{j_k j_k}$ should naturally be non-zero. Therefore, we only need to ask the existence of at least one non-zero chain $\eta_{Ji_1}\eta_{i_1 i_2}\cdots\eta_{i_{v-1}i_v}\eta_{i_v K}$, which implies that φ_J shares n-SP with φ_{i_1}, φ_{i_1} shares n-SP with φ_{i_2}, and so on. With this picture, the notation of n**-SP chain** has been defined in [83]:

- If φ_I and φ_J share an n-SP, we say they are n-SP related, and denote it as $\varphi_I \sim \varphi_J$.
- The n-SP chain is the collection of φ_I's, such that for arbitrary pair of φ_a, φ_b there exists an ordered list $\{\varphi_a, \varphi_{i_1}, \ldots, \varphi_{i_k}, \varphi_b\}$ (notice that $\varphi_a \sim \varphi_b, \varphi_b \sim \varphi_c$ do not give $\varphi_a \sim \varphi_c$, that is why there is a order) such that $\varphi_a \sim \varphi_{i_1} \sim \ldots \sim \varphi_{i_k} \sim \varphi_b$ where every φ belongs to the chain. If φ_a, φ_b belong to a n-SP chain, we denote it as $\varphi_a \sim\sim \varphi_b$.

Using this notation, we can simply say that the condition of $\left(\eta^{-1}\right)_{JK} \neq 0$ is φ_J and φ_K belongs to an n-SP chain. For example, assume there is a case with seven master integrals, whose indices are denoted as $\{J, K, 1, 2, 3, 4, 5\}$, and suppose $\varphi_J \sim \varphi_1 \sim \varphi_2 \sim \varphi_K$, we have $\varphi_J \sim\sim \varphi_K$ and

$$(-1)^{J+K}|\eta^{(KJ)}| = \begin{vmatrix} * & \eta_{J1} & * & * & * & * \\ * & * & \eta_{12} & * & * & * \\ \eta_{2K} & * & * & * & * & * \\ * & * & * & \eta_{33} & * & * \\ * & * & * & * & \eta_{44} & * \\ * & * & * & * & * & \eta_{55} \end{vmatrix} \tag{2.89}$$

could be non-zero due to $\eta_{\mathrm{J}1}\eta_{12}\eta_{2\mathrm{K}}\eta_{33}\eta_{44}\eta_{55}$ could be non-zero.

2.5.5 Conclusion

Having discussed the two factors in (2.14), now we can state the selection rules of the CDE:

- $\left(\hat{\mathrm{d}}\Omega\right)_{IK} = \langle\dot{\varphi}_I|\varphi_J\rangle\left(\eta^{-1}\right)_{JK}$ could be non-zero only when in the summation over J there exists a φ_J share $(n-1)$-SP or n-SP with φ_I, and $\varphi_J \sim\sim \varphi_K$.
- $\left(\hat{\mathrm{d}}\Omega\right)_{IK}$ could be determined via merely LO contribution formula of intersection number. For φ_I and φ_J that share a $(n-1)$-SP, it contributes

$$-\frac{\gamma_j^{(\alpha)}}{\gamma^{(\alpha)}}\,\hat{\mathrm{d}}\int C_I^{(\boldsymbol{b}_I)} C_J^{(\boldsymbol{b}_J)}\,\hat{\mathrm{d}}\rho_j^{(\alpha)} \tag{2.90}$$

to $\langle\dot{\varphi}_I|\varphi_J\rangle$. For φ_I and φ_J share a n-SP, it contributes

$$\frac{C_I^{(\mathbf{-1})}C_J^{(\mathbf{-1})}}{\boldsymbol{\gamma}^{(\alpha)}}\,\hat{\mathrm{d}}\log\left(\bar{u}_\alpha(\boldsymbol{\rho}^{(\alpha)})\right), \tag{2.91}$$

to $\langle\dot{\varphi}_I|\varphi_J\rangle$ with constant $C_I^{(\mathbf{-1})}C_J^{(\mathbf{-1})}$. These two formulas give all symbol letters. For $\eta_{JK}=\langle\varphi_J|\varphi_K\rangle$, each shared n-SP gives

$$\frac{C_J^{(\mathbf{-1})}C_K^{(\mathbf{-1})}}{\boldsymbol{\gamma}^{(\alpha)}} \tag{2.92}$$

with constant $C_J^{(\mathbf{-1})}C_K^{(\mathbf{-1})}$.

- The canonical-form differential equations emerge naturally. To see it, let's assign the original hypersurface-powers β_i in $u=\prod_i\left[P_i(\boldsymbol{z})\right]^{\beta_i}$ to have the transcendental weight-(-1). For our application in Feynman integrals, $\beta_i\sim a\epsilon$ or $\beta_i\sim\delta_k$ with constant a (typically, integer or half-integer). This means that ϵ and δ_i have the transcendental weight-(-1), so all hypersurface-powers $\gamma_i^{(\alpha)}$ for each factorization are also weight-(-1)

$$\mathcal{T}(\beta_i)=\mathcal{T}(\epsilon)=\mathcal{T}(\delta_k)=\mathcal{T}(\gamma_i^{(\alpha)})=-1\,. \tag{2.93}$$

  Because of (2.90) and (2.91), $\langle\dot{\varphi}_I|\varphi_J\rangle$ are weight-(n-1) coefficient times $\hat{\mathrm{d}}\log$-form. Similarly, $\left(\eta^{-1}\right)_{JK}$ is weight-$(-n)$ coefficient. Combining them, we have $\left(\hat{\mathrm{d}}\Omega\right)_{IK}$ is weight-(-1) coefficient times $\hat{\mathrm{d}}\log$-form. After taking regulators δ_i to zero, the weight-(-1) coefficient in $\left(\hat{\mathrm{d}}\Omega\right)_{IK}$ could only be proportional to ϵ. Hence we have

$$\left(\hat{\mathrm{d}}\Omega\right)_{IK}=\epsilon\sum_i \mathrm{C}_{IK}^{(i)}\;\hat{\mathrm{d}}\log\mathrm{W}^{(i)}(\boldsymbol{s}) \tag{2.94}$$

  where $\mathrm{W}^{(i)}(\boldsymbol{s})$ are symbol letters and $\mathrm{C}_{IK}^{(i)}$ are the corresponding constant coefficient matrix. This is nothing, but the so-called canonical form or ϵ-form.

Since after factorization, the n-variable multivariate residues are reduced to n univariate residues, and only the $(n_0\geq n-1)$-SP shared by φ_I and φ_J could contribute, naturally one can image that people could take residue of $(n-1)$-SP first. This gives an overall factor

$$\frac{\gamma_j^{(\alpha)}}{\boldsymbol{\gamma}^{(\alpha)}} \tag{2.95}$$

and left an univariate problem, whose u part (denoted as u' here) is

$$u'=\bar{u}_\alpha(\rho_1^{(\alpha)},\cdots,\rho_{j-1}^{(\alpha)},x_j^{(\alpha)},\rho_{j+1}^{(\alpha)},\cdots,\rho_n^{(\alpha)})\left(x_j^{(\alpha)}-\rho_j^{(\alpha)}\right)^{\gamma_j^{(\alpha)}} \tag{2.96}$$

All univariate u-part could be reformed as

$$u'=P_0^{\beta_0}\prod_i(z-c_i)^{\beta_i}\,. \tag{2.97}$$

Since we rescale all β_i to be $a_i\epsilon$, the square root part is absorbed into $\bar{\varphi}$ part. Without loss of generality, the square root part can be expressed as $\prod_{i=1}^m(z-c_i)^{-\frac{1}{2}}$. For the cases we discussed in this paper, $m = 0, 1, 2$, so that there is no elliptic integral and it is the $\mathrm{d}\log$ cases. If $m = 3, 4$, they are the elliptic cases. If $m \geq 5$, it is the beyond elliptic case. Then, the $\mathrm{d}\log$ basis and the CDE of all univariate cases without elliptic integrals ($m = 0, 1, 2$) could be systematically discussed based on this form and of course give the same answer as (2.90) and (2.91). The details of the univariate $\mathrm{d}\log$ cases have been given in [83]. [83] also shows that since univariate rationalization is easy, one could also transform all $m = 0, 1, 2$ cases into the $m = 0$ cases with an algebraic transformation. Let's denote the new u after the transformation as

$$u' = (P_0')^{a_0\epsilon} \prod_i (z' - c_i')^{a_i\epsilon} . \tag{2.98}$$

Then, all the symbol letters are the distance between these univariate poles $c_i' - c_j'$ and pure parameter factor P_0' as has been discussed in [83]. For the $m \geq 3$ cases, which are elliptic or beyond, unfortunately, there are not a $\mathrm{d}\log$ basis then. Meanwhile, the square roots like $\prod_{i=1}^3 (z - c_i)^{-\frac{1}{2}}$ can not be rationalized by an algebraic transformation. These cases are beyond the scope of this paper.

3 The selection rules of the CDE via relative cohomology

3.1 Dual form in twisted relative cohomology

In the last section, we have introduced how to compute the intersection number between forms in twisted cohomology. In this section, we will present simple computational rules for intersection numbers between twisted cohomology and twisted relative cohomology. Recall the integrals take the form

$$\begin{aligned} I[u, \varphi] &\equiv \int u\, \varphi_L \,, \\ \varphi &\equiv \hat{\varphi}(\boldsymbol{z}) \bigwedge_j \mathrm{d}z_j = \frac{Q(\boldsymbol{z})}{\left(\prod_k D_k^{a_k}\right)\left(\prod_i P_i^{b_i}\right)} \bigwedge_j \mathrm{d}z_j \,, \\ u &= \prod_i \left[P_i(\boldsymbol{z})\right]^{\beta_i} \,, \qquad a_k, b_j \in \mathbb{N} \,, \end{aligned} \tag{3.1}$$

where regulators have not been introduced for factors D_k. In Feynman integrals, the denominators D_i with integer power are propagators and determine the "sector" to which the integral belongs to. Here we borrow the concept of "sector" from the Feynman integral by considering denominators in φ_I having negative integer power in $u\varphi_I$,[5] as the "propagator" and defining the sector by the list $\hat{I}$ selecting from the set of propagators. Furthermore, we denote the set of hypersurfaces relating to the sector $\hat{I}$ as

$$\mathcal{B}_{\hat{I}} = \{D_{I_1} = 0, D_{I_2} = 0, \cdots, D_{I_n} = 0\} . \tag{3.2}$$

Selecting the dual space of dual form $|\varphi\rangle$ to be the space of twisted relative cohomology just means selecting dual forms living on the maximal cut of each sector. To distinguish,

[5] The P_i in φ_I will have contribution from u, so the total power is not integer and we should consider them as the "propagator".

we denote these dual forms living on its maximal cut $(\hat{R})$ as $\Delta_{\hat{R}}\varphi_R$. In practice, we do not need to know the exact meaning of "live on the cuts", and all we need to know is the intersection number of φ_L and $\Delta_{\hat{R}}\varphi_R$ obey a simple rule: just applying the maximal cut of the sector of φ_R to both sides, then applying the intersection number between the cut forms as we did in the previous section, i.e.,

$$\begin{aligned}&\langle\varphi_L|\Delta_{\hat{R}}\varphi_R\rangle=\langle\varphi_{L;\hat{R}}|\varphi_{R;\hat{R}}\rangle=\sum_{\boldsymbol{p}_{\hat{R}}}\mathrm{Res}_{\boldsymbol{p}_{\hat{R}}=0}\,\psi_{L;\hat{R}}\varphi_{R;\hat{R}}\,,\quad \nabla_{1;\hat{R}}\cdots\nabla_{n;\hat{R}}\psi_{L;\hat{R}}=\varphi_{L;\hat{R}}\\ &\varphi_{L;\hat{R}}=\mathrm{Res}_{\mathcal{B}_{\hat{R}}}\left(\frac{u\varphi_L}{u_{\hat{R}}}\right)\,,\quad \varphi_{R;\hat{R}}=\mathrm{Res}_{\mathcal{B}_{\hat{R}}}\left(\frac{\varphi_R u_{\hat{R}}}{u}\right)\,,\quad u_{\hat{R}}=u|_{\mathcal{B}_{\hat{R}}}\,,\end{aligned}\tag{3.3}$$

where $\nabla_{i;\hat{R}}$ is defined via the cut integral family corresponding to $u_{\hat{R}}$. $\boldsymbol{p}_{\hat{R}}$ are again isolated intersection points (see (2.5)) of hypersurfaces containing $\mathcal{B}_{\hat{R}}$. The definition of $\varphi_{L;\hat{R}}$ in (3.3) has the property $\mathrm{Res}_{\mathcal{B}_{L;\hat{R}}}u\varphi=u_{\hat{R}}\,\varphi_{L;\hat{R}}$, and the form in (3.3) makes $\varphi_{L;\hat{R}}$ manifest as the single-valued function.

In relative cohomology, the concept of "subsector" is the same as Feynman integrals, i.e., if $\mathcal{B}_{\hat{I}}\subseteq\mathcal{B}_{\hat{J}}$, we say $\mathcal{B}_{\hat{I}}$ is a subsector of $\mathcal{B}_{\hat{J}}$. Obviously, only when the sector of φ_R is a subsector of the sector of φ_L, the intersection number could be non-zero.

3.2 The improved selection rules of the CDE

Now we consider the $\hat{\mathrm{d}}\Omega$ in the CDE using the dual basis in the relative cohomology

$$\begin{aligned}&(\hat{\mathrm{d}}\Omega)_{IK}=\langle\dot{\varphi}_{I;\hat{J}}|\varphi_{J;\hat{J}}\rangle\,(\eta^{-1})_{JK}\,,\\ &\eta_{IJ}(\eta^{-1})_{JK}=\delta_{IK}\,,\quad \eta_{IJ}=\langle\varphi_{I;\hat{J}}|\varphi_{J;\hat{J}}\rangle\,.\end{aligned}\tag{3.4}$$

For $\langle\dot{\varphi}_{I;\hat{J}}|\varphi_{J;\hat{J}}\rangle$, let us denote the number of remaining integration variables of sector $\mathcal{B}_{\hat{J}}$ as $n_{\hat{J}}$. With the knowledge discussed in the previous section, we see immediately that $\langle\dot{\varphi}_{I;\hat{J}}|\varphi_{J;\hat{J}}\rangle$ could be non-zero only when $\dot{\varphi}_{I;\hat{J}}$ and $\varphi_{J;\hat{J}}$ share $n_{\hat{J}}$-SP or $(n_{\hat{J}}-1)$-SP.

For $(\eta^{-1})_{JK}$, things are a little bit different since $\langle\varphi_{I;\hat{J}}|\varphi_{J;\hat{J}}\rangle\neq\langle\varphi_{J;\hat{I}}|\varphi_{I;\hat{I}}\rangle$. To count the anti-symmetry, we need to slightly modify the concept of n-SP chain to a new concept of **cut-n-SP** chain, i.e.,

- If $\varphi_{I;\hat{J}}$ and $\varphi_{J;\hat{J}}$ share $n_{\hat{J}}$-SP, we say φ_I is cut-n-SP related to φ_J, and denoted as $\varphi_I\rightarrow\varphi_J$ or $\varphi_J\leftarrow\varphi_I$. Now, it is an oriented relation, i.e., $\varphi_I\rightarrow\varphi_J$ does not imply $\varphi_J\rightarrow\varphi_I$.

- If $\varphi_I\rightarrow\varphi_J$, we also say that φ_I is linked to φ_J via a cut-n-SP chain. If $\varphi_I\rightarrow\varphi_J$ and $\varphi_J\rightarrow\varphi_K$, then we say φ_I is linked to φ_K via the cut-n-SP chain $\varphi_I\rightarrow\varphi_J\rightarrow\varphi_K$, we denote it as $\varphi_I\rightarrow\!\!\rightarrow\varphi_J$. Similar understanding for more forms φ.

With the above definition the condition of $(\eta^{-1})_{JK}$ could be non-zero is $\varphi_J\rightarrow\!\!\rightarrow\varphi_K$. Obviously, if $\mathcal{B}_{\hat{I}}\subset\mathcal{B}_{\hat{J}}$, $\varphi_I\rightarrow\!\!\rightarrow\varphi_J$ could not be true.

The contributions for $\langle\dot{\varphi}_{I;\hat{J}}|\varphi_{J;\hat{J}}\rangle$ are still calculated via (2.90) and (2.91). However, every element in these expressions should be replaced by the cut one, corresponding to the maximal cut $\mathcal{B}_{\hat{J}}$. One difference the cut leads to is the transcendent weight of coefficient $\gamma_{\hat{J}}^{(\alpha)}/\boldsymbol{\gamma}^{(\alpha)}$, which changes from $n-1$ to $n_{\hat{J}}-1$. Similarly, contributions for $(\eta^{-1})_{JK}$ is calculated via (2.92), but the weight of coefficient becomes $-n_{\hat{J}}$. One point that needs to be emphasized is that since we avoid regulators at the beginning, only $a\epsilon$ appears in β_i and hypersurface-powers. Weight-n coefficient could only be proportional to ϵ^{-n}.

With the above discussions, we have the selection rules of the CDE (improved and exact version):

- $(\hat{\mathrm{d}}\Omega)_{IK} = \langle \dot{\varphi}_{I;\hat{J}} | \varphi_{J;\hat{J}} \rangle \, (\eta^{-1})_{JK}$ could be non-zero only when there exist at least one φ_J that $\dot{\varphi}_{I;\hat{J}}$ and $\varphi_{J;\hat{J}}$ share $n_{\hat{J}}$-SP or $(n_{\hat{J}}-1)$-SP and $\varphi_J \to\to \varphi_K$.
- $(\hat{\mathrm{d}}\Omega)_{IK}$ could be determined via merely LO contribution formula of intersection number. It gives cut version of (2.90) and (2.91) for $n_{\hat{J}}$-SP or $(n_{\hat{J}}-1)$-SP contribution in $\langle \dot{\varphi}_{I;\hat{J}} | \varphi_{J;\hat{J}} \rangle$ and cut vertion of (2.92) for $(\eta^{-1})_{JK}$.
- $\langle \dot{\varphi}_{I;\hat{J}} | \varphi_{J;\hat{J}} \rangle \, (\eta^{-1})_{JK}$ is weight-(-1) coefficient, which could only be propotional to ϵ, times $\hat{\mathrm{d}}\log$-form. Thus canonical differential equation emerges.

We know that the differential of a Feynman integral belonging to a subsector will not rely on a higher sector, so

$$\text{If } \mathcal{B}_{\hat{I}} \subset \mathcal{B}_{\hat{K}}\,, \text{ then } \hat{\mathrm{d}}\Omega_{IK} = 0\,. \tag{3.5}$$

Unfortunately, with regulators the structure of the sector has been broken, i.e., there is also no explicit reason to tell us if $\mathcal{B}_{\hat{I}} \subset \mathcal{B}_{\hat{K}}$. Thus the statement (3.5) is not true and $\hat{\mathrm{d}}\Omega_{IK}$ vanishes when and only when taking regulators to zero after the computation of intersection number. This is one of the manifestations of redundancy when using the regulator method. However, with dual form in relative cohomology, $\langle \dot{\varphi}_{I;\hat{J}} | \varphi_{J;\hat{J}} \rangle$ could be non-zero only when $\mathcal{B}_{\hat{J}} \subseteq \mathcal{B}_{\hat{I}}$. Furthermore, $(\eta^{-1})_{JK}$ could be non-zero only when $\mathcal{B}_{\hat{K}} \subseteq \mathcal{B}_{\hat{J}}$. Therefore, $(\hat{\mathrm{d}}\Omega)_{IK}$ could be non-zero only when $\mathcal{B}_{\hat{K}} \subseteq \mathcal{B}_{\hat{I}}$ and (3.5) holds true as well.

4 Univariate example: $u = z^\delta (z-c_1)^\epsilon (z-c_2)^\epsilon$

In this section, we consider the case $u = z^\delta (z-c_1)^\epsilon (z-c_2)^\epsilon$, where δ is a regulator. This structure appears in many cases of Feynman integral. For example, all n-point one-loop integrals with their $n-1$ propagator cut will give such a u in their Baikov representation. We will compute its CDE via both methods, with and without regulators.

For the given u, the related hypersurfaces are

$$\mathcal{B} = \{z=0, z-c_1=0, z-c_2=0, z=\infty\}\,. \tag{4.1}$$

Its $\mathrm{d}\log$ basis can be easily constructed as

$$\varphi_1 = \frac{\mathrm{d}z}{z} = \mathrm{d}\log z\,, \quad \varphi_2 = \frac{\mathrm{d}z}{z-c_1} - \frac{\mathrm{d}z}{z-c_2} = \mathrm{d}\log\left(\frac{z-c_1}{z-c_2}\right)\,. \tag{4.2}$$

4.1 With regulator

With a regulator, we have

$$\omega = \mathrm{d}\log u = \left(\frac{\delta}{z} + \frac{\epsilon}{z-c_1} + \frac{\epsilon}{z-c_2}\right)\mathrm{d}z\,. \tag{4.3}$$

Poles of ω are

$$\{\boldsymbol{p}\} = \{0, c_1, c_2, \infty\} \tag{4.4}$$

and the hypersurface powers corresponding to each pole are

$$\gamma_1 = \delta\,, \quad \gamma_2 = \epsilon\,, \quad \gamma_3 = \epsilon\,, \quad \gamma_4 = -2\epsilon - \delta\,. \tag{4.5}$$

Now we compute $\eta_{IJ} = \langle \varphi_I | \varphi_J \rangle$ using the result (2.18) and (2.24). For η_{11}, φ_1 shares 1-SP with itself at the points 0 and ∞. For η_{12}, φ_1 do not share any pole with φ_2. For φ_2, it shares 1-SP with itself at the points c_1 and c_2. Thus we have we have

$$\begin{aligned} \eta = \langle \varphi_I | \varphi_J \rangle &= \begin{pmatrix} \frac{1}{\gamma_1} + \frac{1}{\gamma_4} & 0 \\ 0 & \frac{1}{\gamma_2} + \frac{1}{\gamma_3} \end{pmatrix} = \begin{pmatrix} \frac{1}{\delta} - \frac{1}{\delta + 2\epsilon} & 0 \\ 0 & \frac{2}{\epsilon} \end{pmatrix}, \\ \eta^{-1} &= \begin{pmatrix} \delta(\delta + 2\epsilon)/(2\epsilon) & 0 \\ 0 & \epsilon/2 \end{pmatrix}. \end{aligned} \tag{4.6}$$

Let's turn to consider $\langle \dot{\varphi}_I | \varphi_J \rangle$. Using (2.12) we have $\dot{\varphi}_I$

$$\begin{aligned} \dot{\varphi}_1 &= \mathrm{D}\left(\log \bar{u}_1(z) + \delta \log z\right) \wedge \mathrm{D} \log z \\ &= \mathrm{D} \log \bar{u}_1(z) \wedge \mathrm{D} \log z = \mathrm{d} \log \bar{u}_1(z) \wedge \hat{\mathrm{d}} \log z + \hat{\mathrm{d}} \log \bar{u}_1(z) \wedge \mathrm{d} \log z\,, \\ \dot{\varphi}_2 &= \mathrm{D} \log \bar{u}_2(z) \wedge \mathrm{D} \log(z - c_1) - \mathrm{D} \log \bar{u}_3(z) \wedge \mathrm{D} \log(z - c_2) \\ &= \delta\ \mathrm{D} \log z \wedge \varphi_2 + 2\epsilon\ \mathrm{D} \log(z - c_2) \wedge \mathrm{D} \log(z - c_1)\,. \end{aligned} \tag{4.7}$$

As a reminder, the $\hat{\mathrm{d}}$ here is a formal differential operator with respect to arbitrary one parameter. Since it could act on c_1 as well as c_2, we should keep both $\hat{\mathrm{d}} \log(z - c_1)$ and $\hat{\mathrm{d}} \log(z - c_2)$. Notice that even for $\hat{\mathrm{d}} \log z$, we still keep it, because we regard $\hat{\mathrm{d}} \log z$ as $\hat{\mathrm{d}} \log(z - c_0)$ with $c_0 = 0$ and $\hat{\mathrm{d}}$ could act on c_0 as well. In (4.7), the $\bar{u}_i(z)$ is defined in (2.16) and for current example they are

$$\bar{u}_1(z) = (z - c_1)^\epsilon (z - c_2)^\epsilon\,, \bar{u}_2(z) = z^\delta (z - c_2)^\epsilon\,, \bar{u}_3(z) = z^\delta (z - c_1)^\epsilon\,. \tag{4.8}$$

As a pedagogical example, we show the computation details of $\langle \dot{\varphi}_1 | \varphi_1 \rangle$ here. First as the d-log form, φ_1 shares 1-SP with itself only, i.e., there is no contribution from 0-SP. For pole $z = 0$, the first term of $\dot{\varphi}_1$ in (4.7) gives zero contribution, while the second term gives the contribution

$$\frac{1}{\delta} \hat{\mathrm{d}} \log\left((z - c_1)^\epsilon (z - c_2)^\epsilon |_{z=0}\right) = \frac{\epsilon}{\delta} \hat{\mathrm{d}} \log(c_1 c_2)\,. \tag{4.9}$$

For the pole $z = \infty$ the second term gives the contribution

$$\frac{1}{-\delta - 2\epsilon} \hat{\mathrm{d}} \log\left((1 - c_1 t)^\epsilon (1 - c_2 t)^\epsilon |_{t=0}\right) = \frac{\epsilon}{-\delta - 2\epsilon} \hat{\mathrm{d}} \log 1 = 0\,, \tag{4.10}$$

where $t = 1/z$. For $\langle \dot{\varphi}_1 | \varphi_2 \rangle$, φ_1 shares the 0-SP with φ_2 at the poles c_1, c_2. The contribution is $\hat{\mathrm{d}} \log\left(c_2/c_1\right)$, which can be seen from the first term of $\dot{\varphi}_1$ in (4.7). After finishing all these computations, we have

$$\left(\langle \dot{\varphi}_I | \varphi_J \rangle \right) = \hat{\mathrm{d}} \begin{pmatrix} \frac{\epsilon}{\delta} \log\left(c_1 c_2\right) & \log\left(c_2/c_1\right) \\ \log\left(c_2/c_1\right) & \frac{\delta}{\epsilon} \log\left(c_1 c_2\right) + 4 \log(c_1 - c_2) \end{pmatrix}. \tag{4.11}$$

Then, we have

$$\hat{\mathrm{d}}\Omega(c_i;\epsilon,\delta) = \hat{\mathrm{d}}\begin{pmatrix} \frac{\delta+2\epsilon}{2}\log(c_1c_2) & \frac{\epsilon}{2}\log\left(\frac{c_2}{c_1}\right) \\ \frac{\delta(\delta+2\epsilon)}{2\epsilon}\log\left(\frac{c_2}{c_1}\right) & \frac{\delta}{2}\log(c_1c_2)+2\epsilon\log(c_1-c_2) \end{pmatrix}. \tag{4.12}$$

At this moment, one sees that $\hat{\mathrm{d}}\Omega_{21}\neq 0$, which is not physical since it means that the decomposition of the differential of the subsector integral will rely on the higher sector. This is the consequence of involving a regulator, just as we have discussed in the last two sections. Take the regulator δ to be zero, we get the final result

$$\hat{\mathrm{d}}\Omega(c_i;\epsilon,0) = \epsilon\,\hat{\mathrm{d}}\begin{pmatrix} \log(c_1c_2) & \frac{1}{2}\log(c_2/c_1) \\ 0 & 2\log(c_1-c_2) \end{pmatrix}. \tag{4.13}$$

.

4.2 With relative cohomology

In this case, we have

$$\begin{aligned} u &= (z-c_1)^{\epsilon}(z-c_2)^{\epsilon}\,, \quad D_1 = z\,, \\ \omega &= \mathrm{d}\log u = \left(\frac{\epsilon}{z-c_1}+\frac{\epsilon}{z-c_2}\right)\mathrm{d}z\,. \end{aligned} \tag{4.14}$$

The dual basis in relative cohomology is now $\Delta_{\hat{1}}\varphi_1, \Delta_{\hat{2}}\varphi_2$. Again we need to consider contributions from the following locations

$$\{\boldsymbol{p}\} = \{0, c_1, c_2, \infty\}\,. \tag{4.15}$$

Computations of $\langle\varphi_I|\Delta_{\hat{1}}\varphi_1\rangle$ will be changed due to cutting[6]

$$\begin{aligned} \langle\varphi_1|\Delta_{\hat{1}}\varphi_1\rangle &= \langle\varphi_{1;\hat{1}}|\varphi_{1;\hat{1}}\rangle = \langle 1|1\rangle = 1 \\ \langle\varphi_2|\Delta_{\hat{1}}\varphi_1\rangle &= \langle\varphi_{2;\hat{1}}|\varphi_{1;\hat{1}}\rangle = \langle 0|1\rangle = 0 \\ \langle\varphi_1|\Delta_{\hat{2}}\varphi_2\rangle &= \langle\varphi_1|\varphi_2\rangle = 0 \\ \langle\varphi_2|\Delta_{\hat{2}}\varphi_2\rangle &= \langle\varphi_2|\varphi_2\rangle = \frac{2}{\epsilon}\,. \end{aligned} \tag{4.16}$$

For $\langle\dot{\varphi}_I|\Delta_{\hat{J}}\varphi_J\rangle$, $\dot{\varphi}_1$ is same as the one given in (4.7) while

$$\begin{aligned} \dot{\varphi}_2 &= \mathrm{D}\log u\wedge\mathrm{D}\log(z-c_1) - \mathrm{D}\log u\wedge\mathrm{D}\log(z-c_2) \\ &= 2\epsilon\,\mathrm{D}\log(z-c_2)\wedge\mathrm{D}\log(z-c_1) \end{aligned} \tag{4.17}$$

thus we have

$$\begin{aligned} \langle\dot{\varphi}_1|\Delta_{\hat{1}}\varphi_1\rangle &= \langle\dot{\varphi}_{1;\hat{1}}|\varphi_{1;\hat{1}}\rangle = \langle\hat{\mathrm{d}}\log u|_{z=0}|1\rangle = \epsilon\hat{\mathrm{d}}\log(c_1c_2) \\ \langle\dot{\varphi}_2|\Delta_{\hat{1}}\varphi_1\rangle &= \langle\dot{\varphi}_{2;\hat{1}}|\varphi_{1;\hat{1}}\rangle = \langle 0|1\rangle = 0 \\ \langle\dot{\varphi}_2|\Delta_{\hat{2}}\varphi_2\rangle &= \langle\dot{\varphi}_2|\varphi_2\rangle = 4\epsilon\hat{\mathrm{d}}\log(c_1-c_2) \\ \langle\dot{\varphi}_1|\Delta_{\hat{2}}\varphi_2\rangle &= \langle\dot{\varphi}_1|\varphi_2\rangle = \epsilon\hat{\mathrm{d}}\log(c_2/c_1) \end{aligned} \tag{4.18}$$

[6]For this example, z corresponds to the propagator, so there is no cut available for φ_2, so we have $\langle\star|\Delta_{\hat{2}}\varphi_2\rangle = \langle\star|\varphi_2\rangle$.

Putting all together we have

$$\eta = \langle \varphi_{I;\hat{J}} | \varphi_{J;\hat{J}} \rangle = \begin{pmatrix} 1 & 0 \\ 0 & 2/\epsilon \end{pmatrix}, \quad \eta^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon/2 \end{pmatrix},$$
$$\Big(\langle \dot{\varphi}_{I;\hat{J}} | \varphi_{J;\hat{J}} \rangle \Big) = \hat{\mathrm{d}} \begin{pmatrix} \epsilon \log(c_1 c_2) & \epsilon \log(c_2/c_1) \\ 0 & 4\log(c_1 - c_2) \end{pmatrix}$$
$$\hat{\mathrm{d}}\Omega = \Big(\langle \dot{\varphi}_{I;\hat{J}} | \varphi_{J;\hat{J}} \rangle \Big) . \eta^{-1} = \epsilon\, \hat{\mathrm{d}} \begin{pmatrix} \log(c_1 c_2) & \frac{1}{2}\log(c_2/c_1) \\ 0 & 2\log(c_1 - c_2) \end{pmatrix}. \tag{4.19}$$

We get the same result as (4.13), without extra terms in the intermediate steps (4.12). Cut also makes the computation easier.

5 2-loop example: kite topology

Now we consider the kite topology defined by

$$\begin{aligned} &z_1 = l_1^2 - m^2\,,\ z_2 = (l_2 - p)^2 - m^2\,,\ z_3 = (l_1 - l_2)^2\,, \\ &z_4 = l_2^2\,, \quad z_5 = (l_1 - p)^2\,, \quad p^2 = s\,, \end{aligned} \tag{5.1}$$

with cut z_1, z_2, z_3. We are going to compute it using the relative cohomology. For comparison, we also present the computation of η using the regulator method.

5.1 Computation of η with regulator

In the Baikov representation, u of this integral family with regulator is

$$\begin{aligned} u(z_4, z_5) &= z_4^{\delta_1} z_5^{\delta_2} \left[\mathcal{G}(z_4, z_5)\right]^{-\epsilon} \\ \mathcal{G}(z_4, z_5) &\equiv 4\mathrm{G}(l_1, l_2, p)|_{z_{1,2,3}=0} = -2m^6 + m^4(s + z_4 + z_5) \\ &\qquad + m^2(2z_4 z_5 - s z_4 - s z_5) + z_4 z_5 (s - z_4 - z_5)\,, \end{aligned} \tag{5.2}$$

where $\mathrm{G}(l_1, l_2, p)$ is Gram determinant defined by

$$\mathrm{G}(q_1, q_2, \cdots, q_n) \equiv |q_i \cdot q_j| \tag{5.3}$$

If ignoring the exchange symmetry $z_4 \leftrightarrow z_5$, there are four master integrals f_i in this integral family and their integrands can be constructed as $\mathrm{d}\log$-forms

$$\begin{aligned} f_i &= \int u \varphi_i \\ \varphi_1 &= \mathrm{d}\log(z_4) \wedge \mathrm{d}\log(z_5) = \frac{\mathrm{d}z_4 \mathrm{d}z_5}{z_4 z_5}\,, \\ \varphi_2 &= \mathrm{d}\log(\tau[z_4, m^2; r_{1;\pm}]) \wedge \mathrm{d}\log\left(\frac{z_5 - r_{5+}}{z_5 - r_{5-}}\right) = \frac{\sqrt{s(s-4m^2)}}{\mathcal{G}} \mathrm{d}z_4 \mathrm{d}z_5, \\ \varphi_3 &= -\mathrm{d}\log(\tau[z_4, \infty; r_{1;\pm}]) \wedge \mathrm{d}\log\left(\frac{z_5 - r_{5+}}{z_5 - r_{5-}}\right) = \frac{z_4 - m^2}{\mathcal{G}} \mathrm{d}z_4 \mathrm{d}z_5, \\ \varphi_4 &= -\mathrm{d}\log(\tau[z_5, \infty; r_{1;\pm}]) \wedge \mathrm{d}\log\left(\frac{z_4 - r_{4+}}{z_4 - r_{4-}}\right) = \frac{z_5 - m^2}{\mathcal{G}} \mathrm{d}z_4 \mathrm{d}z_5, \end{aligned} \tag{5.4}$$

where

$$\mathcal{G}_1(z_5) \equiv -4\mathrm{G}(l_1,p)|_{z_{1,2,3}=0} = (z_5-s)^2 + m^4 - 2m^2(z_5+s) \tag{5.5}$$

and various roots of quadratic polynomials are defined by

$$\begin{aligned} &r_\pm[ax^2+bx+c;x] \equiv \frac{-b\pm\sqrt{b^2-4ac}}{2a}\,, \\ &r_{1;\pm} \equiv r_\pm[\mathcal{G}_1(z_5);z_5]\,, \quad r_{4\pm}(z_5) \equiv r_\pm[\mathcal{G};z_4]\,, \quad r_{5\pm}(z_4) \equiv r_\pm[\mathcal{G};z_5]\,. \end{aligned} \tag{5.6}$$

One can see that with the above definition

$$r_{5+}(\infty) = \infty\,, \quad r_{5-}(\infty) = 0\,, \quad r_{5\pm}(m^2) = m^2\,. \tag{5.7}$$

Notice that if taking exchange symmetry $z_4 \leftrightarrow z_5$ into consideration, there are only three master integrals remaining and $\int u\varphi_3 = \int u\varphi_4$.

For the d log basis we have chosen, the location (z_4,z_5) of poles are

$$\boldsymbol{p} \in \Big\{(0,0),(m^2,m^2),(\infty,0),(0,\infty),(\infty,\infty)\Big\}\,. \tag{5.8}$$

To describe the behavior around $z_4 \to \infty$ and $z_5 \to \infty$, we use variables $t_4 = 1/z_4$ and $t_5 = 1/z_5$. With the changing of variables, u around poles $(\infty,0),(0,\infty),(\infty,\infty)$ becomes

$$u_{\infty 0} = t_4^{2\epsilon-\delta_1} z_5^{\delta_2}\, \mathcal{G}_{\infty 0}^{-\epsilon}, \quad u_{0\infty} = z_4^{\delta_2} t_5^{2\epsilon-\delta_2}\, \mathcal{G}_{0\infty}^{-\epsilon}, \quad u_{\infty\infty} = t_4^{2\epsilon-\delta_1} t_5^{2\epsilon-\delta_2}\, \mathcal{G}_{\infty\infty}^{-\epsilon}\,, \tag{5.9}$$

where

$$\begin{aligned} \mathcal{G}_{\infty 0}(t_4,z_5) \equiv t_4^2\, \mathcal{G}(1/t_4,z_5) &= -2m^6t_4^2+m^4st_4^2+m^4t_4+m^4t_4^2z_5-m^2st_4 \\ &\quad -m^2st_4^2z_5+2m^2t_4z_5+st_4z_5-t_4z_5^2-z_5\,, \\ \mathcal{G}_{0\infty}(z_4,t_5) \equiv t_5^2\, \mathcal{G}(z_4,1/t_5) &= \mathcal{G}_{\infty 0}(t_4\to t_5, z_5\to z_4)\,, \\ \mathcal{G}_{\infty\infty}(t_4,t_5) \equiv t_4^2t_5^2\, \mathcal{G}(1/t_4,1/t_5) &= (-2m^6+m^4s)t_4^2t_5^2+(m^4-m^2s)t_4t_5^2 \\ &\quad +(m^4-m^2s)t_4^2t_5+2m^2t_4t_5+st_4t_5-t_4-t_5\,. \end{aligned} \tag{5.10}$$

Let us analysis the pole $t_4=0, z_5=0$ for $u_{\infty 0}$. From the explicit expression of $\mathcal{G}_{\infty 0}(t_4,z_5)$, we can see the leading terms are $m^4t_4 - m^2st_4 - z_5$. Thus three hypersurfaces $z_5 = 0$, $t_4 = 0$ and $m^4t_4 - m^2st_4 - z_5$ meet at $(z_4,z_5) = (\infty,0)$, so $(\infty,0)$ is a degenerate pole. Using the relation $\mathcal{G}_{0\infty}(z_4,t_5) = \mathcal{G}_{\infty 0}(t_4 \to t_5, z_5 \to z_4)$, we see that $(z_4,z_5) = (0,\infty)$ is a degenerate pole. Finally, the leading term of $\mathcal{G}_{\infty\infty}(t_4,t_5)$ is $-t_4-t_5$, so $(z_4,z_5) = (\infty,\infty)$ is a degenerate pole. In [83], we have studied the polynomial $\mathcal{G}$ around each pole using the Newton polytopes of $\mathcal{G}$, where each monomial corresponds to a vertex of the polytope. Here we will not discuss this point further.

With the above analysis, for poles given in (5.8), there are a total 12 contours used for the computation. For the last three degenerated poles, each pole has used 3 contours. The residue of pole (m^2,m^2) is the composite residue [94–96], for which we use two contours to compute it. Now we list the contour and the corresponding factorization transformations for the computation of residue when computing the intersection number. For $\boldsymbol{p}=(0,0)$, it does not need the factorization. However, for the sake of formal uniformity, we denote identity transformation as $\mathrm{T}^{(1)}$

$$\mathrm{T}^{(1)}:(\{z_4\},\{z_5\})\,,\quad z_4\to x_1^{(1)}\,,\ z_5\to x_2^{(1)}\,. \tag{5.11}$$

For $\boldsymbol{p}=(m^2,m^2)$, involved contours and factorizations could be chosen as

$$\begin{aligned}
&\mathrm{T}^{(2)}:(\{z_4-m^2,z_5-r_{5+}\},\{z_5-r_{5-}\}),\\
&\qquad z_4\to x_1^{(2)}x_2^{(2)}+r_+[\mathcal{G}(x_1^{(2)},x_2^{(2)});x_1^{(2)}]\,,\quad z_5\to x_2^{(2)}\,,\\
&\mathrm{T}^{(3)}:(\{z_4-m^2,z_5-r_{5-}\},\{z_5-r_{5+}\}),\\
&\qquad z_5\to x_1^{(3)}x_2^{(3)}+r_+[\mathcal{G}(x_1^{(3)},x_2^{(3)});x_2^{(3)}]\,,\quad z_4\to x_2^{(3)}\,.
\end{aligned} \tag{5.12}$$

For $\boldsymbol{p}=(\infty,0)$, they are

$$\begin{aligned}
&\mathrm{T}^{(4)}:(\{t_4\},\{z_5,z_5-r_-[\mathcal{G}_{\infty 0};z_5]\}),,\\
&\qquad t_4\to x_1^{(4)}x_2^{(4)}\,,\quad z_5\to x_2^{(4)}\,,\\
&\mathrm{T}^{(5)}:(\{t_4,z_5-r_-[\mathcal{G}_{\infty 0};z_5]\},\{z_5\}),\\
&\qquad t_4\to x_1^{(5)}\,,\quad z_5\to x_1^{(5)}x_2^{(5)}\,,\\
&\mathrm{T}^{(6)}:(\{z_5-r_-[\mathcal{G}_{\infty 0};z_5]\},\{t_4,z_5\}),\\
&\qquad t_4\to x_1^{(6)}x_2^{(6)}+r_+[\mathcal{G}_{\infty 0}(x_1^{(6)},x_2^{(6)});x_1^{(6)}]\,,\quad z_5\to x_2^{(6)}\,.
\end{aligned} \tag{5.13}$$

For $\boldsymbol{p}=(0,\infty)$, contours are

$$\begin{aligned}
&\mathrm{T}^{(7)}:(\{t_5\},\{z_4,z_4-r_-[\mathcal{G}_{0\infty};z_4]\}),\\
&\mathrm{T}^{(8)}:(\{t_5,z_4-r_-[\mathcal{G}_{0\infty};z_4]\}\{z_4\}),\\
&\mathrm{T}^{(9)}:(\{z_4-r_-[\mathcal{G}_{0\infty};z_4]\},\{z_4,t_5\}),
\end{aligned} \tag{5.14}$$

and their factorization could be obtained from $\mathrm{T}^{(4,5,6)}$ via $z_4\leftrightarrow z_5$ symmetry. For $\boldsymbol{p}=(\infty,\infty)$, contours and factorizations are

$$\begin{aligned}
&\mathrm{T}^{(10)}:(\{t_4\},\{t_5,t_4-r_+[\mathcal{G}_{\infty\infty};t_4]\}),\\
&\qquad t_4\to x_1^{(10)}x_2^{(10)}\,,\quad t_5\to x_2^{(10)}\,,
\end{aligned} \tag{5.15}$$

$$\begin{aligned}
&\mathrm{T}^{(11)}:(\{t_4,t_4-r_+[\mathcal{G}_{\infty\infty};t_4]\},\{t_5\}),\\
&\qquad t_4\to x_1^{(11)}\,,\quad t_5\to x_1^{(11)}x_2^{(11)}\,,
\end{aligned} \tag{5.16}$$

$$\begin{aligned}
&\mathrm{T}^{(12)}:(\{t_4-r_+[\mathcal{G}_{\infty\infty};t_4]\},\{t_4,t_5\})\\
&\qquad t_4\to x_1^{(12)}x_2^{(12)}+r_+[\mathcal{G}_{\infty\infty}(x_1^{(12)},x_2^{(12)});x_1^{(12)}]\,,\quad t_5\to x_2^{(12)}\,.
\end{aligned} \tag{5.17}$$

From these factorizations, we can read out hypersurface powers as

$$\begin{aligned}
&\boldsymbol{\gamma}^{(1)}=\delta_1\delta_2\,,\quad \boldsymbol{\gamma}^{(2,3)}=(-2\epsilon)(-\epsilon)\,,\quad && \boldsymbol{\gamma}^{(7,8,9)}=\boldsymbol{\gamma}^{(4,5,6)}\Big|_{\delta_1\leftrightarrow\delta_2}\,,\\
&\boldsymbol{\gamma}^{(4)}=(\epsilon-\delta_1+\delta_2)\,(2\epsilon-\delta_1)\,, && \boldsymbol{\gamma}^{(10)}=(3\epsilon-\delta_1-\delta_2)\,(2\epsilon-\delta_1)\,,\\
&\boldsymbol{\gamma}^{(5)}=(\epsilon-\delta_1+\delta_2)\,\delta_2\,, && \boldsymbol{\gamma}^{(11)}=(3\epsilon-\delta_1-\delta_2)\,(2\epsilon-\delta_2)\,,\\
&\boldsymbol{\gamma}^{(6)}=(\epsilon-\delta_1+\delta_2)\,(-\epsilon)\,, && \boldsymbol{\gamma}^{(12)}=(3\epsilon-\delta_1-\delta_2)\,(-\epsilon)\,.
\end{aligned} \tag{5.18}$$

One can notice that the hypersurface power $\beta_i(\beta_1+\beta_2+\beta_3)$ in (2.34) appears here for these degenerate poles again since they have the same degenerate structure (two dimensions, three hypersurfaces). $C_I^{(\mathbf{-1})}$ for each factorization are

$$\begin{aligned}
&\varphi_1:\{1,0,0,-1,1,0,-1,1,0,1,-1,0\}\,,\\
&\varphi_2:\{0,-1,-1,0,0,0,0,0,0,0,0,0\}\,,\\
&\varphi_3:\{0,0,0,1,0,-1,0,0,0,-1,0,1\}\,,\\
&\varphi_4:\{0,0,0,0,0,0,1,0,-1,0,1,-1\}\,.
\end{aligned} \tag{5.19}$$

We see that given a basis φ_1 some contours give a non-zero contribution and some contours, zero. This phenomenon can be easily understood. First, the basis should have the corresponding pole for the contour. Secondly, the poles should be properly grouped. For example, φ_3 at pole $(\infty,0)$ has two factors t_4 and $z_5-r_-[\mathcal{G}_{\infty 0};z_5]$ in the denominator, thus only the contour with the grouping $(\{t_4,\star\},\{z_5-r_-[\mathcal{G}_{\infty 0};z_5],\star\})$ (the $\star$ could be no element or several elements) can have a non-zero contribution. One can see that $\mathrm{T}^{(4)}$ and $\mathrm{T}^{(6)}$ satisfy these conditions, thus the related $C_3^{(\mathbf{-1})}$ is non-zero. Similar understanding for other bases.

Collecting everything together by (2.24) we have

$$\begin{aligned}
\eta=&\begin{pmatrix}
\sum_{\alpha=1,4,5,7,8,10,11}\frac{1}{\gamma^{(\alpha)}} & 0 & -\frac{1}{\gamma^{(4)}}-\frac{1}{\gamma^{(10)}} & -\frac{1}{\gamma^{(7)}}-\frac{1}{\gamma^{(11)}}\\
0 & \frac{1}{\gamma^{(2)}}+\frac{1}{\gamma^{(3)}} & 0 & 0\\
-\frac{1}{\gamma^{(4)}}-\frac{1}{\gamma^{(10)}} & 0 & \sum_{\alpha=4,6,10,12}\frac{1}{\gamma^{(\alpha)}} & -\frac{1}{\gamma^{(12)}}\\
-\frac{1}{\gamma^{(7)}}-\frac{1}{\gamma^{(11)}} & 0 & -\frac{1}{\gamma^{(12)}} & \sum_{\alpha=7,9,11,12}\frac{1}{\gamma^{(\alpha)}}
\end{pmatrix}\\
=&\begin{pmatrix}
\frac{\epsilon\left(2\delta_1\epsilon+2\delta_2\epsilon-\delta_1^2-\delta_2^2+2\delta_1\delta_2+3\epsilon^2\right)}{\delta_1\delta_2(-\delta_1-\delta_2+3\epsilon)(\delta_1-\delta_2+\epsilon)(-\delta_1+\delta_2+\epsilon)} & 0 & -\frac{2}{(-\delta_1-\delta_2+3\epsilon)(-\delta_1+\delta_2+\epsilon)} & -\frac{2}{(-\delta_1-\delta_2+3\epsilon)(\delta_1-\delta_2+\epsilon)}\\
0 & \frac{1}{\epsilon^2} & 0 & 0\\
-\frac{2}{(-\delta_1-\delta_2+3\epsilon)(-\delta_1+\delta_2+\epsilon)} & 0 & -\frac{2(\epsilon-\delta_1)}{\epsilon(-\delta_1-\delta_2+3\epsilon)(-\delta_1+\delta_2+\epsilon)} & \frac{1}{\epsilon(-\delta_1-\delta_2+3\epsilon)}\\
-\frac{2}{(-\delta_1-\delta_2+3\epsilon)(\delta_1-\delta_2+\epsilon)} & 0 & \frac{1}{\epsilon(-\delta_1-\delta_2+3\epsilon)} & -\frac{2(\epsilon-\delta_2)}{\epsilon(-\delta_1-\delta_2+3\epsilon)(\delta_1-\delta_2+\epsilon)}
\end{pmatrix}
\end{aligned} \tag{5.20}$$

and

$$\eta^{-1}=\begin{pmatrix}
\frac{\delta_1\delta_2(-\delta_1-\delta_2+\epsilon)}{\epsilon} & 0 & -2\delta_1\delta_2 & -2\delta_1\delta_2\\
0 & \epsilon^2 & 0 & 0\\
-2\delta_1\delta_2 & 0 & -2\epsilon\left(\delta_2+\epsilon\right) & -\epsilon\left(\delta_1+\delta_2+\epsilon\right)\\
-2\delta_1\delta_2 & 0 & -\epsilon\left(\delta_1+\delta_2+\epsilon\right) & -2\epsilon\left(\delta_1+\epsilon\right)
\end{pmatrix}. \tag{5.21}$$

5.2 Computation of $\boldsymbol{\eta}$ with relative cohomology

In this case, we have

$$\begin{aligned}
&u(z_4,z_5)=[\mathcal{G}(z_4,z_5)]^{-\epsilon}\,,\quad D_1=z_4\,,\ D_2=z_5\,,\\
&\qquad u_{\infty 0}=t_4^{2\epsilon}\,\mathcal{G}_{\infty 0}^{-\epsilon}\,,\quad u_{0\infty}=t_5^{2\epsilon}\mathcal{G}_{0\infty}^{-\epsilon}\,,\quad u_{\infty\infty}=t_4^{2\epsilon}t_5^{2\epsilon}\,\mathcal{G}_{\infty\infty}^{-\epsilon}
\end{aligned} \tag{5.22}$$

where $\mathcal{G}$'s are given in (5.10). Using the same $\mathrm{d}\log$ integrand we have contructed in (5.4), again we need to consider poles located at

$$\boldsymbol{p}\in\Big\{(0,0),(m^2,m^2),(\infty,0),(0,\infty),(\infty,\infty)\Big\}\,. \tag{5.23}$$

For the pole $(0,0)$ it can only contribute to the top sector. The contour $\mathrm{T}^{(1)}$ in previous subsection should be replaced by the contour $\mathrm{T}^{(13)}$, which transform nothing, i.e., $\mathrm{T}^{(13)}:()$. The evaluation of intersection numbers is just like the first line of (4.16), i.e., constant to constant.

For poles $(\infty,0)$ and $(0,\infty)$, the discussion is a little bit tricky. When we consider $\langle\varphi_{I;\hat{j}}|\varphi_{J;\hat{j}}\rangle\,,\ I,J=2,3,4$, since there is no z_5 in $u_{\infty 0}$ and no z_4 in $u_{0\infty}$, they are no longer degenerate poles between subsectors $\mathcal{B}_{\hat{2}}=\mathcal{B}_{\hat{3}}=\mathcal{B}_{\hat{4}}$ for basis $\varphi_2,\varphi_3,\varphi_4$ (given in (5.4)) containing no z_4,z_5. Thus when compute the intersection number, for pole $(\infty,0)$ we can use the contour and the factorization

$$
\begin{aligned}
&\mathrm{T}^{(14)}:(\{t_4\},\{z_5-r_-[\mathcal{G}_{\infty 0};z_5]\})\,,\\
&\qquad t_4\to x_1^{(4)}\,,\quad z_5\to x_2^{(14)}+r_+[\mathcal{G}_{\infty 0}(x_1^{(14)},x_2^{(14)});x_2^{(14)}]\,,
\end{aligned}
\tag{5.24}
$$

and for pole $(0,\infty)$, the contour and factorization

$$
\begin{aligned}
&\mathrm{T}^{(15)}:(\{t_5\},\{z_4-r_-[\mathcal{G}_{\infty 0};z_4]\})\,,\\
&\qquad t_5\to x_1^{(15)}\,,\quad z_4\to x_2^{(15)}+r_+[\mathcal{G}_{0\infty}(x_1^{(15)},x_2^{(15)});x_2^{(15)}]\,.
\end{aligned}
\tag{5.25}
$$

One can check that the contributions of $\mathrm{T}^{(14)}$ and $\mathrm{T}^{(15)}$ can be obtained via sum the contribution of $\mathrm{T}^{(4,6)}$ and $\mathrm{T}^{(7,9)}$ respectively, for example, $1/\boldsymbol{\gamma}^{(14)}=1/\boldsymbol{\gamma}^{(4)}+1/\boldsymbol{\gamma}^{(6)}$, which is nothing, but the (2.60) in previous section.

However, when we compute $\langle\varphi_{1;\hat{j}}|\varphi_{J;\hat{j}}\rangle\,,\ J=2,3$, since φ_1 contains the denominator z_4z_5, poles $(\infty,0)$ and $(0,\infty)$ are still degenerate and we should use the contours $\mathrm{T}^{(4,5,6)}$ and $\mathrm{T}^{(7,8,9)}$. From (5.19) one can see that the fifth and the eighth components of vector $C_I^{(\mathbf{-1})}$ of $\varphi_{2,3,4}$ are zero, so the contributions from $\mathrm{T}^{(5,8)}$ are zero.

The above two situations can be combined into the statement that we should use the contours $\mathrm{T}^{(4,6,7,9)}$ for the computation of all intersection numbers. Then, all the hypersurface powers are

$$
\begin{aligned}
&\boldsymbol{\gamma}^{(13)}=1\,,\quad \boldsymbol{\gamma}^{(2,3)}=(-2\epsilon)(-\epsilon)\,,\quad \boldsymbol{\gamma}^{(4)}=\boldsymbol{\gamma}^{(7)}=2\epsilon(\epsilon)\,,\quad \boldsymbol{\gamma}^{(6)}=\boldsymbol{\gamma}^{(9)}=-\epsilon(\epsilon)\,,\\
&\boldsymbol{\gamma}^{(10)}=3\epsilon\,(2\epsilon)\,,\quad \boldsymbol{\gamma}^{(11)}=3\epsilon\,(2\epsilon)\,,\quad \boldsymbol{\gamma}^{(12)}=3\epsilon(-\epsilon)\,.
\end{aligned}
\tag{5.26}
$$

The $C_I^{(\mathbf{-1})}$ for $\mathrm{T}^{(13,2,3,4,6,7,9,10,11,12)}$ are

$$
\begin{aligned}
&\varphi_1:\{1,0,0,-1,0,-1,0,1,-1,0\}\,,\\
&\varphi_2:\{0,-1,-1,0,0,0,0,0,0,0\}\,,\\
&\varphi_3:\{0,0,0,1,-1,0,0,-1,0,1\}\,,\\
&\varphi_4:\{0,0,0,0,0,1,-1,0,1,-1\}\,.
\end{aligned}
\tag{5.27}
$$

Using (2.24) one could easily compute η and η^{-1} as

$$\eta = \begin{pmatrix} 1 & 0 & -\frac{1}{\gamma^{(4)}} - \frac{1}{\gamma^{(10)}} & -\frac{1}{\gamma^{(7)}} - \frac{1}{\gamma^{(11)}} \\ 0 & \frac{1}{\gamma^{(2)}} + \frac{1}{\gamma^{(3)}} & 0 & 0 \\ 0 & 0 & \sum_{\alpha=4,6,10,12} \frac{1}{\gamma^{(\alpha)}} & -\frac{1}{\gamma^{(12)}} \\ 0 & 0 & -\frac{1}{\gamma^{(12)}} & \sum_{\alpha=7,9,11,12} \frac{1}{\gamma^{(\alpha)}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{2}{3\epsilon^2} & -\frac{2}{3\epsilon^2} \\ 0 & \frac{1}{\epsilon^2} & 0 & 0 \\ 0 & 0 & -\frac{2}{3\epsilon^2} & \frac{1}{3\epsilon^2} \\ 0 & 0 & \frac{1}{3\epsilon^2} & -\frac{2}{3\epsilon^2} \end{pmatrix}$$

$$\eta^{-1} = \begin{pmatrix} 1 & 0 & -2 & -2 \\ 0 & \epsilon^2 & 0 & 0 \\ 0 & 0 & -2\epsilon^2 & -\epsilon^2 \\ 0 & 0 & -\epsilon^2 & -2\epsilon^2 \end{pmatrix} \tag{5.28}$$

Compared to the computation (5.20) with regulator, one could see that we have simpler expressions in intermediate steps.

5.3 $\mathrm{d}\Omega_{1J}$ with relative cohomology

Let's notice that φ_2 and φ_3 belong to subsector whose loop-by-loop Baikov representation corresponds to an univariate problem with $u = (z-m^2)^{-2\epsilon} z^{\epsilon}(z^2+m^4+s^2-2zm^2-2sm^2-2zs)^{-1/2-\epsilon}$, and $\varphi_4{=}\varphi_3$ by symmetry. Since such univariate cases have been systematically discussed and are easily computed using techniques for univariate problems [83], we do not compute them using 2-variable $u(z_4, z_5)$ here. Hence in this subsection, we compute $\mathrm{d}\Omega_{1J}$ only.

For $\mathrm{d}\Omega_{11}$, from the non zero terms in $(\eta^{-1})_{J1}$ (5.28), only $\langle\dot\varphi_{1;\hat 1}|\varphi_{1;\hat 1}\rangle\,(\eta^{-1})_{11}$ could contribute. Thus we only need to compute

$$\mathrm{d}\Omega_{11} = \langle\dot\varphi_{1;\hat 1}|\varphi_{1;\hat 1}\rangle = \hat{\mathrm{d}}\log u(0,0) = -\epsilon(2\hat{\mathrm{d}}\log m^2 + \hat{\mathrm{d}}\log(s-2m^2))\,, \tag{5.29}$$

where (2.13) has been used for the maximal cut $z_4 = z_5 = 0$, i.e.,

$$\hat{\mathrm{d}}u(0,0) = (\hat{\mathrm{d}}\Omega)\; u(0,0) \to \hat{\mathrm{d}}\Omega = \hat{\mathrm{d}}\log u(0,0) \tag{5.30}$$

as a trivial 0-variable problem. If we compute with regulator, it will be more complicated, since by (5.21), $(\eta^{-1})_{11}$, $(\eta^{-1})_{31}$ and $(\eta^{-1})_{41}$ are all non-zero and the corresponding computations could not apply maximal cut as did for $\mathcal{B}_{\hat 1}$.

Now we move to $\mathrm{d}\Omega_{1J}$, $J = 2,3,4$. To get $\mathrm{d}\Omega_{12}$, we only need compute $\langle\dot\varphi_{1;\hat 2}|\varphi_{2;\hat 2}\rangle$. However, φ_1 and φ_2 do not share 2-SP or 1-SP. We have

$$\mathrm{d}\Omega_{12} = \epsilon^2\,\langle\dot\varphi_{1;\hat 2}|\varphi_{2;\hat 2}\rangle = 0 \tag{5.31}$$

To get $\mathrm{d}\Omega_{13}$, we need to compute $\langle\dot\varphi_{1;\hat\jmath}|\varphi_{J;\hat\jmath}\rangle$, $J = 1,3,4$. $J=1$ has been computed in (5.29). Due to the $z_4 \leftrightarrow z_5$ symmetry, $\langle\dot\varphi_{1;\hat 3}|\varphi_{3;\hat 3}\rangle = \langle\dot\varphi_{1;\hat 4}|\varphi_{3;\hat 4}\rangle$. For $\langle\dot\varphi_{1;\hat 3}|\varphi_{3;\hat 3}\rangle$, $\mathrm{T}^{(4,10)}$ lead to 2-SP, whose contributions vanish due to $\hat{\mathrm{d}}\log C = 0$. $\mathrm{T}^{(11,12)}$ lead to shared 1-SP, whose contributions also vanish due to $\hat{\mathrm{d}}\log C = 0$. Only $\mathrm{T}^{(5,6)}$ lead to the same shared 1-SP $(\{t_4,\star\},\{\star\})$ which give the only non-zero contribution

$$\langle\dot\varphi_{1;\hat 3}|\varphi_{3;\hat 3}\rangle = \frac{1}{\epsilon}\left(\hat{\mathrm{d}}\log(m^2) + \hat{\mathrm{d}}\log(s-m^2)\right)\,. \tag{5.32}$$

Then we have

$$
\begin{aligned}
\mathrm{d}\Omega_{13} &= \left(\eta^{-1}\right)_{13} \langle\dot{\varphi}_{1;\hat{1}}|\varphi_{1;\hat{1}}\rangle + \left[\left(\eta^{-1}\right)_{33} + \left(\eta^{-1}\right)_{43}\right)] \langle\dot{\varphi}_{1;\hat{3}}|\varphi_{3;\hat{3}}\rangle \\
&= (-2)(-\epsilon)\hat{\mathrm{d}}(2\log(m^2) + \log(s-2m^2)) + (-2\epsilon^2-\epsilon^2)\frac{1}{\epsilon}\hat{\mathrm{d}}\left(\log(m^2)+\log(s-m^2)\right) \\
&= \epsilon\ \hat{\mathrm{d}}(\log(m^2) + 2\log(s-2m^2) - 3\log(s-m^2))
\end{aligned}
\tag{5.33}
$$

Due to the $z_4 \leftrightarrow z_5$ symmetry, we have

$$
\mathrm{d}\Omega_{13} = \mathrm{d}\Omega_{14}\,. \tag{5.34}
$$

Now we have all $\hat{\mathrm{d}}\Omega_{1J}$.

6 Summary and outlook

In this paper, we focus on the computation of the CDE matrix with d log-form basis φ. By writing the $\dot{\varphi}$ in D log-form (2.13), all computations of intersection numbers can be reduced to the LO contribution form given in (2.24). Thus we can show that the CDE matrix is the $\hat{\mathrm{d}}$ log-form and if the powers of u are proportional to ϵ, then the differential equation is canonical. We have also shown that relative cohomology can simplify the computation of intersection numbers by giving a simple computation rule. Our selection rules, i.e., n-SP and (n-1)-SP contributions, have provided better insights into the structure of the CDE. We provide also careful treatment on the factorization of multivariate poles. To demonstrate the utility of the above analysis, we have presented detailed computations using two examples. Our results will help the understanding and application of intersection numbers.

Applying results in this paper to more complex examples necessitates an automatic and efficient algorithm for identifying all regions of poles and their factorization, which remains an open question to the best of our knowledge. However, our application of the analysis does not need to be so rigid. For instance, if we care about the symbol only, the factorization transformations corresponding to the contours of all multivariable poles do not need to be complete, i.e., like the case (2.56) without getting the full region. The even worse case is that we do not have a d log integrand basis, but the analysis in this paper can still provide us information about alphabet $\mathrm{W}^{(i)}(\boldsymbol{s})$. For example, one could apply one of the factorization transformation

$$
u(\mathrm{T}^{(\alpha)}[\boldsymbol{z}]) = \bar{u}_\alpha(\boldsymbol{x}) \prod_i \left[x_i^{(\alpha)} - \rho_i^{(\alpha)}\right]^{\gamma_i^{(\alpha)}}, \tag{6.1}
$$

and take $n-1$ variables to values of poles to arrive at a univariate form (without loss of generality, we choose $x_1^{(\alpha)}$)

$$
u_1^{(\alpha)} \equiv \bar{u}_\alpha\left(x_1^{(\alpha)}, \rho_2^{(\alpha)}, \rho_3^{(\alpha)}, \cdots\right) \left[x_1^{(\alpha)} - \rho_1^{(\alpha)}\right]^{\gamma_1^{(\alpha)}}. \tag{6.2}
$$

As detail discussed in [83], the symbol letters of the univariate problem (if not elliptic) could be read out immediately from u. These letters are also the letters that could appear in the full multivariate problem. Thus, figure out $u_i^{(\alpha)}$ for all α and i could provide letters even without the construction of d log integrand. These less stringent problems will be explored in the future.

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Towards systematic evaluation of de Sitter correlators via Generalized Integration-By-Parts relations

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Abstract: We generalize Integration-By-Parts (IBP) and differential equations methods to de Sitter correlators related to inflation. While massive correlators in de Sitter spacetime are usually regarded as highly intricate, we find they have remarkably hidden concise structures from the perspective of IBP. We find the factorization of the IBP relations of each vertex integral family corresponding to $\mathrm{d}\tau_i$ integration. Furthermore, with a smart construction of master integrals, the universal formulas for iterative reduction and $\mathrm{d}\log$-form differential equations of arbitrary vertex integral family are presented and proved. These formulas dominate all tree-level de Sitter correlators and play a kernel role at the loop-level as well.

Keywords: de Sitter space, Early Universe Particle Physics, Scattering Amplitudes

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Contents

1 Introduction

The quantum field theory in curved spacetime has a rich history and holds paramount significance in various physical applications. As the maximally symmetric spacetime, de Sitter (dS) spacetime as well as anti-de Sitter (AdS) spacetime should be the simplest curved spacetime to study. The quantum field theory in dS and AdS has gained large interest among researchers. In the phenomenology of particle physics and cosmology, correlators in dS spacetime play a crucial role in the study of inflationary cosmologies and possible signals of new physics, known as the cosmological collider program [1–3]. These correlators are then related to experimental observations [4] such as Cosmic Microwave Background (CMB) radiation and Large-Scale Structure (LSS). Utilizing the in-in formalism [5] in cosmology [6, 7] and its diagrammatic approach, the Schwinger-Keldysh (SK) path integral [5, 8, 9], one can derive Feynman-like diagrams and rules for n-point correlators [10]. The computation and analysis of these inflation correlators is an ongoing research topic. People have considered this problem from many aspects such as cosmological bootstrap [11–21], cutting rule [22–28], Mellin-Barnes (MB) space [29–34], partial MB transformation [35–38], integrand issues [39–45]. Nevertheless, this topic, to the best of our current knowledge, has not been systematically solved and comprehended.

On the other side, the methods developed for computing amplitudes in flat spacetime are well-established in last two decades. Similar methods have recently been applied for dS correlators. For example, MB transformation has been applied in both flat [46, 47] and dS [29–38]. In dS case, people find the (high-order) differential equation of correlator and solve it in terms of hypergeometric functions [37]. Meanwhile, in flat space, similar method also has been considered, using the more systematic mathematical tool GKZ [48–52]. Thus, examining the distinct features of amplitudes in flat spacetime and correlators in dS spacetime and drawing insights from the computation methods for flat spacetime could provide guidance for developing computation methods of dS correlators.

In the beginning, let's highlight the main difference between these two cases. The integrals appearing in flat amplitude take the form

$$\int_{\mathcal{C}} \prod_i P_i(\boldsymbol{z})^{\alpha_i} \mathrm{d}\boldsymbol{z}, \tag{1.1}$$

where P_i is polynomial and the integration region $\mathcal{C}$ is properly chosen. For example, the conventional form of Feynman integral take this form, with $P_i = (l_i + q_i)^2 - m_i^2$, as well as their corresponding Feynman parametrization or Baikov representation [53]. On the contrary, the integrals appearing in dS correlator takes the form

$$\int_{\mathcal{C}} \prod_i P_i(\boldsymbol{z})^{\alpha_i} \prod_j F_j(\boldsymbol{z})^{\beta_j} \mathrm{d}\boldsymbol{z}\,, \tag{1.2}$$

where F_i are functions related to Hankel functions and step functions [10]. This poses substantial challenges in the computation of such dS correlators. Successful experience for higher precision computations of amplitudes in flat space in the past decade leads us to the Integration-By-Parts (IBP) reduction [54] and (first-order) Differential Equations (DE) [55–58] obtained by IBP. These are what we want to introduce to dS correlators. The IBP method establishes linear relations among different integrals within a integral family by integrating total differential with zero boundary. For example, for the integral family

$$G(a) \equiv \int \mathrm{d}^d l \frac{1}{(l^2 - m^2)^a} \tag{1.3}$$

the IBP relation gives

$$0 = \int \mathrm{d}^d l \, \frac{\partial}{\partial l} \cdot l \frac{1}{(l^2 - m^2)^a} \quad \to \quad (d - 2a)G(a) - 2am^2 G(a+1) = 0\,. \tag{1.4}$$

With such relations, one can reduce any integral in the integral family into a linear combination of a selected finite set of integrals (called Master Integrals (MI)), which are the only part need to be really calculated. If the partial derivatives of the master integrals still belong to the same integral family, reducing them back to the MI yields a system of first-order differential equations (DE) of the MI. With the boundary condition, one could solve them analytically or numerically. Moreover, when people encounter the same integral family in another scattering process, redundant computations could be avoid. The introduction of the canonical differential equation (CDE) method in [59] has significantly facilitated the advancement of high-loop analytical calculations, whose DE are d log-form. Then, numerical DE methods gained attention and generalized power series expansions [60–62] and Auxiliary Mass Flow (AMFlow) were developed [63–66]. Meanwhile, automation packages for IBP [67–72] and numerical DE [73–75] have been continuously refined and widely applied in recent years and provided immense convenience. Due to all these developments, IBP and DE have become prioritized tools especially for computing the most complicated amplitudes. For instance, the recent result of a two-loop five-point process [76] is obtained using IBP, numerical DE and the related packages. IBP and DE are used to be applied on the integrals take the expression (1.1). This also implies that if same forms of integrands are identified in other physical scenarios, the computational techniques developed for flat spacetime Feynman integrals may be directly applied to those situations. For instance, in dS correlator, calculations are comparatively more tractable in the case of conformally coupled scalar due to their integrals also take the form (1.1), and IBP and DE have been studied [77, 78] in this case recently.

In this paper, the power and significant success of IBP and DE in flat spacetime have motivated us to promote their application to dS integrals with the form (1.2). This extension aims to develop a systematic, efficient, and automatic technique for computing general (not only massless or conformally coupled) dS correlators. It provides also a new perspective and tool to analyze the structure and properties of dS correlator.

The structure of the paper is following. In section 2, we review the Feynman rule of the dS correlator, then construct IBP of this case. In section 3.1, we indicate that each $\mathrm{d}\tau_i$'s IBP are factorized, which leads to great simplification of the computation. For the so-called vertex integral family (with respect to integration of each τ_i), we have constructed a smart set of bases, which will lead to further significant simplifications in the iterative reduction and DE of this integral family. In section 3.2, we give a pedagogical example of iteratively reducing 1-fold Hankel function vertex integral family, to sketch the features of general cases. In section 3.3, we find it coincidentally that the DE of the MI we construct automatically are $\mathrm{d}\log$-form. Subsequently, we systematically present and prove the universal formula of iterative reduction for arbitrary n-fold vertex integral families in section 3.4 and its $\mathrm{d}\log$-form DE in section 3.5. In section 3.6, we quickly cover the simplest massless case. In section 3.7, we give discussion of IBP and DE when they involving remaining terms come form two types of propagator $G_{\pm\pm}$ and belong to the so-called sub-sector. Finally, a brief summary and discussions are presented in the section 4.

2 Generalized IBP relations of dS correlators

In this section, we review the Feynman rules of in-in formalism dS correlators in cosmology and inflation. We will find dS correlators take the form of (1.2) and determine what terms and functions could appear as P_i and F_j. Then we will generalize the IBP method to this case. For simplifying the discussion without losing key features, we focus on scalar field theory.

Following the standard notation in the inflation physics, we choose the metric with the so called conformal time τ

$$\mathrm{d}s^2 = a^2(\tau)\big(-\mathrm{d}\tau^2 + \mathrm{d}\boldsymbol{x}^2\big), \tag{2.1}$$

where the scale factor is chosen to be $a(\tau) = 1/(-H\tau)$ with H the Hubble parameter. Feynman rules of dS correlators (see [10] for review) are similar to the case of flat space, i.e., with three basic building blocks: external wave functions, propagators and vertexes. For simplicity, let us take scalar field as an example. The mode expansion of field operator is

$$\varphi_a(\tau,\boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\Big[u_a(\tau,\boldsymbol{k})b_a(\boldsymbol{k}) + u_a^*(\tau,-\boldsymbol{k})b_a^\dagger(-\boldsymbol{k})\Big]e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{2.2}$$

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where the wave function $u_a(\tau, \boldsymbol{k})$ satisfies the equation of motion

$$u_a''(\tau, \boldsymbol{k}) - \frac{2}{\tau} u_a'(\tau, \boldsymbol{k}) + \left(\boldsymbol{k}^2 + \frac{m_a^2}{H^2 \tau^2} \right) u_a(\tau, \boldsymbol{k}) = 0 \tag{2.3}$$

with m_a the mass of the field. The solution is given by

$$u(\tau; k) = -i \frac{\sqrt{\pi}}{2} e^{i\pi(\nu/2+1/4)} H^{(d-1)/2} (-\tau)^{d/2} \mathrm{H}_\nu^{(1)}(-k\tau), \tag{2.4}$$

where $\mathrm{H}_\nu^{(1)}(-k\tau)$ is the Hankel function and other parameters are $k = |\boldsymbol{k}|$, $v = \sqrt{\frac{d^2}{4} - \frac{m^2}{H^2}}$ and $d = 3$. The Hankel functions satisfy[1]

$$\begin{aligned}
&\partial_\tau^2 \mathrm{H}_\nu^{(1,2)}(-k\tau) + \frac{1}{\tau} \partial_\tau \mathrm{H}_\nu^{(1,2)}(-k\tau) + \left(k^2 - \frac{\nu^2}{H^2 \tau^2} \right) \mathrm{H}_\nu^{(1,2)}(-k\tau) = 0\,, \\
&\mathrm{H}_\nu^{(1)}(-k\tau) = \left(\mathrm{H}_{\nu^\star}^{(2)}(-k^\star \tau^\star) \right)^\star, \\
&\tau \to 0_+ : \mathrm{H}_\nu^{(1)}(-k\tau) \sim -\frac{i 2^\nu \Gamma[\nu]}{\pi} (-k\tau)^{-\nu}, \text{ for } \mathrm{Re}[\nu] > 0 \\
&\tau \to +\infty : \mathrm{H}_\nu^{(1)}(-k\tau) \sim \sqrt{\frac{2}{\pi}} (-k\tau)^{\frac{1}{2}} e^{-ik\tau - i\pi(\nu/2+1/4)}
\end{aligned} \tag{2.5}$$

Using (2.2) we can get various propagators. The bulk-to-bulk propagators are given by

$$\begin{aligned}
G_>(k; \tau_1, \tau_2) &\equiv u(\tau_1, k) u^*(\tau_2, k), \\
G_<(k; \tau_1, \tau_2) &\equiv u^*(\tau_1, k) u(\tau_2, k).
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
G_{++}(k; \tau_1, \tau_2) &= G_>(k; \tau_1, \tau_2) \theta(\tau_1 - \tau_2) + G_<(k; \tau_1, \tau_2) \theta(\tau_2 - \tau_1), \\
G_{+-}(k; \tau_1, \tau_2) &= G_<(k; \tau_1, \tau_2), \\
G_{-+}(k; \tau_1, \tau_2) &= G_>(k; \tau_1, \tau_2), \\
G_{--}(k; \tau_1, \tau_2) &= G_<(k; \tau_1, \tau_2) \theta(\tau_1 - \tau_2) + G_>(k; \tau_1, \tau_2) \theta(\tau_2 - \tau_1).
\end{aligned} \tag{2.7}$$

The bulk-to-boundary propagators are non-vanish only when the field is massless

$$\begin{aligned}
G_+(k; \tau) &\equiv G_{+\pm}(k; \tau_1, 0) = \frac{H^2}{2k^3} (1 + ik\tau) e^{-ik\tau}, \\
G_-(k; \tau) &\equiv G_{-\pm}(k; \tau_1, 0) = \frac{H^2}{2k^3} (1 - ik\tau) e^{+ik\tau}.
\end{aligned} \tag{2.8}$$

It is necessary to notice that when people consider the asymptotic past $\tau \to -\infty$, iϵ-prescription [10] should be taken into consideration for good behavior. Practically, it is equivalent to attach an external factor $e^{\epsilon k\tau}$ to $G_>$ and $G_<$ making propagators to be exponentially suppressed to zero when $\tau \to -\infty$. For Feynman rules of vertices, some

[1]The definition of $\mathrm{H}_\nu^{(1,2)}$ in textbook is $\mathrm{H}_\nu^{(1)}(z) = \mathrm{J}_\nu(z) + i\mathrm{Y}_\nu(z)$ and $\mathrm{H}_\nu^{(2)}(z) = \mathrm{J}_\nu(z) - i\mathrm{Y}_\nu(z)$. When z, ν are real number, it is obviously that $\mathrm{H}_\nu^{(2)}(z) = (\mathrm{H}_\nu^{(1)}(z))^*$. The second line of (2.5) is a generalization of above result when z, ν are complex number.

examples are listed as follows:

$$
\begin{aligned}
-\frac{\lambda}{24}a^4(\tau)\varphi^4 &\to \mp i\lambda\int_{-\infty}^{0}\mathrm{d}\tau\, a^4(\tau)\cdots \\
-\frac{\lambda}{6}a^2(\tau)\varphi(\partial_i\varphi)(\partial_i\varphi) &\to \pm\frac{i\lambda}{3}(k_{12}+k_{23}+k_{13})\int_{-\infty}^{0}\mathrm{d}\tau\, a^2(\tau)\cdots\,, \qquad k_{ij}\equiv \boldsymbol{k}_i\cdot\boldsymbol{k}_j \\
-\frac{\lambda}{6}a^2(\tau)\varphi\varphi'^2 &\to \mp\frac{i\lambda}{3}\int_{-\infty}^{0}\mathrm{d}\tau\, a^2(\tau)G_{+a_1}(k_1;\tau,\tau_1)\prod_{i=2,3}\left[\partial_\tau G_{+a_i}(k_i;\tau,\tau_i)\right] \\
&\quad + 2\text{ permutations}.
\end{aligned} \tag{2.9}
$$

From above discussions, one can see that dS correlators are indeed the form of (1.2) with following correspondence:

$$
\begin{aligned}
&\int_{\mathcal{C}}\mathrm{d}\boldsymbol{z} = \int_{-\infty}^{0}\mathrm{d}\tau_i\int_{-\infty}^{\infty}\mathrm{d}\boldsymbol{l}_i\,, \qquad \boldsymbol{l}_i\text{ for loop momentum}\,, \\
&\quad P_i = \tau_j,\quad \text{polynomial of loop momentum}\,, \\
&\quad F_i = e^{ik\tau},\quad \mathrm{H}_\nu^{(1,2)}(-k\tau),\quad \partial_\tau\mathrm{H}_\nu^{(1,2)}(-k\tau),\quad \theta(\tau_j-\tau_k)\,.
\end{aligned} \tag{2.10}
$$

We call a integral family defined by (2.10) a **dS integral family**.

To apply IBP method, we need to check the dS integral family satisfying the proper boundary conditions. For the loop integration, just like the flat space, we will use the dimension regularization, thus the IBP relations involving $\boldsymbol{l}_i$ satisfy automatically. The new feature is the conformal time integration. First with the $i\epsilon$-prescription, the integrands equal zero at the boundary $\tau_i=-\infty$. For the boundary $\tau_i=0$, one could add a regulator $\tau_i^{\delta_i}$ if it is divergence, then it is again zero. The regulator for the divergent case is necessary for the IBP method to apply, since only with it the integration is well defined. Physical information is coded at the series expansion of δ (just like dimension regulator ϵ for flat space), for example, $1/\delta$ term in the expansion. In IBP-based differential equations method for flat spacetime, obtaining result in the expansion of regulator is also a usual practice. Since $a(\tau)\sim\frac{1}{\tau}$, power function of τ automatically appears in the integrand. With these considerations, we conclude that the IBP method can be applied to dS integral family. Let us do one simple example

$$
0=\int_{\mathcal{C}}\mathrm{d}\left(\tau^{\nu_0}\mathrm{H}_\nu^{(1)}(-k\tau)\right)\,, \tag{2.11}
$$

gives

$$
0=\nu_0\int_{\mathcal{C}}\tau^{\nu_0-1}\mathrm{H}_\nu^{(1)}(-k\tau)\mathrm{d}\tau+\int_{\mathcal{C}}\tau^{\nu_0}\partial_\tau\mathrm{H}_\nu^{(1)}(-k\tau)\mathrm{d}\tau. \tag{2.12}
$$

One may notice that applying the derivative operator to $\partial_\tau\mathrm{H}_\nu^{(1,2)}(-k\tau)$ will give $\partial_\tau^2\mathrm{H}_\nu^{(1,2)}(-k\tau)$, which is not in the dS integral family (2.10). However, using the relation (2.5) we can express $\partial_\tau^2\mathrm{H}_\nu^{(1,2)}(-k\tau)$ as the linear combination of $\mathrm{H}_\nu^{(1,2)}(-k\tau)$ and $\partial_\tau\mathrm{H}_\nu^{(1,2)}(-k\tau)$. For ∂_{k_i}, similar things also happen. For θ-function, the derivative leads to δ-function which can be easily integrated out. Overall, IBP relations obtained from a defined dS integrals family will be a closed set of equations, which can be solved systematically.

3 IBP of $\mathrm{d}\tau_i$: vertex integral family

The integral family of $\mathrm{d}\tau_i$ plays a key role in IBP of dS integrals and we will focus on its special properties in this section.

3.1 Vertex integral family and factorization of $\mathrm{d}\tau_i$ IBP

Before going to details, let us notice that from (2.10) for most terms, (at most times) different τ_i's are factorized. There is only one exception, i.e., the $\theta(\tau_i - \tau_j)$, which comes from G_{++} and G_{--}. Thus we define a **vertex integral family** corresponding to these good integrands without θ-function:

$$\begin{aligned} V(\nu_0, a_1, a_2, \cdots, a_n) = V(\nu_0, \boldsymbol{a}) &= \int \hat{V}(\nu_0, \boldsymbol{a}; \tau)\mathrm{d}\tau \\ &\equiv \int_{-\infty}^{0} \tau^{\nu_0} e^{ik_0\tau} \prod_i h(\nu_i, a_i; -k_i\tau)\mathrm{d}\tau, \qquad h \equiv h^{(1,2)}\,,\ a_i = 0, 1\,, \\ h^{(1 \text{ or } 2)}(\nu, 0; -k\tau) &\equiv (-k\tau)^{-\nu}\mathrm{H}_{\nu}^{(1)}(-k\tau)\ (\text{or } \mathrm{H}_{\nu^\star}^{(2)}) \propto \tau^{-\frac{3}{2}-\nu}u\ (\text{or } u^*), \\ h^{(1,2)}(\nu, 1; -k\tau) &\equiv -\frac{1}{k}\partial_\tau h^{(1,2)}(\nu, 0; -k\tau)\,. \end{aligned} \tag{3.1}$$

In the definition we have omitted the dependence on k_0, k_i, ν_i for simplicity. We introduce also a factors $(-k\tau)^{-\nu}$ in front of Hankel function to the definition of h. This factor is one of the most important construction in this paper. One can check that the differential equation of h

$$h''(\nu, 0; -k\tau) + \frac{1}{\tau}(2\nu + 1)h'(\nu, 0; -k\tau) + k^2 h(\nu, 0; -k\tau) = 0, \tag{3.2}$$

can be rewritten as

$$\partial_{\tilde{\tau}}^2 h(\nu, 0; \tilde{\tau}) + \frac{1}{\tilde{\tau}}(2\nu + 1)\partial_{\tilde{\tau}} h(\nu, 0; \tilde{\tau}) + h(\nu, 0; \tilde{\tau}) = 0, \quad \tilde{\tau} = -k\tau \tag{3.3}$$

and $h(\nu, 1; -k\tau)$ is defined by $\partial_{\tilde{\tau}} h(\nu, 0; -k\tau)$. To cancel the $1/\tau^2$ term in the differential equation (2.5), one need multiply a prefactor $(-k\tau)^{\pm\nu}$ in the defination of h. Here we choose $(-k\tau)^{-\nu}$. Another key point of (3.3) is that h depends only on the combination of $k\tau$, not individual k and τ. As it will be seen, this construction extremely simplify the IBP relations of vertex integral family. We even find coincidentally that this definition automatically gives $\mathrm{d}\log$-form DE of all vertex integral family. We will show the details later.

Due to the fact that u and $u^\star$, or equivalently, $\mathrm{H}_{\nu}^{(1)}(-k\tau)$ and $\mathrm{H}_{\nu^\star}^{(2)}(-k\tau)$ satisfy the same differential equation for the natural condition $k \in \mathbb{R}$ and $\nu = \pm\nu^\star$ (or says ν are real or pure imaginary),[2] they also share the same properties under IBP. Thus we use the same symbol to denote both of them here for convenience. We call the integral family consists of these $V(\nu_0 + a_0, a_1, a_2, \cdots, a_n)$s for selected ν_i and k_i an **n-fold** (Hankel function) vertex integral family.

[2] One can see it easily by checking the first line of (2.5) where only ν^2 appears in the differential equation. With this condition in the definition of $h(\nu, 0; -k\tau)$, the prefactor $(-k\tau)^{-\nu}$ will be same for both $\mathrm{H}_{\nu}^{(1)}(-k\tau)$ and $\mathrm{H}_{\nu^\star}^{(2)}$. Otherwise, the prefactor needs to be changed accordingly.

To apply IBP method, we need know the action of differential operators on $h(\nu, a, -k\tau)$. For ∂_τ, we have

$$
\begin{aligned}
\partial_\tau h(\nu, 0, -k\tau) &= -kh(\nu, 1, -k\tau)\,, \\
\partial_\tau h(\nu, 1, -k\tau) &= -k\Big[\frac{1}{k\tau}(2\nu+1)h(\nu, 1; -k\tau) - h(\nu, 0; -k\tau)\Big]\,.
\end{aligned} \tag{3.4}
$$

To construct IBP relation at loop-level corresponding to $\mathrm{d}l_i$, and also construct DE with respect to k_i, we also need know the result of applying ∂_{k_i} to $h(\nu, a, -k\tau)$. One can use the properties of Hankel function to get the result, but a much simpler way is to notice that $h(\nu, a; \tilde{\tau})$'s dependence on k and τ are only mediated through $\tilde{\tau}$ as given in (3.3). Thus one immediately get

$$
\begin{aligned}
\partial_k h(\nu, 0, -k\tau) &= -\tau h(\nu, 1, -k\tau)\,, \\
\partial_k h(\nu, 1, -k\tau) &= -\tau\Big[\frac{1}{k\tau}(2\nu+1)h(\nu, 1; -k\tau) - h(\nu, 0; -k\tau)\Big]\,.
\end{aligned} \tag{3.5}
$$

With (3.4) and (3.5), IBP relations and DE corresponding to ∂_k could be easily constructed.

Now we want to show some important properties of this IBP system. Let us consider expressions involving only G_{+-} or/and G_{-+} type propagators. In this case, the correlator factorizes to

$$
\int \prod_j \Big[\hat{V}_j(\cdots; \tau_j)\mathrm{d}\tau_j\Big]\, f(l_1, \cdots, l_L) \prod_i \mathrm{d}l_i \tag{3.6}
$$

where f is polynomial of l_i and $\hat{V}$ is given in (3.1). The IBP relation with respect to a selected τ_k is given by

$$
\int \Big(\partial_{\tau_k}\hat{V}_k(\cdots; \tau_k)\mathrm{d}\tau_k\Big) \times \prod_{j\neq k}\Big(\hat{V}_j(\cdots; \tau_j)\mathrm{d}\tau_j\Big)\, f(l_1, \cdots, l_L; k_1 \cdots, k_E) \prod_i \mathrm{d}l_i \tag{3.7}
$$

which does not affect other parts. We say the IBP relations with respect to different τ_i are factorized.

Now let's consider G_{++} and G_{--} type propagators. Its factorization is not as good as $G_{\pm\mp}$, since derivative acting on θ-function leads to the term

$$
\int \hat{V}_i(\cdots; \tau_i)\delta(\tau_i - \tau_j)\hat{V}_j(\cdots; \tau_j)\mathrm{d}\tau_i\mathrm{d}\tau_j \times \cdots = \int \hat{V}_i(\cdots; \tau_i)\hat{V}_j(\cdots; \tau_i)\mathrm{d}\tau_i \times \cdots\,, \tag{3.8}
$$

where naively $(n_i + n_j)$-fold Hankel function emerges. However, as we will show later, in fact, it becomes $(n_i + n_j - 2)$-fold Hankel function or vanishes completely. Fortunately, the new term is belong a sub-sector with less vertex integral family, by which we means the IBP of the sub-sector will not invovle the orignal higher sector. So we can still reduce each vertex integral family individually. And these remaining terms belong to sub-sector are regarded as known results in this part of reduce which have been reduced using the IBP-reduction of sub-sector.

Now we consider the merging happened in (3.8) by considering the symmetry in $G_{\pm\pm}$ more carefully. There are four cases could appear in the IBP system which involves massive $G_{\pm\pm}$:

$$\begin{aligned}
&\left[h^{(a)}(\nu,0,-k\tau_i)(\partial_{\tau_i}\theta_{ij})h^{(3-a)}(\nu,0,-k\tau_j)+h^{(3-a)}(\nu,0,-k\tau_i)(\partial_{\tau_i}\theta_{ji})h^{(a)}(\nu,0,-k\tau_j)\right]\times\cdots\\
&\left[h^{(a)}(\nu,1,-k\tau_i)(\partial_{\tau_i}\theta_{ij})h^{(3-a)}(\nu,0,-k\tau_j)+h^{(3-a)}(\nu,1,-k\tau_i)(\partial_{\tau_i}\theta_{ji})h^{(a)}(\nu,0,-k\tau_j)\right]\times\cdots\\
&\left[h^{(a)}(\nu,0,-k\tau_i)(\partial_{\tau_i}\theta_{ij})h^{(3-a)}(\nu,1,-k\tau_j)+h^{(3-a)}(\nu,0,-k\tau_i)(\partial_{\tau_i}\theta_{ji})h^{(a)}(\nu,1,-k\tau_j)\right]\times\cdots\\
&\left[h^{(a)}(\nu,1,-k\tau_i)(\partial_{\tau_i}\theta_{ij})h^{(3-a)}(\nu,1,-k\tau_j)+h^{(3-a)}(\nu,1,-k\tau_i)(\partial_{\tau_i}\theta_{ji})h^{(a)}(\nu,1,-k\tau_j)\right]\times\cdots\\
&\quad\theta_{ij}\equiv\theta(\tau_i-\tau_j)\,,\quad a=1,2\,.
\end{aligned}\tag{3.9}$$

The integration $\int \mathrm{d}\tau_i\mathrm{d}\tau_j$ of first and fourth cases will vanish due to

$$\begin{aligned}
&\int \mathrm{d}\tau_i\mathrm{d}\tau_j\left[h^{(a)}(\nu,0,-k\tau_i)h^{(3-a)}(\nu,0,-k\tau_j)-h^{(3-a)}(\nu,0,-k\tau_i)h^{(a)}(\nu,0,-k\tau_j)\right]\delta(\tau_i-\tau_j)\times\cdots\\
&=\int \mathrm{d}\tau_i\left[h^{(a)}(\nu,0,-k\tau_i)h^{(3-a)}(\nu,0,-k\tau_i)-h^{(3-a)}(\nu,0,-k\tau_i)h^{(a)}(\nu,0,-k\tau_i)\right]\times\cdots=0\\
&\int \mathrm{d}\tau_i\mathrm{d}\tau_j\left[h^{(a)}(\nu,1,-k\tau_i)h^{(3-a)}(\nu,1,-k\tau_j)-h^{(3-a)}(\nu,1,-k\tau_i)h^{(a)}(\nu,1,-k\tau_j)\right]\delta(\tau_i-\tau_j)\times\cdots\\
&=\int \mathrm{d}\tau_i\left[h^{(a)}(\nu,1,-k\tau_i)h^{(3-a)}(\nu,1,-k\tau_i)-h^{(3-a)}(\nu,1,-k\tau_i)h^{(a)}(\nu,1,-k\tau_i)\right]\times\cdots=0\,.
\end{aligned}\tag{3.10}$$

The integration $\int \mathrm{d}\tau_i\mathrm{d}\tau_j$ of second and third cases give

$$\begin{aligned}
&-(-1)^a\int \mathrm{d}\tau_i\,[F(-k\tau_i)]\times\cdots=+\int \mathrm{d}\tau_i\;C_\nu\frac{4i}{\pi}(-k\tau_i)^{-2\nu-1}\times\cdots\\
&+(-1)^a\int \mathrm{d}\tau_i\,[F(-k\tau_i)]\times\cdots=-\int \mathrm{d}\tau_i\;C_\nu\frac{4i}{\pi}(-k\tau_i)^{-2\nu-1}\times\cdots\\
&F(-k\tau_i)=h^{(1)}(\nu,1,-k\tau_i)h^{(2)}(\nu,0,-k\tau_i)-h^{(2)}(\nu,1,-k\tau_i)h^{(1)}(\nu,0,-k\tau_i)
\end{aligned}\tag{3.11}$$

for real or imaginary ν, where we have used the property of Hankel function

$$\begin{aligned}
&F(-k\tau_i)=C_\nu\frac{4i}{\pi}(-k\tau_i)^{-2\nu-1}\,,\\
&C_\nu=\begin{cases}1 & \text{for real }\nu\\ e^{-i\pi\nu} & \text{for imaginary }\nu\end{cases}\,.
\end{aligned}\tag{3.12}$$

One can also check it that the derivative of $F(-k\tau_i)$ gives $\partial_{-k\tau}F(-k\tau)=-\frac{2\nu+1}{-k\tau}F(-k\tau)$, then, $F(-k\tau_i)=C(-k\tau_i)^{-2\nu-1}$ and C can be determined by asymptotic behavior of Hankel functions on the boundary. Then, two vertex are "pinched" together and give a (n_i+n_j-2)-fold vertex integral family, not (n_i+n_j)-fold from the naive observation in (3.8).

For massless $G_{\pm\pm}$, if one choose to keep the combination of the two terms in these propagator, there are also "vanishing" case and "pinched" case. "Vanishing" case comes from

$$\int \mathrm{d}\tau_i\mathrm{d}\tau_j\left[e^{ik\tau_i}(\partial_{\tau_i}\theta_{ij})e^{-ik\tau_j}+e^{-ik\tau_i}(\partial_{\tau_i}\theta_{ji})e^{ik\tau_j}\right]\times\cdots=0\tag{3.13}$$

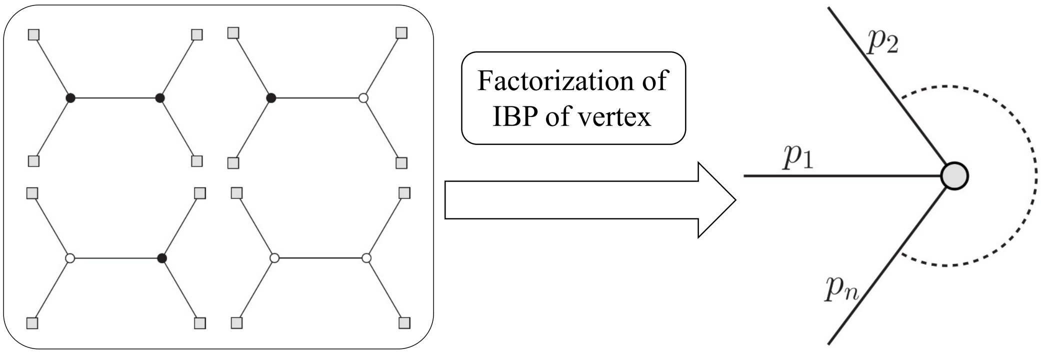

Figure 1. Factorization of IBP relations of vertex: all reductions with respect to $\mathrm{d}\tau$ can be achieved by reducing individual families of vertex integrals separately.

and "pinched" cases comes from

$$\begin{aligned}
&\int \mathrm{d}\tau_i \mathrm{d}\tau_j \left[(\partial_{\tau_i} e^{ik\tau_i})(\partial_{\tau_i}\theta_{ij})e^{-ik\tau_j} + (\partial_{\tau_i} e^{-ik\tau_i})(\partial_{\tau_i}\theta_{ji})e^{ik\tau_j}\right] \times \cdots = \int \mathrm{d}\tau_i 2ik \times \cdots \\
&\int \mathrm{d}\tau_i \mathrm{d}\tau_j \left[e^{ik\tau_i}(\partial_{\tau_i}\theta_{ij})\partial_{\tau_i} e^{-ik\tau_j} + e^{-ik\tau_i}(\partial_{\tau_i}\theta_{ji})\partial_{\tau_i} e^{ik\tau_j}\right] \times \cdots = -\int \mathrm{d}\tau_i 2ik \times \cdots
\end{aligned} \tag{3.14}$$

One can also choose to handle the two terms in massless propagator individually.

With the above discussion, we can reduce each vertex integral family individually:

$$\int \prod_i \mathrm{d}\tau_i \hat{V}_i \times \cdots = \left[\int \prod_i \left(\mathrm{d}\tau_i \sum_j c_j^{(i)} \hat{f}_j^{(i)}\right) \times \cdots\right] + R, \tag{3.15}$$

where $\hat{f}_j^{(i)}$ is the integrand of the MI of the integral family V_i belong to, "$\cdots$" part could contain $\theta(\tau_i - \tau_j)$, and we use R to denote remaining terms. R terms come from pinching two vertex connected via $G_{\pm\pm}$ together. Since the reduction of these R terms will not involve the family before pinching, it belongs to a sub-sector and can treat as known in the reduction of family V_i in (3.15). This form leads following simplifications in the computation:

- **Factorization of IBP relations of vertex.** We say that the IBP relations of different $\mathrm{d}\tau_i$ are factorized in the $(t, \boldsymbol{k})$ space of dS integral, since we can perform the IBP reduction individually for each of them, as shown in (3.15). Since all vertex integral family take the similar expression (3.1), the reduction result can be used repeatedly for different vertex, correlator and theory. For example, once people get the IBP reduction of a n-fold vertex integral family, it can be directly applied to all n-fold vertex appears in other place (if without external symmetry).

- At loop-level, one need to further consider the IBP relations of $\mathrm{d}l_i$. Notice that terms outside of $\hat{V}_i$ and θ-function are independent of τ_i. Due to this, one can reduce the family of all $\mathrm{d}\tau$ integrals to MI first, to get $\mathrm{d}\tau_i$-reduced IBP relations. Then, solving these $\mathrm{d}\tau_i$-reduced IBP relations of $\mathrm{d}l_i$ will give the complete result of reduction. In this progress, differential equations of $\mathrm{d}\tau$ integrals family could serve for directly giving

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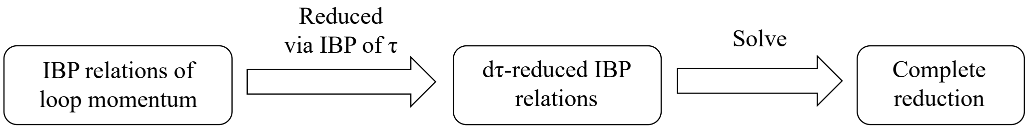

Figure 2. Steps of loop-level reduction.

$\mathrm{d}\tau_i$-reduced IBP relations. For example, consider total derivative with respect to l for the following loop integrals

$$\int \mathrm{d}\left(\frac{1}{|l||l-p_1|\cdots}\boldsymbol{f}\right)=0\,,\quad \boldsymbol{f}=\{f_1,f_2,\cdots\} \tag{3.16}$$

where the $\boldsymbol{f}$ are MI of the integrals family of all $\mathrm{d}\tau_i$. Then, we have a series of $\mathrm{d}\tau_i$-reduced IBP relations

$$\int\left(\mathrm{d}\frac{1}{|l||l-p_1|\cdots}\right)\boldsymbol{f}+\frac{1}{|l||l-p_1|\cdots}(\Omega.\boldsymbol{f}\mathrm{d}l)=0\,,\quad \mathrm{d}\boldsymbol{f}=\Omega.\boldsymbol{f}\mathrm{d}l \tag{3.17}$$

where the Ω is the differential equation matrix of $\boldsymbol{f}$ with respect to l. (3.17) is directly the so called $\mathrm{d}\tau$-reduced IBP relation, because the $\mathrm{d}\tau$ integration part of each integral in these relations are kept as master integrals.

In the rest of this section, we will show some examples for n-fold vertex integral family and sketch more features of this IBP system. Following the idea similar to iterative reduction [79, 80] in flat space, we will further see that the infinite number of integrals in a vertex integral family can be reduced iteratively to MI once we have solved the iterative relations from a finite linear system. Furthermore, we get the universal expression of these iterative reductions for any n-fold vertex integral family, as well as $\mathrm{d}\log$-form DE of MI. We also will give formulas and discussion for reduction with remaining terms come from $G_{\pm\pm}$.

3.2 1-fold vertex integral family: iterative reduction

Consider 1-fold vertex integral family

$$V(\nu_0,a_1)=\int_{-\infty}^{0}\tau^{\nu_0}e^{ik_0\tau}h(\nu_1,a_1;-k_1\tau)\mathrm{d}\tau,\qquad a_i=0,1\,. \tag{3.18}$$

From now on we set H to be 1 for convenience. We use $eq[\nu_0,\boldsymbol{a}]$ to denote the IBP relation

$$\int \mathrm{d}\hat{V}(\nu_0,\boldsymbol{a};\tau)=0, \tag{3.19}$$

then we have IBP relations

$$\begin{aligned}
&eq[\nu_0,0]:\ ik_0V(\nu_0,0)+\nu_0V(\nu_0-1,0)-k_1V(\nu_0,1)=0\\
&eq[\nu_0,1]:\ ik_0V(\nu_0,1)+\nu_0V(\nu_0-1,1)+k_1V(\nu_0,0)+(-2\nu_1-1)V(\nu_0-1,1)=0.
\end{aligned} \tag{3.20}$$

We have numerically evaluated these integrals at their convergent region and verified these two IBP relations to be right. Defining

$$\boldsymbol{f}^{(a_0)}=\{V(\nu_0+a_0,0)\,,V(\nu_0+a_0,1)\}, \tag{3.21}$$

the IBP relation corresponding to a selected ν_0 can be expressed in the form of matrix as

$$
\begin{aligned}
&\left(M_1^{(1)}+\nu_0\mathbb{I}_2\right).\boldsymbol{f}^{(-1)}+\left(M_0^{(1)}+ik_0\mathbb{I}_2\right).\boldsymbol{f}^{(0)}=0\\
&M_1^{(j)}=-\frac{2\nu_j+1}{2}\left(\mathbb{I}_2-\sigma_3\right)=\begin{pmatrix}0 & 0\\ 0 & -2\nu_j-1\end{pmatrix},\quad M_0^{(j)}=-ik_j\sigma_2=\begin{pmatrix}0 & -k_j\\ k_j & 0\end{pmatrix},
\end{aligned}
\tag{3.22}
$$

where $\sigma_{1,2,3}$ are the Pauli matrices. Traditionally, taking ν_0 to be ν_0+a_0 with different a_0, we can get a series of IBP relations. Solving them give the reduction of this integral family. As the consequence, we find there are two MI in this system, which can be chosen as $\boldsymbol{f}^{(0)}$. However, here we obviously have a simpler method. People only need to solve iterative relation

$$
\boldsymbol{f}^{(-1)}=A_-(\nu_0).\boldsymbol{f}^{(0)}. \tag{3.23}
$$

Then, all integrals in the family can be reduced iteratively by

$$
\begin{aligned}
\boldsymbol{f}^{(-n)}&=\left(\prod_{i=n-1}^{0}A_-(\nu_0-i)\right).\boldsymbol{f}^{(0)},\\
\boldsymbol{f}^{(n)}&=\left(\prod_{i=n-1}^{0}A_+(\nu_0+i)\right).\boldsymbol{f}^{(0)},\qquad A_+(\nu_0)\equiv\left(A_-(\nu_0+1)\right)^{-1}.
\end{aligned}
\tag{3.24}
$$

For 1-fold vertex integral family, $A_\pm$ can be immediately solved and given as following

$$
\begin{aligned}
A_-(\nu_0)&=\begin{pmatrix}-\frac{ik_0}{\nu_0} & \frac{k_1}{\nu_0}\\ \frac{k_1}{-\nu_0+2\nu_1+1} & -\frac{ik_0}{\nu_0-2\nu_1-1}\end{pmatrix}\\
A_+(\nu_0)&=\begin{pmatrix}\frac{ik_0(\nu_0+1)}{k_0^2-k_1^2} & \frac{k_1(\nu_0-2\nu_1)}{k_0^2-k_1^2}\\ \frac{k_1(\nu_0+1)}{k_1^2-k_0^2} & \frac{ik_0(\nu_0-2\nu_1)}{k_0^2-k_1^2}\end{pmatrix}
\end{aligned}
\tag{3.25}
$$

For $k_0=0$, they will be simplified to

$$
A_-(\nu_0)=\begin{pmatrix}0 & \frac{k_1}{\nu_0}\\ \frac{k_1}{-\nu_0+2\nu_1+1} & 0\end{pmatrix},\qquad A_+(\nu_0)=\begin{pmatrix}0 & -\frac{\nu_0-2\nu_1}{k_1}\\ \frac{\nu_0+1}{k_1} & 0\end{pmatrix}, \tag{3.26}
$$

3.3 1-fold vertex integral family: d log-form DE

Using above reduction we can establish the DE for the MI by acting ∂_{k_0} and ∂_{k_1} on $\boldsymbol{f}^{(0)}$. They gives

$$
\begin{aligned}
\partial_{k_0}\boldsymbol{f}^{(0)}&=\begin{pmatrix}iV(\nu_0+1,0)\\ iV(\nu_0+1,1)\end{pmatrix}=iA_+(\nu_0+1)\boldsymbol{f}^{(0)},\\
\partial_{k_1}\boldsymbol{f}^{(0)}&=\begin{pmatrix}-V(\nu_0+1,1)\\ \frac{(-2\nu_1-1)V(\nu_0,1)}{k_1}+V(\nu_0+1,0)\end{pmatrix}=\left(\frac{1}{k_1}M_1^{(1)}-i\sigma_2.A_+(\nu_0+1)\right).\boldsymbol{f}^{(0)}.
\end{aligned}
\tag{3.27}
$$

Evidently, the partial derivatives do not give rise to any special functions beyond the original family of functions, this implies that DE method corresponding to ∂_{k_i} can be applied for such integral family. Let's reduce the right hand side of these equations with the iterative relation we get in (3.25). It gives

$$\begin{aligned}
\partial_{k_i} \boldsymbol{f}^{(0)} &= \Omega_{k_i}.\boldsymbol{f}^{(0)},\\
\Omega_{k_0} &= \begin{pmatrix} -\frac{k_0(\nu_0+1)}{k_0^2-k_1^2} & -\frac{ik_1(\nu_0+1)}{k_0^2-k_1^2} \\ \frac{ik_1(\nu_0-2\nu_1)}{k_0^2-k_1^2} & -\frac{k_0(\nu_0-2\nu_1)}{k_0^2-k_1^2} \end{pmatrix},\\
\Omega_{k_1} &= \begin{pmatrix} \frac{k_1(\nu_0+1)}{k_0^2-k_1^2} & -\frac{ik_0k_1(\nu_0+1)}{k_1^3-k_0^2k_1} \\ -\frac{ik_0(\nu_0-2\nu_1)}{k_0^2-k_1^2} & \frac{k_0^2(2\nu_1+1)-k_1^2(\nu_0+1)}{k_1^3-k_0^2k_1} \end{pmatrix}.
\end{aligned} \tag{3.28}$$

In flat space, canonical DE play an important role in analytical calculation of loop integral. It takes the form

$$\mathrm{d}\boldsymbol{f} = \varepsilon(\mathrm{d}\Omega).\boldsymbol{f} \tag{3.29}$$

with $\mathrm{d}\Omega$ in $\mathrm{d}\log$-form. It is interesting that we find in 1-fold vertex integral family, in the MI we defined, the DE automatically is in $\mathrm{d}\log$-form. Although it is subtly different from canonical DE, we report this elegant result as it may offer potential assistance in understanding the mathematical structure of dS integrals in the future. The DE can be written in $\mathrm{d}\log$-form as

$$\begin{aligned}
&\mathrm{d}\boldsymbol{f}^{(0)} = \mathrm{d}\Omega \boldsymbol{f}^{(0)}\\
&\mathrm{d}\Omega = \sum_{i=0,1} \tilde{\Omega}_{k_i}\mathrm{d}k_i = C_1\mathrm{d}\log(k_1) + C_2\mathrm{d}\log[(k_0-k_1)(k_0+k_1)] + C_3\mathrm{d}\log\left(\frac{k_0+k_1}{k_0-k_1}\right)\\
&C_1 = \begin{pmatrix} 0 & 0 \\ 0 & -2\nu_1-1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} \frac{1}{2}(-\nu_0-1) & 0 \\ 0 & \frac{1}{2}(2\nu_1-\nu_0) \end{pmatrix}\\
&C_3 = \begin{pmatrix} 0 & -\frac{1}{2}i(\nu_0-2\nu_1) \\ \frac{1}{2}i(\nu_0+1) & 0 \end{pmatrix},
\end{aligned} \tag{3.30}$$

where $\mathrm{d}f \equiv \sum_i \partial_{k_i} f \ \mathrm{d}k_i$, and $\mathrm{d}\nu_i$ is not included (i.e., we treat ν_i as fixed parameters).

3.4 n-fold vertex integral family: universal formula of iterative reduction

Now we consider n-fold vertex integral family

$$V(\nu_0, a_1, \cdots, a_n) = \int_{-\infty}^{0} \tau^{\nu_0} e^{ik_0\tau} \prod_{i=1}^{n} h(\nu_i, a_i; -k_i\tau)\mathrm{d}\tau, \qquad a_i = 0, 1\,. \tag{3.31}$$

It has 2^n MI, which can be chosen as

$$f_{\boldsymbol{a}}^{(0)} = V(\nu_0, \boldsymbol{a}), \quad \boldsymbol{a} = a_1, \cdots, a_n, \quad \forall a_i =, 0, 1, \tag{3.32}$$

and together denoted as $\boldsymbol{f}^{(0)}$. As a vector with 2^n components, the ordering is given by n indices a_i according to the natural binary number j

$$j = 1 + \sum_i a_i 2^{n-i}. \tag{3.33}$$

For example, when $n = 2$, we have

$$f_1^{(0)} = f_{0,0}^{(0)}\,, \quad f_2^{(0)} = f_{0,1}^{(0)}\,, \quad f_3^{(0)} = f_{1,0}^{(0)}\,, \quad f_4^{(0)} = f_{1,1}^{(0)}\,. \tag{3.34}$$

It can be derived from (3.4) that all IBP relations $eq[\nu_0, \boldsymbol{a}]$ corresponding to a selected ν_0 can be expressed in the form

$$(M_1)_{\boldsymbol{ba}}\, f_{\boldsymbol{a}}^{(-1)} + (M_0)_{\boldsymbol{ba}}\, f_{\boldsymbol{a}}^{(0)} = 0 \tag{3.35}$$

where the matrix elements are given by

$$\begin{aligned}
(M_1)_{\boldsymbol{ba}} &= \sum_{j=1}^{n} \left[\left(M_1^{(j)} \right)_{b_j a_j} \prod_{i \neq j} \delta_{b_i a_i} \right] + \nu_0 \delta_{\boldsymbol{ba}} \\
(M_0)_{\boldsymbol{ba}} &= \sum_{j=1}^{n} \left[\left(M_0^{(j)} \right)_{b_j a_j} \prod_{i \neq j} \delta_{b_i a_i} \right] + i k_0 \delta_{\boldsymbol{ba}}
\end{aligned} \tag{3.36}$$

with $M_0^{(j)}, M_1^{(j)}$ given in (3.22). The matrix can be compactly represented as

$$\begin{aligned}
M_1 &= \sum_{j=1}^{n} \left(\nu_j + \frac{1}{2} \right) \Lambda_3^{(j)} + \left(\nu_0 - \frac{n}{2} - \sum_{i=1}^{n} \nu_i \right) \mathbb{I}_{2^n} \\
M_0 &= -i \sum_{j=1}^{n} k_j \Lambda_2^{(j)} + i k_0 \mathbb{I}_{2^n}
\end{aligned} \tag{3.37}$$

where

$$\left(\Lambda_k^{(j)} \right)_{\boldsymbol{ba}} \equiv (\sigma_k)_{b_j, a_j} \prod_{i \neq j} \delta_{b_i, a_i}, k = 1, 2, 3 \tag{3.38}$$

is direct product of a series 2×2 identity matrices except the j-th one as Pauli σ_k matrix.

From the representation (3.37), one can see M_1 is a diagonal matrix

$$(M_1)_{\boldsymbol{ba}} = \begin{cases} \nu_0 - \sum_i a_i (2\nu_i + 1), & \boldsymbol{b} = \boldsymbol{a} \\ 0, & \boldsymbol{b} \neq \boldsymbol{a} \end{cases} \tag{3.39}$$

Here we show IBP relation $eq[\nu_0, \boldsymbol{a}]$ for $n = 2$ and all $\boldsymbol{a}$ in matrix form as example

$$\begin{aligned}
&\left(M_1 \middle| M_0 \right) . \boldsymbol{f}^{\mathsf{T}} \\
&= \left(\begin{array}{cccc|cccc}
\nu_0 & 0 & 0 & 0 & ik_0 & -k_2 & -k_1 & 0 \\
0 & \nu_0 - 2\nu_2 - 1 & 0 & 0 & k_2 & ik_0 & 0 & -k_1 \\
0 & 0 & \nu_0 - 2\nu_1 - 1 & 0 & k_1 & 0 & ik_0 & -k_2 \\
0 & 0 & 0 & \nu_0 - 2\nu_1 - 2\nu_2 - 2 & 0 & k_1 & k_2 & ik_0
\end{array} \right) . \boldsymbol{f}^{\mathsf{T}} = 0\,, \\
&\boldsymbol{f} = \{\boldsymbol{f}^{(-1)}, \boldsymbol{f}^{(0)}\}, \qquad \boldsymbol{f}^{(i)} = \{f_{0,0}^{(i)}, f_{0,1}^{(i)}, f_{1,0}^{(i)}, f_{1,1}^{(i)}\}\,,
\end{aligned} \tag{3.40}$$

where

$$\begin{aligned}
M_1 &= M_1^{(1)} \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes M_1^{(2)} + \nu_0 \mathbb{I}_2 \otimes \mathbb{I}_2 \\
M_0 &= M_0^{(1)} \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes M_0^{(2)} + i k_0 \mathbb{I}_2 \otimes \mathbb{I}_2.
\end{aligned} \tag{3.41}$$

Obviously,

$$A_-(\nu_0) = -M_1^{-1}.M_0\,, \qquad A_+(\nu_0-1) = -M_0^{-1}.M_1\,. \tag{3.42}$$

In above expression, the inverse of diagonal matrix M_1 is trivial, but the inverse of M_0 is non-trivial. Thus we are looking for different representation. Noticing that M_0 is formed by the direct product of $\mathbb{I}_2$ and σ_2, diagonalizing σ_2 by redefining each $h(\nu_i, a_i; -k\tau)$ immediately leads to the diagonalization of M_0:

$$\tilde{h}(\nu_i, a_i; -k\tau) = \sum_{b_i=0,1} \mathrm{T}_{a_i b_i} h(\nu_i, b_i; -k\tau) \tag{3.43}$$

with

$$\mathrm{T} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \quad \mathrm{T}^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tag{3.44}$$

This leads to

$$\begin{aligned} &\tilde{V}(\nu_0, \boldsymbol{a}) = \int_{-\infty}^{0} \tau^{\nu_0} e^{ik_0\tau} \prod_i \tilde{h}(\nu_i, a_i; -k_i\tau)\mathrm{d}\tau \\ &\tilde{\boldsymbol{f}}^{(a_0)} = \mathrm{T}_n.\boldsymbol{f}^{(a_0)}\,, \qquad (\mathrm{T}_n)_{\boldsymbol{ba}} = \prod_{i=1}^{n} T_{b_i a_i}\,, \quad \left(\mathrm{T}_n^{-1}\right)_{\boldsymbol{ba}} = \prod_{i=1}^{n} T^{-1}_{b_i a_i} \end{aligned} \tag{3.45}$$

Using the properties

$$\begin{aligned} &\mathrm{T}.\sigma_2.\mathrm{T}^{-1} = \sigma_3\,, \quad \mathrm{T}.\sigma_3.\mathrm{T}^{-1} = -\sigma_2\,, \quad \mathrm{T}.\sigma_1.\mathrm{T}^{-1} = \sigma_1\,, \quad \mathrm{T}.\mathbb{I}_2.\mathrm{T}^{-1} = \mathbb{I}_2\,, \\ &\mathrm{T}_n.\Lambda_2^{(j)}.\mathrm{T}_n^{-1} = \Lambda_3^{(j)}\,, \quad \mathrm{T}_n.\Lambda_3^{(j)}.\mathrm{T}_n^{-1} = -\Lambda_2^{(j)}\,, \end{aligned} \tag{3.46}$$

after applying the transformation (3.44) and (3.45) to (3.22), we have

$$\begin{aligned} &\tilde{M}_1^{(j)} = -\frac{2\nu_i+1}{2}\left(\mathbb{I}_2+\sigma_2\right)\,, \qquad \tilde{M}_0^{(j)} = -ik_j\sigma_3\,, \\ &\tilde{M}_1 = -\sum_{j=1}^{n}\left(\nu_j+\frac{1}{2}\right)\Lambda_2^{(j)} + \left(\nu_0 - \frac{n}{2} - \sum_{i=1}^{n}\nu_i\right)\mathbb{I}_{2^n}\,, \\ &\tilde{M}_0 = -\sum_{j=1}^{n} ik_j\Lambda_3^{(j)} + ik_0\mathbb{I}_{2^n}\,. \end{aligned} \tag{3.47}$$

Now, on the contrary, $\tilde{M}_0$ is a diagonal matrix

$$\left(\tilde{M}_0\right)_{\boldsymbol{ba}} = \begin{cases} i\left(k_0 + \sum_{i=1}^{n}(2a_i-1)k_i\right), & \boldsymbol{b}=\boldsymbol{a} \\ 0, & \boldsymbol{b}\neq\boldsymbol{a} \end{cases} \tag{3.48}$$

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while $\tilde{M}_1$ is not. For example, when $n = 2$, we have

$$
\begin{aligned}
&\left(\tilde{M}_1\middle|\tilde{M}_0\right).\tilde{\boldsymbol{f}}^{\mathsf{T}} = 0\,, \qquad \tilde{\boldsymbol{f}} = \{\tilde{\boldsymbol{f}}^{(-1)}, \tilde{\boldsymbol{f}}^{(0)}\}\,,\\
&\tilde{M}_1 = \begin{pmatrix} \nu_0-\nu_1-\nu_2-1 & \frac{1}{2}i\,(2\nu_2+1) & \frac{1}{2}i\,(2\nu_1+1) & 0\\ -\frac{1}{2}i\,(2\nu_2+1) & \nu_0-\nu_1-\nu_2-1 & 0 & \frac{1}{2}i\,(2\nu_1+1)\\ -\frac{1}{2}i\,(2\nu_1+1) & 0 & \nu_0-\nu_1-\nu_2-1 & \frac{1}{2}i\,(2\nu_2+1)\\ 0 & -\frac{1}{2}i\,(2\nu_1+1) & -\frac{1}{2}i\,(2\nu_2+1) & \nu_0-\nu_1-\nu_2-1 \end{pmatrix}\\
&\tilde{M}_0 = i\begin{pmatrix} k_0-k_1-k_2 & 0 & 0 & 0\\ 0 & k_0-k_1+k_2 & 0 & 0\\ 0 & 0 & k_0+k_1-k_2 & 0\\ 0 & 0 & 0 & k_0+k_1+k_2 \end{pmatrix}
\end{aligned}
\tag{3.49}
$$

Putting all together finally we have

$$
\begin{aligned}
&A_-(\nu_0) = -M_1^{-1}.M_0\,,\\
&A_+(\nu_0-1) = -\mathrm{T}_n^{-1}.\tilde{M}_0^{-1}.\tilde{M}_1.\mathrm{T}_n = -\mathrm{T}_n^{-1}.\tilde{M}_0^{-1}.\mathrm{T}_n.M_1\,,
\end{aligned}
\tag{3.50}
$$

where M_1^{-1}, $\tilde{M}_0^{-1}$ and their inverse are diagonal matrices, with M_1 merely relying on ν_i, $\tilde{M}_0$ only relying on k_i and T_n a constant matrix.

Before ending this subsection, let us give a remark. Let's see what will happen if we haven't asked the definition of $h(\mu, \boldsymbol{a})$ to cancel the $1/\tau^2$ in (3.2). For example, define $h(\nu_0, \boldsymbol{a}) = (-k\tau)^{\frac{1}{2}}\mathrm{H}_\nu^{(1,2)}$ in (3.1) which satisfy

$$
\partial_{\tilde{\tau}}^2 h(\mu, 0; \tilde{\tau}) + \left(1 + \frac{\mu^2}{\tilde{\tau}^2}\right) h(\mu, 0; \tilde{\tau}) = 0, \quad \tilde{\tau} = -k\tau, \quad \mu^2 = \frac{m^2}{H^2} - 2 \tag{3.51}
$$

we will have

$$
\begin{aligned}
&M_2^{(1)}.\boldsymbol{f}^{(-2)} + \nu_0\mathbb{I}_2.\boldsymbol{f}^{(-1)} + \left(M_0^{(1)} + ik_0\mathbb{I}_2\right)).\boldsymbol{f}^{(0)}\\
&M_2^{(j)} = \begin{pmatrix} 0 & 0\\ \frac{\mu_j^2}{k_j} & 0 \end{pmatrix}, \qquad M_0^{(j)} = \begin{pmatrix} 0 & -k_j\\ k_j & 0 \end{pmatrix}.
\end{aligned}
\tag{3.52}
$$

The formula of reduction will not be as explicit as it is now due to the emergence of $\boldsymbol{f}^{(-2)}$ from the $1/\tau^2$ term. That's why we request this term to be canceled in the definition.

3.5 n-fold vertex integral family: universal formula of d log-form DE

For n-fold vertex integral family, with (3.5), acting ∂_{k_0} and ∂_{k_i} on $\boldsymbol{f}^{(0)}$ give

$$
\begin{aligned}
&\partial_{k_0}\boldsymbol{f}^{(0)} = i\boldsymbol{f}^{(1)} = iA_+(\nu_0)\boldsymbol{f}^{(0)}\,,\\
&\partial_{k_i}\boldsymbol{f}^{(0)} = \left(-\frac{1}{k_i}\frac{2\nu_i+1}{2}(\mathbb{I}_{2^n} - \Lambda_3^{(i)}) - i\Lambda_2^{(i)}.A_+(\nu_0)\right).\boldsymbol{f}^{(0)}\,, \text{ for } i > 0\,.
\end{aligned}
\tag{3.53}
$$

Defining

$$
\begin{aligned}
&\left(\tilde{\Omega}_0\right)_{\boldsymbol{ba}} \equiv \begin{cases} -i\log\left[k_0 + \sum_i(2a_i-1)k_i\right], & \boldsymbol{b} = \boldsymbol{a}\\ 0, & \boldsymbol{b} \neq \boldsymbol{a} \end{cases},\\
&(\Omega_{ex})_{\boldsymbol{ba}} \equiv \begin{cases} -\sum_i a_i(2\nu_i+1)\log k_i, & \boldsymbol{b} = \boldsymbol{a}\\ 0, & \boldsymbol{b} \neq \boldsymbol{a} \end{cases},
\end{aligned}
\tag{3.54}
$$

we have d log-form DE

$$
\begin{aligned}
\mathrm{d}\boldsymbol{f}^{(0)} = (\mathrm{d}\Omega).\boldsymbol{f}^{(0)} &= \sum_{i=0}^{n} \Omega_{k_i}.\boldsymbol{f}^{(0)} \mathrm{d}k_i \,, \\
\Omega &= \Omega_{ex} - i\mathrm{T}_n^{-1}.\tilde{\Omega}_0.\mathrm{T}_n.M_1(\nu_0+1)\,,
\end{aligned} \tag{3.55}
$$

where $M_1(\nu_0+1)$ is shifting the ν_0 in the original M_1 to ν_0+1.

Proof:

$$
\begin{aligned}
\partial_{k_0}\Omega &= -i\mathrm{T}_n^{-1}.\partial_{k_0}\tilde{\Omega}_0.\mathrm{T}_n.M_1(\nu_0+1) = -i\mathrm{T}_n^{-1}.\tilde{M}_0^{-1}.\mathrm{T}_n.M_1(\nu_0+1) = iA_+(\nu_0) \\
\partial_{k_i}\Omega &= \partial_{k_i}\Omega_{ex} - i\mathrm{T}_n^{-1}.\partial_{k_i}\tilde{\Omega}_0.\mathrm{T}_n.M_1(\nu_0+1) \\
&= -\frac{1}{k_i}\frac{2\nu_i+1}{2}(\mathbb{I}_{2^n} - \Lambda_3^{(i)}) + i\mathrm{T}_n^{-1}.\Lambda_3^{(i)}.\tilde{\Omega}_0.\mathrm{T}_n.M_1(\nu_0+1) \\
&= -\frac{1}{k_i}\frac{2\nu_i+1}{2}(\mathbb{I}_{2^n} - \Lambda_3^{(i)}) + i\Lambda_2^{(i)}.\mathrm{T}_n.\tilde{\Omega}_0.\mathrm{T}_n.M_1(\nu_0+1) \\
&= -\frac{1}{k_i}\frac{2\nu_i+1}{2}(\mathbb{I}_{2^n} - \Lambda_3^{(i)}) - i\Lambda_2^{(i)}.A_+(\nu_0)\,.
\end{aligned} \tag{3.56}
$$

Comparing these equations with (3.53), the proof is complete.

Then, once people have got boundary condition of $\boldsymbol{f}^{(0)}$, $\boldsymbol{f}^{(0)}(k_0^0, k_1^0, \cdots)$ for example, $\boldsymbol{f}^{(0)}(k_0', k_1', \cdots)$ can be got by

$$
\begin{aligned}
\boldsymbol{f}^{(0)}(k_0', k_1', \cdots) &= \boldsymbol{f}^{(0)}(k_0^0, k_1^0, \cdots) \\
&+ \mathcal{P}\exp\left[\int_{(k_0^0,k_1^0,\cdots)}^{(k_0',k_1',\cdots)} \sum_i \Omega_{k_i}(\nu_0,\nu_1,\cdots;k_0,k_1,\cdots)\mathrm{d}k_i\right].\,\boldsymbol{f}^{(0)}(k_0^0, k_1^0, \cdots)
\end{aligned} \tag{3.57}
$$

for a integration path starts from $(k_1^0, k_2^0, \cdots)$ and end at $(k_1', k_2', \cdots)$. $\mathcal{P}$ for path ordering.

At loop-level, these differential equations with respect to loop momentum will also directly give the $\mathrm{d}\tau_i$-reduced IBP relations we have mentioned near the end of section 3.1.

3.6 0-fold vertex integral family: massless cases

For a vertex only connect to massless propagator, the vertex function family is given by

$$
V(\nu_0) = \int \tau^{\nu_0} e^{ik_0\tau}. \tag{3.58}
$$

It has only $2^0 = 1$ MI and the iterative reduction relations are given by

$$
\begin{aligned}
&eq[\nu_0] : ik_0 V(\nu_0) + \nu_0 V(\nu_0-1) = 0, \\
&A_+(\nu_0) = i\frac{\nu_0+1}{k_0}\,, \quad A_-(\nu_0) = -i\frac{k_0}{\nu_0}\,,
\end{aligned} \tag{3.59}
$$

and d log-form DE is given by

$$
\begin{aligned}
&\partial_{k_0} V(\nu_0) = iV(\nu_0+1) = -(\nu_0+1)\frac{1}{k_0}V(\nu_0)\,, \\
&\mathrm{d}\Omega = \Omega_{k_0}\mathrm{d}k_0 = -(\nu_0+1)\mathrm{d}\log(k_0)\,.
\end{aligned} \tag{3.60}
$$

3.7 Discussion on remaining terms come from $G_{\pm\pm}$

3.7.1 Iterative reduction

Consider an simple example with remaining terms. Suppose $f^{(a_0)}_{\boldsymbol{a};1}$ are integrals belong to vertex integral family V_1, corresponding to the integration of $\mathrm{d}\tau_1$, and $f^{(b_0)}_{\boldsymbol{b};2}$ are integrals belong to vertex integral family V_2. We use $\hat{f}$ to denote their integrand. For example

$$\hat{f}^{(a_0)}_{\boldsymbol{a};1} = \tau_1^{\nu_{0;1}+a_0} \prod_i h(\nu_{i;1}, a_i, -k_{i;1}\tau_1)\,. \tag{3.61}$$

We also denote the combination of two terms in propagator as

$$\begin{aligned}
&h(\nu_{i;1}, a_i, -k_{i;1}\tau_1)\theta^{(i,j)}_{1,2} h(\nu_{j;2}, b_j, -k_{j;2}\tau_2) \\
&\equiv h^{(1)}(\nu_{i;1}, a_i, -k_{i;1}\tau_1)\theta_{12} h^{(2)}(\nu_{j;2}, b_j, -k_{j;2}\tau_2) + h^{(2)}(\nu_{i;1}, a_i, -k_{i;1}\tau_1)\theta_{21} h^{(1)}(\nu_{j;2}, b_j, -k_{j;2}\tau_2)\,, \\
&\nu_{i;1} = \nu_{j;2}\,, \quad k_{i;1} = k_{j;2}\,.
\end{aligned} \tag{3.62}$$

Then, if there is a G_{++}, we denote the integrals as

$$\boldsymbol{f}^{(a_0,b_0)}_{\boldsymbol{a},\boldsymbol{b}} \equiv \int \mathrm{d}\tau_1 \mathrm{d}\tau_2 \hat{f}^{(a_0)}_{\boldsymbol{a};1} \theta^{(i,j)}_{1,2} \hat{f}^{(b_0)}_{\boldsymbol{b};2} \tag{3.63}$$

Without loss of generality, consider IBP of $\mathrm{d}\tau_1$ and $a_0 = b_0 = 0$, one could have IBP relations

$$(M_1)_{\boldsymbol{c}\boldsymbol{a}}\, \boldsymbol{f}^{(-1,0)}_{\boldsymbol{a},\boldsymbol{b}} + (M_0)_{\boldsymbol{c}\boldsymbol{a}}\, \boldsymbol{f}^{(0,0)}_{\boldsymbol{a},\boldsymbol{b}} + \delta_{\boldsymbol{c}\boldsymbol{a}} \boldsymbol{R}^{(0,0)}_{\boldsymbol{a},\boldsymbol{b}} = 0\,, \tag{3.64}$$

which is just adding remaining terms of sub-sector and the factor $\theta^{(i,j)}_{1,2} \hat{f}^{(0)}_{\boldsymbol{b};2}$ to (3.35). According to (3.10) and (3.11),

$$\begin{aligned}
&\boldsymbol{R}^{(a_0,b_0)}_{\boldsymbol{a},\boldsymbol{b}} = \begin{cases} 0, & a_i = b_j \\ (-1)^{a_i} \frac{4i}{\pi} C_\nu(-k_{i;1})^{-2\nu_{i;1}-1} f^{(a_0+b_0-2\nu_{i;1}-1)}_{\boldsymbol{a}_{\hat{i}},\boldsymbol{b}_{\hat{j}}}, & a_i \neq b_j \end{cases}, \\
&\boldsymbol{a}_{\hat{i}}, \boldsymbol{b}_{\hat{j}} = a_1, a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_{n_1}, b_1, \cdots, b_{j-1}, b_{j+1}, \cdots, b_{n_2}\,,
\end{aligned} \tag{3.65}$$

where $f^{(a_0+b_0-2\nu_{i;1}-1)}_{\boldsymbol{a}_{\hat{i}},\boldsymbol{b}_{\hat{j}}}$ is integral belong to (n_1+n_2-2)-fold vertex integral family. Then, reduction of these sector with two vertex integral family could be achieved via using

$$\begin{aligned}
\boldsymbol{f}^{(-1,0)}_{\boldsymbol{c},\boldsymbol{b}} &= (A_{-;1}(\nu_0))_{\boldsymbol{c}\boldsymbol{a}}\, \boldsymbol{f}^{(0,0)}_{\boldsymbol{a},\boldsymbol{b}} - \left(M^{-1}_{1;1}\right)_{\boldsymbol{c}\boldsymbol{a}} \boldsymbol{R}^{(0,0)}_{\boldsymbol{a},\boldsymbol{b}}\,, \\
\boldsymbol{f}^{(1,0)}_{\boldsymbol{c},\boldsymbol{b}} &= (A_{+;1}(\nu_0))_{\boldsymbol{c}\boldsymbol{a}}\, \boldsymbol{f}^{(0,0)}_{\boldsymbol{a},\boldsymbol{b}} - \left(\mathrm{T}^{-1}_{n_1}.\tilde{M}^{-1}_{0;1}.\mathrm{T}_{n_1}\right)_{\boldsymbol{c}\boldsymbol{a}} \boldsymbol{R}^{(0,0)}_{\boldsymbol{a},\boldsymbol{b}}
\end{aligned} \tag{3.66}$$

iteratively. In these equations, the subscript $;1$ means they are the expression with respect to family V_1. In other words, the $A_{\pm;1}$, $M_{1;1}$, and $\tilde{M}_{0;1}$ here are just given by the same formulas as (3.50), (3.37), and (3.47). These formulas give the iterative reduction relations without computing inverse of large matrix. For more than one $G_{\pm\pm}$, the reduction is similar to (3.66), but with more remaining terms corresponding to each $G_{\pm\pm}$. Since all remaining terms are also product of vertex integral families, one could reduce them via formula like (3.66) as well, thus the reduction is complete. We have got the universal formula for reducing all tree-level dS integrals.

3.7.2 d log-form DE

Consider the example (3.63) again, DE is given by

$$\begin{aligned}
\left(\partial_{k_{0;1}}\boldsymbol{f}^{(0,0)}\right)_{\boldsymbol{c},\boldsymbol{b}} &= i\boldsymbol{f}^{(1,0)}_{\boldsymbol{c},\boldsymbol{b}}\,,\\
\left(\partial_{k_{i;1}}\boldsymbol{f}^{(0,0)}\right)_{\boldsymbol{c},\boldsymbol{b}} &= -\frac{1}{k_i}\frac{2\nu_i+1}{2}\left(\mathbb{I}_{2^n}-\Lambda_3^{(i)}\right)_{\boldsymbol{ca}}\boldsymbol{f}^{(0,0)}_{\boldsymbol{a},\boldsymbol{b}} - i\left(\Lambda_2^{(i)}\right)_{\boldsymbol{ca}}\boldsymbol{f}^{(1,0)}_{\boldsymbol{a},\boldsymbol{b}}\,,\ \text{for } i>0\,,
\end{aligned}\tag{3.67}$$

where the $\boldsymbol{f}^{(1,0)}_{\boldsymbol{a},\boldsymbol{b}}$ is reduced by (3.66). As expected, the additional term arises from the remaining terms $\boldsymbol{R}^{(0,0)}_{\boldsymbol{a},\boldsymbol{b}}$ that emerge during its reduction. Obviously, if we choose these $\frac{4i}{\pi}C_\nu(-k_{i;1})^{-2\nu_{i;1}-1}f^{(-2\nu_{i;1}-1)}_{\boldsymbol{a}_{\hat{i}},\boldsymbol{b}_{\hat{j}}}$ in remaining terms (3.65) to be master integrals of this sub-sector, the dependence of the original sector on the sub-sector in the differential equation matrix is

$$\Omega_{(\boldsymbol{a},\boldsymbol{b})(\boldsymbol{c}_{\hat{i}},\boldsymbol{d}_{\hat{j}})} = \begin{cases} 0\,, & \text{for } a_i=b_j \mid\mid (a_i\neq b_j \ \&\ (\boldsymbol{c}_{\hat{i}},\boldsymbol{d}_{\hat{j}})\neq(\boldsymbol{a}_{\hat{i}},\boldsymbol{b}_{\hat{j}}))\,;\\ -i\sum_{c_i}\left(\mathrm{T}_n^{-1}.\tilde{\Omega}_{0;1}.\mathrm{T}_n\right)_{\boldsymbol{ac}}(-1)^{c_i} - i\sum_{d_i}\left(\mathrm{T}_n^{-1}.\tilde{\Omega}_{0;2}.\mathrm{T}_n\right)_{\boldsymbol{bd}}(-1)^{d_i}\,, & \\ \qquad \text{for } a_i\neq b_j \ \&\ (\boldsymbol{c}_{\hat{i}},\boldsymbol{d}_{\hat{j}})=(\boldsymbol{a}_{\hat{i}},\boldsymbol{b}_{\hat{j}})\,. \end{cases}$$

$$\left(\tilde{\Omega}_{0;j}\right)_{\boldsymbol{ba}} \equiv \begin{cases} -i\log\left[k_{0;j}+\sum_i(2a_i-1)k_{i;j}\right], & \boldsymbol{b}=\boldsymbol{a}\\ 0, & \boldsymbol{b}\neq\boldsymbol{a}\end{cases}\,,\tag{3.68}$$

where for the subsript of $\Omega_{(\boldsymbol{a},\boldsymbol{b})(\boldsymbol{c}_{\hat{i}},\boldsymbol{d}_{\hat{j}})}$, $\boldsymbol{a},\boldsymbol{b}$ are the indices for original sector and $\boldsymbol{c}_{\hat{i}},\boldsymbol{d}_{\hat{j}}$ are indices for the sub-sector. Then, it is again d log-form DE. Also notice that $\nu_{i;1}=\nu_{j;2}$, $k_{i;1}=k_{j;2}$, so the master integral $\frac{4i}{\pi}C_\nu(-k_{i;1})^{-2\nu_{i;1}-1}f^{(-2\nu_{i;1}-1)}_{\boldsymbol{a}_{\hat{i}},\boldsymbol{b}_{\hat{j}}}$ will be consistent for ;1 and ;2. For more than one $G_{\pm\pm}$, the discussion is similar.

4 Summary and outlook

In this paper, we define the dS integral family, which naturally incorporates the case of time derivatives interaction. We generalize IBP method to dS integral family with respect to $d\tau_i$ and dk_i by (3.4) and (3.5), whose integrands involve special functions. With (3.5) and result of IBP reduction, people also can construct DE with respect to ∂_{k_i} of dS integrals. We indicate the factorization of IBP relations of vertex in the dS correlator. For the vertex integral families, we derive an universal iterative reduction formula for arbitrary n-fold Hankel vertex function families, along with the d log-form DE satisfied by the MI we selected, as listed in (3.50) and (3.55). And the remaining terms come from $G_{\pm\pm}$ are also discussed. Since the tree-level dS correlators only involve integrals over τ_i, we have obtained the reduction and DE of arbitrary tree-level diagrams equivalently.

This paper in fact has presented an alternative pathway toward systematically and efficiently computing dS correlators. Once we have IBP relations, drawing from the experience in flat spacetime, the number of integrals people need to compute will be significantly reduced. Once we have DE, with proper boundary conditions, the remaining steps do not differ from the case of flat spacetime, and numerical result can be efficiently obtained via numerical DE method. Its effectiveness has been validated in flat spacetime, and many existing packages designed for flat spacetime can also be readily applied. For example, Kira could solve IBP

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reduction of user-defined system [81], AMFlow [73] and DiffExp [74] can numerically solving DE with given boundary condition and DE.

This paper also suggest many interesting open questions. We merely list a part of them as follows to inspire people's future research. Firstly, while we mainly focus on the IBP linear system, how to give a boundary condition of DE is beyond the scope of this paper and haven't been discussed. One can work like the flat space DE, select a simple boundary and analytically or numerically compute it. But whether there is a boundary-determine method like AMFlow [63, 66] in flat cases needs more consideration. For example, one may consider the boundary $k_0 \to -i\infty$ in dS cases, which shrink all vertex integrals to zero. Secondly, AdS correlator could be considered in the future and the relation of dS and AdS integral could be examined in the perspective of IBP and DE. Thirdly, follow the idea that reclassifying Feynman integrals itself as special function via DE [82], to bring dS integrals closer to a so called analytical result, people need to develop systematical method to analyze this integral family. For example, people may consider how to distinguish the different signal and background in the dS correlator using DE formalism like people have done using partial MB transformation [36] in the future.

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Module intersection and uniform formula for iterative reduction of one-loop integrals

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Abstract: In this paper, we develop an iterative sector-level reduction strategy for Feynman integrals, which bases on module intersection in the Baikov representation and auxiliary vector for tensor structure. Using this strategy we have studied the reduction of general one-loop integrals, i.e., integrals having arbitrary tensor structures and arbitrary power for propagators. Inspired by these studies, a uniform and compact formula that iteratively reduces all one-loop integrals has been written down, where messy polynomials in integration-by-parts (IBP) relations have organized themselves to Gram determinants.

Keywords: Automation, Scattering Amplitudes, Electroweak Precision Physics

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Contents

1 Introduction

Particle physics is highly developed and comes to the era of precise measurement, which calls for high precision theoretical predictions. One of the most important parts of theoretical predictions is perturbation computations in quantum field theory. For such computations, reducing general Feynman integrals into master integrals plays an important role [1]. It not only greatly reduces the number of integrals to be calculated, but also produces the differential equations [2–5] for solving the master integrals. Based on differential equations, many analytical or numerical methods and packages for solving master integrals have been developed [6–17], which can be widely applied to multi-loop and multi-scale processes.

The study of reduction has a long history, such as the famous Passarino-Veltman reduction (PV-reduction) method [18] and the well used Integration-by-Part (IBP) method [1, 19]. When combining with the Laporta algorithm [20], several widely used packages have been developed such as [21–24] for the IBP method. When using the IBP method, one will face some crucial steps, such as the generation of IBP relations and the solving of them. IBP relations tell us that there are linear relations among various integrals, thus they are not independent to each other. By solving them, we obtain the reduction, i.e., expressing all integrals as the linear combinations of a set of integrals with minimum number (i.e., the master integrals). When applying the IBP method to multi-loop and multi-scale integrals required by higher precision theoretical predictions, the size of linear equations generated by IBP method increases dramatically. Solving them becomes more and more difficult.

Motivated by this, many alternative reduction methods have been explored recently for higher loops[1] [30–51].

A crucial difficulty of IBP method is that one need to solve a hugely **redundant** linear system with many equations. Although with the help of computation programs, one can do it automatically, but with the increase of the complexity, even the computer can not handle it. Thus we really need to fully understand the origin of these redundancies and inefficiencies and find a way to avoid them.

One of the most obvious redundancies is that we write down more equations than we need. This redundancy is not hard to avoid. For example, we can just take values of parameters and numerically found the set of independent equations we need. Unfortunately, even if we have done this, there are still inefficiency in this linear system: the matrix is hugely **sparse**. While the whole system of IBP relations contains many integrals, a particular relation involves only a very small part of these integrals. Thus, naively solving the linear system by computing the inverse of the hugely sparse matrix is not efficient. To a certain extent, Laporta algorithm [20] serves to avoid such a difficulty. Even though, by our observation, another aspect of inefficiencies remains: **the similar structure appears in the linear system repeatedly**. If a calculation can use the iterative structure nicely, the efficiency has the potential to be improved considerably. Such iterative structure is the main topic of this paper.

To convince people that such an iterative structure exists wildly in any Feynman integral family, let us consider a simple example, i.e., the bubble topology with propagators:

$$z_1 = l^2 - m_1^2, \quad z_2 = (l-p)^2 - m_2^2. \tag{1.1}$$

In the family, there are three master integrals defined by

$$\vec{f} = \{\mathrm{I}_{1,0}, \mathrm{I}_{0,1}, \mathrm{I}_{1,1}\}, \qquad \mathrm{I}_{a_1,a_2} \equiv \int \frac{d^d l}{i(\pi)^{d/2}} \frac{1}{z_1^{a_1} z_2^{a_2}} \,. \tag{1.2}$$

One kind of differential equations of master integrals is given by

$$\partial_{m_1} \vec{f} = A_{m_1} \vec{f} \,. \tag{1.3}$$

It is easy to see when taking ∂_{m_1} on $\vec{f}$ iteratively we have

$$\partial_{m_1}^n \vec{f} = \{(n-1)!\ \mathrm{I}_{n+1,0},\ 0,\ (n-1)!\ \mathrm{I}_{n+1,1}\} = A_{m_1}^{(n)} \vec{f} \tag{1.4}$$

where

$$A_{m_1}^{(n)} = \partial_{m_1} A_{m_1}^{(n-1)} + A_{m_1}^{(n-1)} A_{m_1} \tag{1.5}$$

When talking about the reduction of integral $\mathrm{I}_{n+1,1}$ with arbitrary large integer n, from (1.4) one can see that the result can be read out from the matrix $A_{m_1}^{(n)}$. Now the iterative structure of $A_{m_1}^{(n)}$ in (1.5) gives a very simple way to compute it from the simple object A_{m_1} given in (1.3).

[1] For one-loop integrals, two very efficient methods are the unitarity cut method [25–28] for reduction at the integral level and OPP method [29] for reduction at the integrand level.

Since any Feynman integral family has master integrals and related differential equations, from the example we see that some kinds of iterative structures must exist for the reduction. Once we find one kind of iterative structures in IBP relation (not necessary in the form of differential equations), it can be used to reduce a part of integrals iteratively, thus only several polynomials that appear in the iterative relations can arise in the final reduction coefficient. Such property not only takes full usage of the iterative structure to accelerate the reduction program, but also has the potential to provide the analytic structure of reduction coefficients for integrals in the family by using the polynomials appearing in the iterative structure.

Based on above discussions, in this paper, we will try to develop an iterative method to speed up the reduction procedure by identifying a nice iterative structure. Before doing so, we will clarify two related but different concepts: the reduction relation at the **topology-level** and the reduction at the **sector-level**. When we say a reduction is at the topology-level, we mean that the integral is written as the combination of master integrals of the same topology and master integrals of the sub-topologies. When we say the reduction is at the sector-level, we mean that the integral is written as the combination of master integrals of the top-sector and integrals (not need to be master integrals) of the sub-sector for the selected top-sector.

Using above clarification, one can see that (1.4) and (1.5) are an iterative reduction relation at topology-level. While the iterative relation at the topology-level is powerful, it is also not easy to get. As in this example, one usually need to use the traditional IBP method to compute the A_{m_1}, which can become more difficult for more complicated examples. The complexity of getting such topology-level iterative reduction inspires us to find some wiser strategy for iterative reduction, which leads to the iteration at the sector-level, as we will explain now.

Let us start by noticing another well-known phenomenon of reduction: the reduction can be organized into different sectors. Usually, the reduction in the top sector can be done easily using some tricks, for example, the maximum cut for the one-loop box in [27]. However, such a trick lost the information of sub-sectors. Thus if there is a way such that the reduction to its top sector is not much harder than the maximal cut, but the complete information of sub-sectors has been kept, we can carry out the whole reduction top-down (or triangulated) by treating each sub-sector as another "top sector". Since in general the information of the sub-sectors are not in the form of master integrals, the iterative structure is not the topology-level. To emphasize this difference, we name it as the reduction at the sector-level.

In this paper, we will explore the iterative structure for general one-loop integrals, i.e., with arbitrary quadratic or linear propagators, arbitrary high power of propagators, arbitrary tensor structures and arbitrary multi-point multi-scale integrals. People are still concerning such one-loop integrals as shown in the recent work [52]. Our method mainly bases on syzygy and module intersection [30, 41–50, 53] in Baikov representation [54]. We will give a quick review in section 2 for Baikov representation and module intersection. One main point of our results comparing with the module intersection in [48] is that we will not generate all IBP relations with basis obtained by the module intersection and solve these redundant linear equations as usually. On the contrary, as illustrated in section 3, we pick smartly some elements in the module intersection. These elements generate iterative reduction relations for corresponding sector. Using just a few relations, one can iteratively reduce any integral in this sector to master integrals of this sector, and keep the complete information of sub-sectors. In such a strategy of **iteration at the sector-level**, iterative reduction relations are easier to obtain and they avoid the huge redundancy in linear system of traditional IBP methods.

Roughly, the reduction problem can be divided into the reduction of loop momenta in the numerator (called "**tensor reduction**" or **TR**) and the reduction of propagators in the denominator with general powers (called "**denominator reduction**" or **DR**). These two kinds of reductions are found to be tightly connected in one-loop cases.[2] To demonstrate our method, we will present one TR example and one DR example in section 4 and discuss the similarity of these two examples. Inspired by examples in the previous section, we construct an uniform iterative reduction formula at the sector-level in section 5. It solves both TR and DR for general one-loop integrals, and naturally keeps the expression of reduction coefficients in the form of Gram determinant, which is much more compact than these from traditional methods, especially for multi-point multi-scale cases. Formula presented in section 5 can be degenerated for kinematics and masses take some specific values, such as null momenta or on-shell momenta. In section 6, we give a brief discussion for these situations. We will find that naively applying the uniform formula laid out in section 5 sometimes does not lead to the simplest iterative relation (although it does still work). On the other side, the method introduced in section 3 still works well and giving the simplest iterative relation. Finally, we give a brief discussion and summary in the section 7.

2 Baikov representation and module intersection

In this section, we will review the Baikov representation of integrals [54]. In this frame, it is easier to implement the module intersection as will be discussed shortly. The Baikov representation transforms integrals in the standard form obtained from Feynman rules by changing integral variables from $\prod_i d^d l_i$ to $\prod_j dz_j$, where each z_i represents a propagator (or related Lorentz invariant scalar product involving loop momenta). For one-loop integrals, which are the focus of our current paper, we denote propagators and integrals as

$$\begin{aligned}
&z_1 = l^2 - m_1^2, \qquad z_2 = (l+p_1)^2 - m_2^2, \qquad z_3 = (l+p_1+p_2)^2 - m_3^2, \cdots \\
&z_n = (l+p_1+\cdots+p_E)^2 - m_n^2. \\
&\mathrm{I}_{\{a_i\}} \equiv \mathrm{I}_{a_1,a_2,\cdots,a_n} \equiv \int \frac{d^d l}{i(\pi)^{d/2}} \frac{1}{\prod_{i=1}^n z_i^{a_i}},
\end{aligned} \tag{2.1}$$

where E is the number of independent external momenta and $n = E+1$. The Baikov representation of integrals is

$$\mathrm{I}_{a_1,a_2,\cdots,a_{E+1}} = \int C_n(d) \mathcal{K}^{-(d-n)/2} \mathcal{G}(\boldsymbol{z})^{(d-n-1)/2} \frac{dz_i}{\prod_{i=1}^n z_i^{a_i}} \tag{2.2}$$

[2] For example, in [55] one has used the tensor reduction to solve reduction with general denominators for one-loop integrals.

where the constant-coefficient $C_n(d)$ and the Gram determinant $\mathcal{K}$ of external momenta do not involve z_i, so they do not affect our later discussions and can be ignored. The $\mathcal{G}(\boldsymbol{z})$ is another Gram determinant depending on both loop momentum and external momenta, i.e.,

$$\mathcal{G}(\boldsymbol{z}) = G(l, p_1, \cdots, p_E) \tag{2.3}$$

with G defined as

$$G(q_1, \ldots, q_n) \equiv \det(q_i \cdot q_j) \equiv \det \begin{pmatrix} q_1 \cdot q_1 & q_1 \cdot q_2 & \cdots & q_1 \cdot q_n \\ q_2 \cdot q_1 & q_2 \cdot q_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ q_n \cdot q_1 & \cdots & \cdots & q_n \cdot q_n \end{pmatrix}. \tag{2.4}$$

The well-established IBP relations can also be easily implemented in the Baikov representation. However, the differentiation on $\mathcal{G}$ will change its power, which is equivalent to shifting the space-time dimension d to different values.

To avoid such a situation, the syzygy module is introduced [41]. Let us consider the IBP relation

$$C \int \sum_{i=1}^{n} \left[\partial_{z_i} \left(P_i \frac{1}{\prod_{i=1}^{n} z_i^{a_i}} \mathcal{G}(\boldsymbol{z})^{(d-n-1)/2} \right) \right] \prod_{i=1}^{n} dz_i \tag{2.5}$$

with P_js being polynomials of z_i, one can see that if these P_i's are properly chosen, i.e., they satisfy

$$\sum_{i}^{n} (P_i \partial_{z_i} \mathcal{G}) + P_0 \mathcal{G} = 0, \tag{2.6}$$

the power of $\mathcal{G}$ will not be shifted. The relation (2.6) is a syzygy equation for the set of $(n+1)$ polynomials

$$\langle \partial_{z_1} \mathcal{G}, \cdots, \partial_{z_n} \mathcal{G}, \mathcal{G} \rangle \tag{2.7}$$

All solutions of (2.6) give the syzygy module of the set (2.7). Putting every solution back to (2.5) we get an IBP relation with a given $\{a_i\}$ set, which does not involve dimension shift.

For later convenience we define the following notations:

$$\begin{aligned} \langle P \rangle &= \langle P_1, P_2, \cdots, P_n, P_0 \rangle \\ D_{\langle P \rangle} &\equiv \{D_{P_1}, \cdots, D_{P_n}, D_{P_0}\} \equiv \left\{ \partial_{z_1} (P_1 \cdot), \cdots, \partial_{z_n} (P_n \cdot), \frac{d-n-1}{2} P_0 \cdot \right\} \\ D_{\langle P \rangle} \cdot Q &\equiv - \sum_{i=1}^{n} [\partial_{z_i} (P_i \cdot Q)] + \frac{d-n-1}{2} P_0 \cdot Q. \end{aligned} \tag{2.8}$$

Then the IBP relation (2.5) can be written in a more compact form

$$C \int \left\{ D_{\langle P \rangle} \cdot \frac{1}{\prod_{i=1}^{n} z_i^{a_i}} \right\} \mathcal{G}(\boldsymbol{z})^{(d-n-1)/2} \prod_{i=1}^{n} dz_i. \tag{2.9}$$

The syzygy module is a linear space with a basis of generators[3]

$$\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n\} \tag{2.10}$$

and the general solution of (2.6) can be written as $\langle P\rangle = \sum_{i=1}^n f_i \boldsymbol{e}_i$ with f_is being arbitrary polynomials of z_i.

Another well-known phenomenon in IBP relation is the changing of power of propagators in (2.9). The power can be increased or decreased. Among them, only $\partial_{z_i} z_i^{-a_i}$ increases the power. To avoid the increase, we can do a similar thing by requiring $\langle P\rangle$ in (2.8) to be the module generated by the following basis

$$\begin{aligned}
\boldsymbol{d}_1 &= \{z_1, 0, \cdots, 0, 0\}\\
\boldsymbol{d}_2 &= \{0, z_2, \cdots, 0, 0\}\\
&\cdots\\
\boldsymbol{d}_n &= \{0, 0, \cdots, z_n, 0\}\\
\boldsymbol{d}_{n+1} &= \{0, 0, \cdots, 0, 1\}.
\end{aligned} \tag{2.11}$$

Up to now, we have two modules: one is given by (2.6) and avoids the dimension shift of space-time, while another is given by (2.11) and avoids the increase of power of propagators. If we want to avoid both things, we just take the intersection of the above two modules, i.e., $\{\boldsymbol{h}_i\} \equiv \{\boldsymbol{e}_i\} \cap \{\boldsymbol{d}_i\}$. Notice that the syzygy module (2.6) and the module intersection $\boldsymbol{h}_i$ can be solved by computational algebraic geometry [43], and in this work, we use the package Singular [60] to do this. In the examples given in this paper, it takes only seconds or even less to finish the computation. The syzygy of Gram determinant can also easily be obtained by Laplace expansion of the determinant [46].

3 The method

In this paper, we will re-investigate the reduction problem for general one-loop integrals, i.e., with arbitrary tensor structure and arbitrary power of propagators. As one can see, these two different reductions, i.e., the tensor reduction (TR) and denominator reduction (DR) can be treated uniformly by module intersection method [41, 49].

Let us start from the TR first. As pointed out in several papers [51, 55, 56, 59, 61–64], arbitrary tensor structure can be compactly organized using an auxiliary vector R. Thus for TR of one-loop n-point integrals, we enlarge the set of propagators given in (2.1) by adding one new propagator, i.e.,

$$\begin{aligned}
&z_1 = l^2 - m_1^2, \qquad z_2 = (l+p_1)^2 - m_2^2, \qquad z_3 = (l+p_1+p_2)^2 - m_3^2, \cdots\\
&z_n = (l+p_1+\cdots+p_{n-1})^2 - m_n^2, \qquad z_{n+1} = l\cdot R.
\end{aligned} \tag{3.1}$$

[3] For one-loop integrals, it has been proved that the number of generators is exactly the number of propagators.

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with the power of z_{n+1} to be non-positive integer. Now the Baikov representation becomes[4]

$$\begin{aligned}
\mathrm{I}_{a_1,a_2,\cdots,a_n,a_{n+1}} &= C\int \mathcal{G}(\boldsymbol{z})^{(d-n-2)/2}\frac{dz_i}{\prod_{i=1}^{n+1} z_i^{a_i}} \\
\mathcal{G}(\boldsymbol{z}) &= G(l,p_1,\cdots,p_{n-1},R)
\end{aligned} \tag{3.2}$$

Let us denote the basis of syzygy module corresponding to

$$\langle \partial_{z_1}\mathcal{G},\cdots,\partial_{z_{n+1}}\mathcal{G},\mathcal{G}\rangle \tag{3.3}$$

as $\{\boldsymbol{e}_i\}$, while the basis of another module is $\{\boldsymbol{d}_i\}$ with

$$\begin{aligned}
\boldsymbol{d}_1 &= \{z_1,0,\cdots,0,0,0\} \\
&\cdots \\
\boldsymbol{d}_n &= \{0,0,\cdots,z_n,0,0\} \\
\boldsymbol{d}_{n+1} &= \{0,0,\cdots,0,1,0\} \\
\boldsymbol{d}_{n+2} &= \{0,0,\cdots,0,0,1\}
\end{aligned} \tag{3.4}$$

After obtained the module intersection $\{\boldsymbol{h}_i\} \equiv \{\boldsymbol{e}_i\}\cap\{\boldsymbol{d}_i\}$, we use elements in $\{\boldsymbol{h}_i\}$ to generate differential operators as in (2.8) and produce corresponding IBP relations like this:[5]

$$\mathrm{I}_{\boldsymbol{a},-r_{\max}} = \sum_{j=1}^{m} c_j \mathrm{I}_{\boldsymbol{a},-r_{\max}+j} + \text{l.p.p.t.}, \tag{3.5}$$

with $r_{\max} > 0$ and $a_i > 0$ for $i < n$, where the l.p.p.t. denote terms with lower power of propagators. More explicitly, we say $\mathrm{I}_{\boldsymbol{b},-r_b}$ is a l.p.p.t. corresponding to $\mathrm{I}_{\boldsymbol{a},-r_a}$, if it satisfies $b_i \le a_i$ for all $i \le n$, and $\sum_i^n b_i < \sum_i^n a_i$. Notice that when propagators' power is

$$\{\boldsymbol{a},-r\} = \{1,\cdots,1,-r\}, \tag{3.6}$$

the l.p.p.t. are all terms of sub-sectors.

To have a nice tensor reduction relation, there are some requirements for the (3.5). Firstly, the sign of power a_{n+1} of z_{n+1} indicates it is a numerator or a denominator. Since we want to discuss the tensor reduction, a_{n+1} should be a non-positive integer and relation (3.5) should not include any term with a_{n+1} positive. Thus for any $r_{\max} > 0$, we should require

$$c_j = 0 \quad \text{when} \quad j > r_{\max}, \tag{3.7}$$

Secondly, no c_j becomes infinity for any $r_{\max} > 0$ for (3.5) to be well defined.

[4] By (2.2), the C of (3.2) will depend on R also, but it will not influence IBP relations derived later.

[5] It is possible that the IBP relation can not be written to the form (3.5). For such a situation, we just throw away this IBP relation.

Having discussed the TR, let us move to the DR. For the reduction of propagators with arbitrary powers, the idea is similar. Without loss of generality, let us consider how to reduce the general power a_n of the n-th propagator to one. With propagators given by

$$
\begin{aligned}
&z_1 = l^2 - m_1^2, \quad z_2 = (l+p_1)^2 - m_2^2, \quad z_3 = (l+p_1+p_2)^2 - m_3^2, \cdots \\
&z_n = (l+p_1+\cdots+p_{n-1})^2 - m_n^2.
\end{aligned} \tag{3.8}
$$

the Baikov representation is given by

$$
\begin{aligned}
\mathrm{I}_{a_1,a_2,\cdots,a_n} &= C\int \mathcal{G}(\boldsymbol{z})^{(d-n-1)/2}\frac{dz_i}{\prod_{i=1}^n z_i^{a_i}} \\
\mathcal{G}(\boldsymbol{z}) &= G(l,p_1,\cdots,p_{n-1}).
\end{aligned} \tag{3.9}
$$

Noting that for $\mathcal{G}$, when writing using momentum variables, it is the same as the one given in (3.2). However, when writing using the z variables, the linear form in (3.1) and the quadratic form in (3.8) do make some differences. Again first we find the syzygy module for relation (2.6) to avoid the shift of the space-time dimension. However, the second module will be a little different from the one given in (2.11). More explicitly, generators become

$$
\begin{aligned}
\boldsymbol{d}_1 &= \{z_1, 0, \cdots, 0, 0, 0\} \\
&\cdots \\
\boldsymbol{d}_{n-1} &= \{0, 0, \cdots, z_{n-1}, 0, 0\} \\
\boldsymbol{d}_n &= \{0, 0, \cdots, 0, 1, 0\} \\
\boldsymbol{d}_{n+1} &= \{0, 0, \cdots, 0, 0, 1\},
\end{aligned} \tag{3.10}
$$

where the $\boldsymbol{d}_n$ is different. The reason is that now we do not ask to avoid the increase of the power of the n-th propagator. Finding the module intersection $\{\boldsymbol{h}_i\} \equiv \{\boldsymbol{e}_i\} \cap \{\boldsymbol{d}_i\}$ we will get IBP relations of the form

$$
\mathrm{I}_{\boldsymbol{a},a_{n,max}} = \sum_{j=1}^{m} c_j \mathrm{I}_{\boldsymbol{a},a_{n,max}-j} + \text{l.p.p.t.}, \tag{3.11}
$$

where no c_j becomes infinity for any $a_{n,max} > 1$. This equation is similar to (3.5), but with the following difference: in (3.5) it is the smallest power $-r_{\max}$ at the left-hand side while in (3.11) it is the maximum power $a_{n,max}$ at the left-hand side. Another difference is that in (3.11) we do not need to require $c_j = 0$ when $j > a_{n,max}$ since now it becomes the tensor of the sub-sector.

Before ending this section, let us emphasize that the reason we are able to treat the TR and DR uniformly using module intersection is following two key points. First, we have introduced the auxiliary vector R to represent all tensor structures.[6] Secondly, we are not write down all relations coming from module intersection, but select minimum ones, which have nice property, i.e., giving iteration at the sector-level. The meaning of this point will be clear by examples in the section 4.

[6]We want to emphasize that introducing R is different from introducing irreducible scalar products in usual IBP method. For later, if there are m ISP's we need to introduce m factors, but for R, we need to just introduce one for each loop momentum.

4 Pedagogical examples

Having discussed the method in the previous section, we will present two examples to demonstrate our method.

4.1 Tensor reduction of bubbles

In this subsection, we consider the tensor reduction of one-loop bubble integrals. The propagators are

$$z_1 = l^2 - m_1^2, \quad z_2 = (l+p_1)^2 - m_2^2, \quad z_3 = l \cdot R. \tag{4.1}$$

and the Gram determinant in Baikov representation is

$$\mathcal{G} = \det \begin{pmatrix} m_1^2 + z_1 & \frac{1}{2}\left(-m_1^2 + m_2^2 - p_1^2 - z_1 + z_2\right) & z_3 \\ \frac{1}{2}\left(-m_1^2 + m_2^2 - p_1^2 - z_1 + z_2\right) & p_1^2 & R \cdot p_1 \\ z_3 & R \cdot p_1 & R^2 \end{pmatrix}. \tag{4.2}$$

Using the expression in (4.2) we can find

$$\begin{aligned} \partial_{z_1}\mathcal{G} &= \frac{1}{2}R^2\left(-m_1^2 + m_2^2 + p_1^2 - z_1 + z_2\right) - z_3 R \cdot p_1 - (R \cdot p_1)^2 \\ \partial_{z_2}\mathcal{G} &= \frac{1}{2}R^2\left(m_1^2 - m_2^2 + p_1^2 + z_1 - z_2\right) + z_3 R \cdot p_1 \\ \partial_{z_3}\mathcal{G} &= -R \cdot p_1\left(m_1^2 - m_2^2 + p_1^2 + z_1 - z_2\right) - 2p_1^2 z_3 \end{aligned} \tag{4.3}$$

and the solutions of equation

$$\sum_i^3 (P_i \partial_{z_i}\mathcal{G}) + P_0\mathcal{G} = 0 \tag{4.4}$$

can be solved by the syzygy module with three generators

$$\{\boldsymbol{e}_i\} = \begin{pmatrix} 2z_3 & 2\left(R \cdot p_1 + z_3\right) & R^2 & 0 \\ m_1^2 + m_2^2 - p_1^2 + z_1 + z_2 & 2\left(m_2^2 + z_2\right) & R \cdot p_1 + z_3 & -2 \\ -2\left(m_2^2 - p_1^2 + z_2\right) & m_1^2 - 3m_2^2 - p_1^2 + z_1 - 3z_2 & -2R \cdot p_1 - z_3 & 2 \end{pmatrix}. \tag{4.5}$$

Meanwhile, the module $\{\boldsymbol{d}_i\}$ is generated by

$$\mathrm{DM}[z_1, z_2, 1, 1], \tag{4.6}$$

where the DM denotes the diagonal matrix. The module intersection of them is given by $\{\boldsymbol{h}_i\}$ with 10 basis as polynomials of variables $\{z_1, z_2, z_3, m_1^2, m_2^2, p_1^2, R \cdot p_1, R^2\}$[7] when using Singular [60]. Among them, the one with the lowest total power of z_i is given by

$$\begin{aligned} \boldsymbol{h}_{1,1} &= 2z_1\left(R \cdot p_1\left(m_1^2 - m_2^2 + p_1^2 + z_1 - z_2\right) + 2p_1^2 z_3\right) \\ \boldsymbol{h}_{1,2} &= 2z_2\left(R \cdot p_1\left(m_1^2 - m_2^2 + p_1^2 + z_1 - z_2\right) + 2p_1^2 z_3\right) \end{aligned}$$

[7]The reason not using $\{z_1, z_2, z_3\}$ is explained in [48].

$$
\begin{aligned}
\boldsymbol{h}_{1,3} = {} & R^2\left(-p_1^2\left(2m_1^2+2m_2^2+z_1+z_2\right)+\left(m_1^2-m_2^2\right)\left(m_1^2-m_2^2+z_1-z_2\right)+\left(p_1^2\right)^2\right) \\
& +2\left(\left(2m_1^2+z_1\right)(R\cdot p_1)^2+z_3R\cdot p_1\left(2m_1^2-2m_2^2+2p_1^2+z_1-z_2\right)+2p_1^2z_3^2\right) \\
\boldsymbol{h}_{1,4} = {} & -4\left(R\cdot p_1\left(m_1^2-m_2^2+p_1^2+z_1-z_2\right)+2p_1^2z_3\right). \qquad (4.7)
\end{aligned}
$$

Using (4.7), one can check that the IBP relation generated by $D_{\langle \boldsymbol{h}_1\rangle}$ acting on $\mathrm{I}_{a_1,a_2,-r-1}$ can be rewritten as

$$
\begin{aligned}
& \mathrm{I}_{a_1,a_2,-r} = \frac{1}{4p_1^2(a_1+a_2-r-d+1)}\times \\
& \Big[-2\left(m_1^2-m_2^2+p_1^2\right)R\cdot p_1\left(a_1+a_2-d-2r+2\right)\mathrm{I}_{a_1,a_2,-(r-1)} \\
& +(r-1)\left(4m_1^2(R\cdot p_1)^2-2m_1^2R^2\left(m_2^2+p_1^2\right)+R^2\left(m_2^2-p_1^2\right)^2+m_1^4R^2\right)\mathrm{I}_{a_1,a_2,-(r-2)} \\
& +\text{l.p.p.t.}\Big], \qquad (4.8)
\end{aligned}
$$

where

$$
\begin{aligned}
\text{l.p.p.t.} = & -2R\cdot p_1\left(a_1+a_2-r-d+1\right)\mathrm{I}_{a_1-1,a_2,-(r-1)} \\
& +2R\cdot p_1\left(a_1+a_2-r-d+1\right)\mathrm{I}_{a_1,a_2-1,-(r-1)} \\
& -(r-1)R^2\left(m_1^2-m_2^2+p_1^2\right)\mathrm{I}_{a_1,a_2-1,-(r-2)} \\
& +(r-1)\left(R^2\left(m_1^2-m_2^2-p_1^2\right)+2\left(R\cdot p_1\right)^2\right)\mathrm{I}_{a_1-1,a_2,-(r-2)}. \qquad (4.9)
\end{aligned}
$$

Result (4.8) is the iterative relation we are looking for. Notice that when $r=1$, the coefficient of $\mathrm{I}_{a_1,a_2,-(r-2)}=\mathrm{I}_{a_1,a_2,1}$ is zero by the $r-1$ factor, which satisfies the condition (3.5). Since r is the rank of tensor in the numerator, the relation (4.8) tells us that the integrals of tensor rank r can be written as the sum of integrals of tensor rank $(r-1)$ and $(r-2)$ with proper rational coefficients. This kind of relations has been firstly observed in [51] for the case $a_1=a_2=1$ and then has been proved in [64]. Here we give a simple derivation using module intersection with generalization to arbitrary power of propagators, as well as given the analytic l.p.p.t part missed in [51, 64]. It is obvious that applying this kind of second-order iterative relation, one can immediately reduce any tensor integrals of this sector to scalar integrals of the same sector and tensor integrals of sub-sectors.

Another interesting property of relation (4.8) is that in (4.8) the a_1, a_2 of top-sector are invariant, while for l.p.p.t. in (4.9), (a_1+a_2) has changed only by minus one. This observation will be explained in section 5.

Before ending this subsection, let us emphasize that to deal with tensor reduction of bubble topology, we have required $a_3=-r<0$. If we consider the case $a_3>0$, we will get the triangle topology, and the relation becomes IBP relation for triangles, but with the third propagator is not the standard quadratic one. In next subsection, we will consider triangle topology with the standard Feynman propagators, thus it will be useful to compare results in these two subsections.

4.2 Denominator reduction of triangle

In this subsection, we discuss the reduction of triangles with arbitrary powers for propagators. The propagators are

$$z_1 = l^2 - m_1^2, \quad z_2 = (l+p_1)^2 - m_2^2, \quad z_3 = (l+p_1+p_2)^2 - m_3^2. \tag{4.10}$$

and the corresponding Gram determinant $\mathcal{G}$ in Baikov representation is

$$\det \begin{pmatrix} m_1^2 + z_1 & \cdots & \cdots \\ \frac{1}{2}\left(-m_1^2 + m_2^2 - p_1^2 - z_1 + z_2\right) & p_1^2 & \cdots \\ \frac{1}{2}\left(-m_2^2 + m_3^2 - p_2^2 - 2p_1\cdot p_2 - z_2 + z_3\right) & p_1\cdot p_2 & p_2^2 \end{pmatrix} \tag{4.11}$$

where the $\cdots$ denote the terms, which can be obtained by symmetry. With

$$\begin{aligned}
\partial_{z_1}\mathcal{G} &= \frac{1}{2}\left(p_2^2\left(-m_1^2+m_2^2+p_1^2-z_1+z_2\right)+p_1\cdot p_2\left(m_2^2-m_3^2+p_2^2+z_2-z_3\right)\right),\\
\partial_{z_2}\mathcal{G} &= \frac{1}{2}\left(p_1\cdot p_2\left(m_1^2-2m_2^2+m_3^2-p_1^2-p_2^2+z_1-2z_2+z_3\right)+m_3^2p_1^2+m_1^2p_2^2\right.\\
&\quad \left.-m_2^2\left(p_1^2+p_2^2\right)+p_2^2z_1-p_1^2z_2-p_2^2z_2+p_1^2z_3-2\left(p_1\cdot p_2\right)^2\right),\\
\partial_{z_3}\mathcal{G} &= \frac{1}{2}\left(p_1^2\left(m_2^2-m_3^2+p_2^2+z_2-z_3\right)+p_1\cdot p_2\left(-m_1^2+m_2^2+p_1^2-z_1+z_2\right)\right).
\end{aligned} \tag{4.12}$$

one can solve

$$\sum_i^3 (P_i\partial_{z_i}\mathcal{G}) + P_0\mathcal{G} = 0, \tag{4.13}$$

with the basis $\{\boldsymbol{e}_i\}$ of the syzygy module is

$$\begin{pmatrix} m_1^2+m_3^2-s+z_1+z_3 & m_2^2+m_3^2-p_2^2+z_2+z_3 & 2\left(m_3^2+z_3\right) & -2 \\ m_1^2+m_2^2-p_1^2+z_1+z_2 & 2\left(m_2^2+z_2\right) & m_2^2+m_3^2-p_2^2+z_2+z_3 & -2 \\ 2\left(m_1^2+z_1\right) & m_1^2+m_2^2-p_1^2+z_1+z_2 & m_1^2+m_3^2-s+z_1+z_3 & -2 \end{pmatrix} \tag{4.14}$$

where $s=(p_1+p_2)^2$. Another module is generated by (see (3.10))

$$\{\boldsymbol{d}_i\} = \mathrm{DM}[z_1, z_2, 1, 1], \tag{4.15}$$

From them, we can compute $\{\boldsymbol{h}_i\}$ as the module intersection of them. Among them, the one with the lowest total power of z_i is given by

$$\begin{aligned}
\boldsymbol{h}_{1,1} &= 2z_1\left(p_1^2\left(m_2^2-m_3^2+p_2^2+z_2-z_3\right)+p_1\cdot p_2\left(-m_1^2+m_2^2+p_1^2-z_1+z_2\right)\right),\\
\boldsymbol{h}_{1,2} &= 2z_2\left(p_1^2\left(m_2^2-m_3^2+p_2^2+z_2-z_3\right)+p_1\cdot p_2\left(-m_1^2+m_2^2+p_1^2-z_1+z_2\right)\right),\\
\boldsymbol{h}_{1,3} &= -2\left(m_1^2p_2^2z_1-m_1^2p_2^2z_2-m_3^2p_1^2z_2+2\left(p_1\cdot p_2\right)^2\left(2m_2^2+z_2\right)+2m_3^2p_1^2z_3\right.\\
&\quad +m_2^2\left(-2m_3^2p_1^2-2m_1^2p_2^2+p_1^2z_2-2p_1^2z_3-p_2^2z_1+p_2^2z_2\right)\\
&\quad +p_1\cdot p_2\left(-m_2^2\left(2m_3^2-2p_1^2-2p_2^2+z_1-2z_2+2z_3\right)-m_1^2\left(2m_2^2-2m_3^2+2p_2^2+z_2-2z_3\right)\right.\\
&\quad \left.-2m_3^2p_1^2+m_3^2z_1-m_3^2z_2+2m_2^4-p_2^2z_1+p_1^2z_2+p_2^2z_2-2p_1^2z_3+2p_1^2p_2^2+z_1z_3-z_2z_3\right)\\
&\quad +m_1^4p_2^2-2m_1^2p_1^2p_2^2+m_3^4p_1^2-2m_3^2p_1^2p_2^2+m_2^4\left(p_1^2+p_2^2\right)+p_1^2z_3^2-p_1^2p_2^2z_1-2p_1^2p_2^2z_3\\
&\quad \left.-p_1^2z_2z_3+p_1^2p_2^4+p_1^4p_2^2\right),\\
\boldsymbol{h}_{1,4} &= 4p_1\cdot p_2\left(m_1^2-m_2^2-p_1^2+z_1-z_2\right)-4p_1^2\left(m_2^2-m_3^2+p_2^2+z_2-z_3\right).
\end{aligned} \tag{4.16}$$

Using (4.16) the action of $D_{\langle \boldsymbol{h}_1\rangle}$ on $\mathrm{I}_{a_1,a_2,a_3-1}$ gives the wanted iterative IBP relation

$$\begin{aligned}\mathrm{I}_{a_1,a_2,a_3} = &\frac{1}{Q_1}\times\Big[-p_1^2(a_1+a_2+a_3-d-1)\,\mathrm{I}_{a_1,a_2,a_3-2}\\&(2a_3+a_1+a_2-d-2)\left(p_1^2\left(m_2^2-m_3^2+p_2^2\right)+p_1\cdot p_2\left(-m_1^2+m_2^2+p_1^2\right)\right)\mathrm{I}_{a_1,a_2,a_3-1}\\&+\text{l.p.p.t.}\Big]\end{aligned}\tag{4.17}$$

where

$$\begin{aligned}\text{l.p.p.t.} = &(a_3-1)\left(p_2^2\left(-m_1^2+m_2^2+p_1^2\right)+p_1\cdot p_2\left(m_2^2-m_3^2+p_2^2\right)\right)\mathrm{I}_{a_1-1,a_2,a_3}\\&-(a_3-1)\Big(p_1\cdot p_2\left(-m_1^2+2m_2^2-m_3^2+p_1^2+p_2^2\right)-m_3^2p_1^2-m_1^2p_2^2+m_2^2\left(p_1^2+p_2^2\right)\\&+2\left(p_1\cdot p_2\right)^2\Big)\mathrm{I}_{a_1,a_2-1,a_3}\\&-p_1\cdot p_2\,(a_1+a_2+a_3-d-1)\,\mathrm{I}_{a_1-1,a_2,a_3-1}\\&+\left(p_1^2+p_1\cdot p_2\right)(a_1+a_2+a_3-d-1)\,\mathrm{I}_{a_1,a_2-1,a_3-1}\\Q_1 = &(a_3-1)\Big(m_1^4p_2^2-2m_1^2p_1^2p_2^2+4m_2^2\left(p_1\cdot p_2\right)^2+m_3^4p_1^2-2m_3^2p_1^2p_2^2+m_2^4\left(p_1^2+p_2^2\right)\\&-2p_1\cdot p_2\left(m_1^2-m_2^2-p_1^2\right)\left(m_2^2-m_3^2+p_2^2\right)-2m_2^2\left(m_3^2p_1^2+m_1^2p_2^2\right)+p_1^2p_2^4+p_1^4p_2^2\Big).\end{aligned}\tag{4.18}$$

Notice that when $a_3=2$, the coefficient of $\mathrm{I}_{a_1,a_2,a_3-2}=\mathrm{I}_{a_1,a_2,0}$ is a term of sub-sector (here is the bubble topology). Also for $a_3=1$ we don't need to apply this relation for DR since it is already the final goal we want to achieve. The relation (4.17) is also a second-order iterative relation relating I_{a_1,a_2,a_3} to $\mathrm{I}_{a_1,a_2,a_3-1}$, $\mathrm{I}_{a_1,a_2,a_3-2}$ and l.p.p.t.. Similar to (4.8), the a_1,a_2 of triangles in (4.17) are invariant, while for l.p.p.t. in (4.18), (a_1+a_2) has changed only by minus one.

Thus applying the relation (4.17) iteratively one can immediately reduce any higher power of z_3 to one. Similar iterative relations for reducing the power of other propagators can be obtained. Combining them, we can reduce integrals with arbitrary high power of propagators in this sector.

4.3 Discussions

Before going to general treatment in the next section, let us give some discussions for the two iterative relations (4.8) and (4.17) found in this section. These two relations can be interpret as non-homogenous finite difference equations discussed in [20, 57–59]. In [58, 59], treating the space-time dimension as the variable, algebraic relations between integrals of dimension $(d+2)$ and d have been established. In [20, 57], treating the power of a propagator as the variables, algebraic relations between integrals of different powers have been established. If we interpret the a_3 in (4.17) as the variable, it is exactly the type of recurrence relations found in [20, 57]. From this point of view, the (4.8) can be interpreted as a new type of recurrence relations where the tensor rank has been treated as variable.

However, there is some difference for the purpose of these relations. In [20, 57], it is aimed to find analytic expression for the integrals. Thus one must treat the parameter as continuous variable to get the correct results. In our paper, our aim is not as ambitious as theirs and we just want to find the reduction coefficients, therefore we do not need to treat power of propagator and tensor rank as continuous variable. In other words, (4.8) and (4.17) are literally the recurrence relations, where the indices are integer values. With the obvious boundary condition, solving them uses only elementary algebra.

At the technical level, our treatment has some new features too. In traditional reduction method, the redundant IBP relations are generated by unselected differential operator. When each time we want to reduce the power of denominators or nominators by one, we need to solve and combine them into the proper form. Comparing to it, in our method, we just need to solve the composite differential operator $D_{\langle \boldsymbol{h}_1 \rangle}$ only once using the computational algebraic method. The same iterative relation can used repeatedly to finish the reduction procedure.

5 The uniform formula for general one-loop reduction

The method laid out in section 3 and related computations done in section 4 look fresh, but maybe not so surprising. However, as we will show in this section, for one-loop integrals, we can write down explicit iterative relations for both TR and DR uniformly for any general n-point one-loop integrals. In other words, we have solved IBP relations analytically.

The key of our solving is to select particular elements in module intersection $\{\boldsymbol{h}_i\}$. Let's make an observation for two examples in section 4, i.e., both results (4.7) and (4.16), we find that

$$\{\boldsymbol{h}_{1,1}, \boldsymbol{h}_{1,2}, \boldsymbol{h}_{1,4}\} = C \times \{z_1 \partial_{z_3} \mathcal{G}, z_2 \partial_{z_3} \mathcal{G}, -2\partial_{z_3} \mathcal{G}\} \tag{5.1}$$

and then by (4.13), we have

$$\boldsymbol{h}_{1,3} = C \times (2\mathcal{G} - z_1 \partial_{z_1} \mathcal{G} - z_2 \partial_{z_2} \mathcal{G}) \,. \tag{5.2}$$

This pattern indicates that for DR of the N-th propagator of the N-point one-loop integrals or the TR of $(N-1)$-point one-loop integrals (where the $(R \cdot \ell)$ has been considered as the N-th propagator), the wanted element in the intersection module is given by

$$\begin{aligned} P_{ui} &= z_i \partial_{z_N} \mathcal{G} \quad \text{for} \;\; 1 \le i \le N-1, \\ P_{uN} &= 2\mathcal{G} - \sum_{i=1}^{N-1} z_i \partial_{z_i} \mathcal{G}, \qquad P_{u0} = -2\partial_{z_N} \mathcal{G}. \end{aligned} \tag{5.3}$$

It is easy to check that (5.3) satisfies (2.6) and belongs to the module

$$\{\boldsymbol{d}_i\} = \mathrm{DM}[z_1, z_2, \cdots, z_{N-1}, 1, 1], \tag{5.4}$$

then the differential operator $D_{\langle P_u \rangle}$ gives the generic iterative relation for both TR and DR of general one-loop integrals.

To write down explicit relation, let us compute P_u in (5.3). For one-loop integrals, Gram determinant $\mathcal{G}$ is always quadratic polynomial of z_is

$$\mathcal{G}(\boldsymbol{z}) = \sum_{i,j\leq i} C_2^{(ij)} z_i z_j + \sum_i C_1^{(i)} z_i + C_0 \tag{5.5}$$

where

$$\begin{aligned} &C_2^{(ii)} = \partial_{z_i}^2 \mathcal{G}/2, \quad C_1^{(i)} = (\partial_{z_i}\mathcal{G})\,|_{\boldsymbol{z}=0}, \quad C_0 = \mathcal{G}|_{\boldsymbol{z}=0}, \\ &C_2^{(ij)} = C_2^{(ji)} = \partial_{z_i}\partial_{z_j}\mathcal{G} \qquad \text{for} \quad j\neq i. \end{aligned} \tag{5.6}$$

This leads to

$$\begin{aligned} \partial_{z_i}\mathcal{G} &= 2C_2^{(ii)} z_i + \sum_{j\neq i} C_2^{(ij)} z_j + C_1^{(i)}, \\ \sum_{i=1}^{N} z_i \partial_{z_i}\mathcal{G} &= 2\sum_{i,j\leq i} C_2^{(ij)} z_i z_j + \sum_i C_1^{(i)} z_i, \\ P_{uN} &= z_N \partial_{z_N}\mathcal{G} + \sum_i C_1^{(i)} z_i + 2C_0. \end{aligned} \tag{5.7}$$

The rewriting of P_{uN} in (5.7) tells us that P_{uN} depends on $z_i, i = 1, \ldots, N-1$ only linearly.

Combining P_{ui} in (5.3) and $\partial_{z_i}\mathcal{G}$ in (5.7), one can see that the power of z_is will lead all $\mathrm{I}_{a'_1,\cdots,a'_{N-1},a'_N}$ to satisfy $0 \leq \sum_{i=1}^{N-1} a_i - \sum_{i=1}^{N-1} a'_i \leq 1$. To show that, let us do the following explicit computations. For $i \leq N-1$, carrying out

$$-D_{P_{ui}} \cdot \frac{1}{\prod_{i=1}^{N} z_i^{a_i}} = -\partial_{z_i}\left(\frac{z_i \partial_{z_N}\mathcal{G}}{\prod_{i=1}^{N} z_i^{a_i}}\right), \tag{5.8}$$

we find

$$\begin{aligned} &(a_i-1)\left(2C_2^{(NN)}\mathrm{I}_{\cdots,a_N-1} + C_1^{(N)}\mathrm{I}_{\cdots,a_N}\right) \\ &+(a_i-2)C_2^{(Ni)}\mathrm{I}_{\cdots,a_i-1,\cdots,a_N} + (a_i-1)\sum_{j=1,j\neq i}^{N-1} C_2^{(Nj)}\mathrm{I}_{\cdots,a_j-1,\cdots,a_N}. \end{aligned} \tag{5.9}$$

For $i = N$, action of D_{P_N} gives

$$\begin{aligned} &2\left[(a_N-2)C_2^{(NN)}\mathrm{I}_{\cdots,a_N-1} + (a_N-1)C_1^{(N)}\mathrm{I}_{\cdots,a_N} + a_N C_0 \mathrm{I}_{\cdots,a_N+1}\right] \\ &+\left[(a_N-1)\sum_{j\neq N} C_2^{(Nj)}\mathrm{I}_{\cdots,a_j-1,\cdots,a_N} + a_N \sum_{j\neq N} C_1^{(j)}\mathrm{I}_{\cdots,a_j-1,\cdots,a_N-1}\right]. \end{aligned} \tag{5.10}$$

Finally action of D_{P_0} gives

$$-(d-N-1)\left(C_1^{(N)}\mathrm{I}_{\cdots,a_N} + 2C_2^{(NN)}\mathrm{I}_{\cdots,a_N-1} + \sum_{j=1}^{N-1} C_2^{(Nj)}\mathrm{I}_{\cdots,a_j-1,\cdots,a_N}\right). \tag{5.11}$$

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Combining all together, we have

$$
\begin{aligned}
&2a_N C_0 \mathrm{I}_{\cdots,a_N+1} + \left(\sum_{i=1}^{N-1} a_i + 2a_N - d\right) C_1^{(N)} \mathrm{I}_{\cdots,a_N} \\
&+ \left(\sum_{i=1}^{N} a_i - d\right) 2C_2^{(NN)} \mathrm{I}_{\cdots,a_N-1} + \text{l.p.p.t.} = 0
\end{aligned}
\tag{5.12}
$$

where

$$
\text{l.p.p.t.} = \left(\sum_{i=1}^{N} a_i - d\right) \sum_{j=1}^{N-1} C_2^{(Nj)} \mathrm{I}_{\cdots,a_j-1,\cdots,a_N} + a_N \sum_{j=1}^{N-1} C_1^{(j)} \mathrm{I}_{\cdots,a_j-1,\cdots,a_N-1} \,. \tag{5.13}
$$

Expressions (5.12) and (5.13) are our main results for this paper. Let us emphasize again that although to arrive (5.12) and (5.13) we have used the syzygy and module intersection, when we got them, we can forget our method completely and just use the explicit result.

When $a_N < 0$, it corresponds to the numerator and we should use (5.12) to express $\mathrm{I}_{\cdots,a_N-1}$ by others for the TR, while when $a_N > 0$ it corresponds to the denominator with a higher power and we should use (5.12) to express $\mathrm{I}_{\cdots,a_N+1}$ by others for the DR. When $a_N = 0$, the (5.12) will give the relation between $\mathrm{I}_{\cdots,-1}$ and $\mathrm{I}_{\cdots,0}$, which is just the reduction of tensor with rank one.

6 Example of degenerate case

In section 5, we have assumed that the kinematics and masses are general. But when kinematics and masses take some special values, the Gram determinant may be zero and we meet the degenerate situations. For such situations, some master integrals will disappear. This can be seen by counting the number of master integrals using critical point [65] (see also [33, 66] in Baikov representation). The existence of vanished Gram determinant requires some special treatment in many method, see for example, in [30, 67] for traditional momentum representation. The main point we want to show in this section is that relations (5.12) and (5.13) do not need to be modified and directly using them the integrals in this sector can all be reduced to sub-sectors (because there is no master integrals in this sector).

To show that there is no modification needed for (5.12) and (5.13), we will show that there are three different possibilities. The first one is that while it works, it may not give the simplest iterative relation. The second one is that since there are different choices when applying (5.12) and (5.13), while it may not work for some choices, there is at least one choice enough for complete reduction. The third one is that it does not work for all choices, but for this case, the integral is scaleless and can be throw away, so it does not matter at all. Furthermore, no matter which situation one meets, one can always go back to the method of module intersection presented in the section 3 and do some computation, although extra labor is needed comparing to the plain use of (5.12) and (5.13).

Now we give an example to elaborate above claims. Let us consider the reduction of triangles

$$\begin{aligned} &z_1 = l^2, && z_2 = (l+p_1)^2, && z_3 = (l+p_1+p_2)^2, \\ &z_4 = l\cdot R, && p_1^2 = p_2^2 = 0. \end{aligned} \tag{6.1}$$

with specific kinematics. As shown later, by IBP relations one can see that all integrals in this sector can be reduced to sub-sectors.

For TR of (6.1), $\mathcal{G} = G(l, p_1, p_2, R)$. Directly applying the formula (5.12) and (5.13), we will get[8]

$$\mathrm{I}_{a_1,a_2,a_3,-r-1} = \frac{R\cdot p_1 \left(r\ R\cdot p_1\ \mathrm{I}_{a_1,a_2,a_3,1-r} - \left(\sum_i^3 a_i - d - 2r\right) \mathrm{I}_{a_1,a_2,a_3,-r}\right)}{\sum_i^3 a_i - d - r} + \text{l.p.p.t.}\ , \tag{6.2}$$

which is a second-order iterative relation. Using (6.2), one can still reduce all $\mathrm{I}_{1,1,1,-r}$ to $\mathrm{I}_{1,1,1,0}$. However, $\mathrm{I}_{1,1,1,0}$ is not a master integral in this example. To reduce $\mathrm{I}_{1,1,1,0}$ we should use (5.12) and (5.13) for DR, which will be explained shortly.

If we calculate the module intersection follow the method in section 3, one can find another element $\boldsymbol{h}_i$ different from the one given in (5.3), which will give us the following iterative reduction relation

$$\mathrm{I}_{a_1,a_2,a_3,-r} = \frac{rR\cdot p_1 \mathrm{I}_{a_1,a_2,a_3,1-r}}{2a_2 + 2a_3 - d - r} + \text{l.p.p.t.}\ . \tag{6.3}$$

Obviously, as a first-order iterative relation, (6.3) is simpler than (6.2), which supports the claim of the first possibility.

For DR,[9] the Gram determinant in Baikov representation is

$$\mathcal{G} = G(l, p_1, p_2) = \frac{1}{2} p_1\cdot p_2 \left((z_1 - z_2)(z_2 - z_3) - 2p_1\cdot p_2 z_2 \right). \tag{6.4}$$

To determine the number of master integrals, we can consider the maximum cut of this sector in Baikov representation, i.e., setting $z_i = 0$ in $\mathcal{G}$. If $\mathcal{G}|_{z=0}$ is a nonzero constant, the number of master integrals is just one by counting critical points [34, 65, 66]. But if $\mathcal{G}|_{z=0} = 0$, there is no master integral in this sector. It is easy to see that $\mathcal{G}|_{z=0} = 0$ in (6.4), thus $\mathrm{I}_{1,1,1}$ can be reduced to sub-sectors.

One can check that if we regard z_1 or z_3 as the z_N and apply (5.12), the $C_0, C_1^{(N)}, C_2^{(NN)}$ are all zero and only l.p.p.t. is left. In other words, for these cases, (5.12) does not produce the wanted relation for reduction purpose. However, if we regard z_2 as the z_N, the iterative relation reduce to first-order due to $C_0 = 0$, $C_1^{(2)} \neq 0$ and $C_2^{(22)} \neq 0$. Using it, we can reduce $I_{a_1,a_2,a_3,0}$ (including the $I_{1,1,1,0}$ discussed in previous paragraph) to sub-sectors $I_{a_1,0,a_3,0}$ and others. The a_1 and a_3 are left to be reduced by DR of (5.12) in the sub-sector. So, as pointed out for the second possibility, although using (5.12) for the DR does not work for z_1 and z_3, there is z_2 it works.

[8] R^2 does not appear in (6.2) because $G(l, p_1, p_2)|_{z=0} = 0$ when using (5.12) and (5.13).

[9] Now there is no z_4 in (6.1).

Nevertheless, even if we regard z_3 as the z_N, we can do similar module computation proposed in section 3. The syzygy module is generated by

$$\{\boldsymbol{e}_i\} = \begin{pmatrix} z_2 & z_2 & z_3 & -1 \\ 2p_1\cdot p_2 - z_3 & -z_3 & z_2 - 2z_3 & 1 \\ -2p_1\cdot p_2 + z_1 - z_2 + z_3 & z_3 & z_3 & -1 \\ 4p_1\cdot p_2 + z_2 - 2z_3 & z_1 - 2z_3 & -2p_1\cdot p_2 + z_1 - 2z_3 & 1 \end{pmatrix}, \tag{6.5}$$

while the element in the intersection module is taken to be

$$\langle P\rangle = \langle z_1, z_2, z_2, 1\rangle. \tag{6.6}$$

Using (6.6), the iterative relation is

$$\mathrm{I}_{a_1,a_2,a_3,0} = -\frac{2a_3\mathrm{I}_{a_1,a_2-1,a_3+1,0}}{2a_1+2a_2-d} \tag{6.7}$$

where the (a_1+a_2) has been reduced by one at the right-hand side, which is just a l.p.p.t.. Although this relation raises the power of z_3 in denominator, it will lower a_2 to 0 finally (i.e., reduce to sub-sector), so it shows that method of module intersection still works.

From this example, it is easy to see that only for more degenerated case, i.e., all $C_2^{(ii)}$, $C_1^{(i)}$ and C_0 in $\mathcal{G}$ equal zero, (5.12) can not reduce integrals in this sector to sub-sectors. But for this situation, all sub-sectors have no master integrals by counting the critical points. To be more explicitly, the number of master integrals in the corresponding sector is equal to the number of solutions to the equations

$$0 = \partial_{z_i}\log\left(\mathcal{G}^{(d-n-1)/2}\right)|_{\boldsymbol{z}'=0}, \quad \text{for all } z_i \notin \{\boldsymbol{z}'\}, \tag{6.8}$$

Now the $\mathcal{G}$ takes the form

$$\mathcal{G} = \sum_{i,j\neq i} c_{ij}z_iz_j \tag{6.9}$$

and the equations (6.8) take the form

$$\frac{C}{z_i - C_i} = 0, \tag{6.10}$$

which obviously has no solution. It means that integrals in this topology are all scaleless integrals, and this topology has no master integral. One can test this conclusion by taking $p_1\cdot p_2$ to zero in (6.4) and immediately find this topology to be scaleless. This is the third possibility we have mentioned.

7 Summary and outlook

In this paper, motivated by improving the efficiency of reduction program, a natural extension of syzygy and module intersection has been explored to find good iterative structures appearing in the reduction procedure. With a nice observation, powerful iterative relation can be written down even without doing explicit computations using computational algebraic geometry and module intersection. Using it, one can uniformly reduce one-loop integrals with arbitrary tensor (by using auxiliary vector) and propagators with arbitrary high powers. Our iterative relations make not only the reduction straightforward, but also give compact expressions. More explicitly, in this formula the polynomials that look messy in the traditional reduction methods are arranged themselves to Gram determinants. Such a property will not only speed up the analytic reduction of Feynman integrals, but also help the investigation of mathematical structures of Feynman integrals and amplitudes in the integrals' level (or says IBP's level) in the future.

It is obvious that the final goal of our study is the reduction for general high loops. However, If a method is good, it must work well at the one-loop level as the first step. Results in this paper have demonstrate this point. To generalize this method to high loops, although steps of computational algebraic geometry can be easily applied, some nontrivial problems will arise. There are irreducible scalar products of loop momenta and more than one master integrals for a given sector in most multi-loop integrals. This suggests that for multi-loop integrals, elements of module intersection needed may also be more than one. More importantly, could we obtain the general reduction formulas like (5.12) for multi-loops? Are there also some hidden information or structures in the messy polynomials in the IBP relation? These problems are definitely interesting for future exploration.

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Baikov representations, intersection theory, and canonical Feynman integrals

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Abstract: The method of canonical differential equations is an important tool in the calculation of Feynman integrals in quantum field theories. It has been realized that the canonical bases are closely related to d-dimensional $d\log$-form integrands. In this work, we explore the generalized loop-by-loop Baikov representation, and clarify its relation and difference with Feynman integrals using the language of intersection theory. We then utilize the generalized Baikov representation to construct d-dimensional $d\log$-form integrands, and discuss how to convert them to Feynman integrals. We describe the technical details of our method, in particular how to deal with the difficulties encountered in the construction procedure. Our method provides a constructive approach to the problem of finding canonical bases of Feynman integrals, and we demonstrate its applicability to complicated scattering amplitudes involving multiple physical scales.

Keywords: Higher Order Electroweak Calculations, Higher-Order Perturbative Calculations, Scattering Amplitudes

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Contents

1 Introduction

Feynman integrals are central objects in perturbative quantum field theories (QFTs). They are the basic ingredients of correlation functions and scattering amplitudes, which are the essential bridges between fundamental theories and experimental observations. The analytic, algebraic and geometric properties of these integrals provide many new insights on QFTs themselves. In textbooks, Feynman integrals are usually represented as integrals over loop momenta or integrals over Feynman parameters. Techniques based on these representations have been greatly advanced in the past decades (see, e.g., [1–3] and references therein), leading to a proliferation of new results which cannot be obtained using traditional methods.

An important toolset in the calculation of Feynman integrals is the integration-by-parts (IBP) identities [4, 5] combined with the method of differential equations [6–10]. The IBP identities are used to reduce all scalar Feynman integrals appearing in a scattering process to a finite set of master integrals (MIs). Such a reduction can be systematically performed with the Laporta algorithm [11] implemented in various program packages such as `AIR` [12], `FIRE` [13, 14], `LiteRed` [15, 16], `Reduze` [17, 18] and `Kira` [19, 20]. The MIs satisfy a closed system of linear differential equations. If these equations can be solved, one obtains the results for the MIs and hence for all integrals under consideration.

In certain cases, the differential equations can be organized into a nice form called the ϵ-form [2, 21, 22]:

$$d\vec{f}(\vec{x},\epsilon) = \epsilon \, d\boldsymbol{A}(\vec{x}) \, \vec{f}(\vec{x},\epsilon) \,, \tag{1.1}$$

where $\epsilon = (4-d)/2$ is the dimensional regulator with spacetime dimension d, $\vec{x} = \{x_i\}$ is the list of kinematic variables, $\vec{f} = \{f_i\}$ is the list of linear combinations of master integrals, and $d\boldsymbol{A}$ is a matrix of the $d\log$ form independent of ϵ. Once written in the ϵ-form, the solutions to the differential equations can be formally written as Chen iterated integrals [23]. The results can often be written in terms of generalized polylogarithms (GPLs) [24, 25] order-by-order in ϵ, which allow efficient numeric evaluation [26–28]. When an analytic solution is not available, it is straightforward to evaluate them numerically either by numerical integration or by a series expansion [29–31].

The list of master integrals $\vec{f}$ satisfying eq. (1.1) is called a canonical basis. These integrals have the property of *uniform transcendentality* (UT) [21]. Namely, they (with suitable normalization) can be expressed as

$$f_i(\vec{x},\epsilon) = \sum_{n=0}^{\infty} \epsilon^n \, f_i^{(n)}(\vec{x}) \,, \tag{1.2}$$

where $f_i^{(n)}(\vec{x})$ is a function with transcendental weight n. It is conventional to assign weight -1 to ϵ, such that the whole function $f_i(\vec{x},\epsilon)$ has weight 0. In a practical problem, it is crucial to find such a canonical basis of UT integrals. This can be done by starting from an arbitrary set of MIs, and performing linear transformations to reduce the differential equations to the ϵ-form. Algorithms for finding such kind of transformations exist [32–39], and some of which have been implemented as program packages [40–43]. These algorithms are particularly useful when only rational transformations are needed.

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An alternative way to find a canonical basis is to construct UT integrals directly without studying the differential equations. It has been realized that UT integrals are closely related to $d\log$-form integrands in $d = 4$ dimensions [44–49], i.e., the integrands can be written (usually in the momentum representation or in certain dual representations) in the form

$$c\, d\log\alpha_1 \wedge d\log\alpha_2 \wedge \cdots \wedge d\log\alpha_n\,, \tag{1.3}$$

where α_i are functions of the integration variables and c is constant. Integrals with such integrands are also dubbed as having constant leading singularities. However, these 4-dimensional $d\log$ integrands are not guaranteed to give rise to UT integrals in $d = 4 - 2\epsilon$ dimensions. Further manipulation is therefore required to arrive at a canonical basis. Construction methods based on the 4-dimensional $d\log$ integrands have been considered in [22, 50–54].

Motivated by the 4-dimensional $d\log$ integrands, it was suggested [50, 52, 55] that one may consider d-dimensional $d\log$ integrands in a suitable representation (where the dimensional regulator ϵ appears as a parameter in the integrand) such as the Baikov representation [56–61]. These $d\log$-forms can be written as

$$c\,[\alpha_0(\boldsymbol{z})]^\epsilon\, d\log\alpha_1(\boldsymbol{z}) \wedge d\log\alpha_2(\boldsymbol{z}) \wedge \cdots \wedge d\log\alpha_n(\boldsymbol{z})\,, \tag{1.4}$$

where $\boldsymbol{z}$ denotes the collection of integration variables (which correspond to coordinates in the base manifold for the differential n-forms). Such d-dimensional $d\log$-forms automatically give rise to UT integrals without further manipulation. This then gives strong hints on the construction of a canonical basis for a given integral family. However, finding a complete set of d-dimensional $d\log$-form integrands is often not a trivial task. In that case one may employ weaker constraints such as looking for integrands having constant leading singularities under certain cuts (which reduce the number of integration variables) [62]. Integrands satisfying such weaker constraints can then be further manipulated to arrive at UT integrals.

In this paper, we develop in more detail the studies of [55], on the construction of d-dimensional $d\log$-form integrands in the Baikov representation as candidates for UT Feynman integrals. We first review the standard and loop-by-loop Baikov representations, and explore the *generalized* loop-by-loop Baikov representation with additional polynomials in the denominators. As will be clear later (and as was mentioned in [62]), the generalized Baikov integrals do not all correspond to Feynman integrals. We introduce the concept of FI-subspace spanned by Feynman integrals within the vector space of generalized Baikov integrals. These vector spaces are studied using the language of intersection theory [63–72]. We demonstrate how to find linear combinations of generalized Baikov integrals that belong to the FI-subspace, and how to convert them to Feynman integrals. We then elaborate on our method of constructing $d\log$-form Baikov integrands and subsequently obtaining the complete canonical basis for a given integral family. We describe how we deal with the technical difficulties encountered in this procedure. We show that our approach can be well applied to complicated problems involving multiple physical scales.

The paper is organized as follows. In section 2, we review the standard and the loop-by-loop Baikov representations, and introduce the concept of generalized loop-by-loop

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Baikov representation. In section 3, we briefly review the concept of intersection theory in the context of Feynman and Baikov integrals. Special focus is put on the correspondence between the dimension of twisted cohomology groups and the number of Baikov integrals. In section 4, we introduce the method for the construction of UT Baikov integrals and for the conversion to canonical Feynman integrals. In section 5 and 6, we demonstrate our method using two non-trivial examples, while technique details and further examples are presented in the appendices. We summarize in section 7.

2 The Baikov representation of Feynman integrals

The Baikov representation was first proposed in [56], and since then were further developed and used to study Feynman integrals [57–61]. In this section, we recap the derivation of the Baikov representation both in the standard and the loop-by-loop approaches. We also propose a generalization of the loop-by-loop representation, that will be useful in our construction of $d\log$-form integrands.

2.1 The standard Baikov representation

We consider L-loop Feynman integrals with $E+1$ external legs in spacetime dimension $d = 4-2\epsilon$. The loop momenta are labelled by k_i ($i = 1, \ldots, L$) and the independent external momenta are p_i ($i = 1, \ldots, E$). For later convenience we collectively refer to them as q_i ($i = 1, \ldots, M$), where $M \equiv L + E$, $q_i \equiv k_i$ ($i = 1, \ldots, L$), and $q_{L+i} \equiv p_i$ ($i = 1, \ldots, E$). Out of these momenta, one can construct $N \equiv L(L + 1)/2 + LE$ independent scalar products involving at least one of the k_i. An integral family is then defined by a given set of N independent propagator denominators z_i ($i = 1, \ldots, N$), which are linear functions of the aforementioned scalar products. A generic integral in such a family is given by

$$F_{a_1,\ldots,a_N} = e^{\epsilon\gamma_E L} \int \left[\prod_{i=1}^{L} \frac{d^d k_i}{i\pi^{d/2}}\right] \frac{1}{z_1^{a_1}\, z_2^{a_2} \cdots z_N^{a_N}}\,, \tag{2.1}$$

where $a_i \in \mathbb{Z}$. A specific topology in the integral family is defined by a chosen subset of the powers $\{a_i\}$ whose values are positive, while the other powers are either zero or negative.

The Baikov representation of the above integral amounts to a change of integration variables from the set $\{k_i^\mu\}$ to the set $\{z_n\}$. For that purpose, we write

$$z_n = \sum_{i=1}^{L}\sum_{j=i}^{M} A_n^{ij}\, s_{ij} + f_n\,, \quad (n = 1, \ldots, N)\,, \tag{2.2}$$

where $s_{ij} \equiv q_i \cdot q_j$, A_n^{ij} are integer constants, and f_n are functions of external momenta and internal masses. Note that the number of the ordered pairs (ij) is N. Therefore A_n^{ij} can be regarded as the elements of an $N \times N$ matrix representing the linear transformation from $\{s_{ij}\}$ to $\{z_n - f_n\}$. We denote this matrix as

$$\boldsymbol{A} = \boldsymbol{A}(z_1, \ldots, z_N; k_1, \ldots, k_L; p_1, \ldots, p_E)\,, \quad A_n^{ij} = \boldsymbol{A}_{n,(ij)}\,. \tag{2.3}$$

With a slight abuse of notation, we denote the elements of the inverse of the matrix $\boldsymbol{A}$ as A^n_{ij}, namely,

$$A^n_{ij} = \left(\boldsymbol{A}^{-1}\right)_{(ij),n}\,, \qquad \sum_{ij} A^m_{ij} A^{ij}_n = \delta_{mn}\,, \qquad \sum_n A^{ij}_n A^n_{kl} = \delta_{(ij),(kl)}\,. \tag{2.4}$$

Therefore we have

$$s_{ij} = \sum_{n=1}^{N} A^n_{ij}(z_n - f_n)\,, \quad (i = 1,\ldots,L;\, j = i,\ldots,M)\,. \tag{2.5}$$

To proceed, we decompose each loop momentum k_i into two parts, $k_i^\mu = k_{i\parallel}^\mu + k_{i\perp}^\mu$, where the parallel components $k_{i\parallel}^\mu$ live in the $(M-i)$-dimensional subspace spanned by q_j $(j = i+1,\ldots,M)$, and the perpendicular components $k_{i\perp}^\mu$ live in the $(d-M+i)$-dimensional orthogonal subspace.[1] The integration measure over the parallel components of k_i is given by

$$d^{M-i}k_{i\parallel} = \left|G(q_{i+1},\ldots,q_M)\right|^{-1/2} \prod_{j=i+1}^{M} ds_{ij}\,, \tag{2.6}$$

where $G(q_1,\ldots,q_n)$ is the Gram determinant defined as

$$G(q_1,\ldots,q_n) \equiv \det(q_i\cdot q_j) \equiv \det \begin{pmatrix} q_1\cdot q_1 & q_1\cdot q_2 & \cdots & q_1\cdot q_n \\ q_2\cdot q_1 & q_2\cdot q_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ q_n\cdot q_1 & \cdots & \cdots & q_n\cdot q_n \end{pmatrix}. \tag{2.7}$$

Note also that $|G(q_1,\ldots,q_n)|^{1/2}$ is the volume of the parallelogram formed by $q_1,\ldots,q_n$ (in the Euclidean sense).

For the perpendicular components $k_{i\perp}^\mu$, only the norm-squared $k_{i\perp}^2$ enters the integrand since $s_{ii} = k_i^2 = k_{i\perp}^2 + k_{i\parallel}^2$. We perform a Wick rotation for the integration contour of $k_{i\perp}^0$ from the real axis to the imaginary axis (during which the value of $k_{i\perp}^2$ is deformed into the complex plane, and in the end gets back to the real axis but with $k_{i\perp}^2 \le 0$). We then change variable to the Euclidean vector k_{iT}^μ as usual with $k_{iT}^2 = -k_{i\perp}^2$. The norm-squared can be expressed in terms of $\{q_i\cdot q_j\}$ through

$$k_{iT}^2 = -\frac{G(q_i,\ldots,q_M)}{G(q_{i+1},\ldots,q_M)} = \frac{|G(q_i,\ldots,q_M)|}{|G(q_{i+1},\ldots,q_M)|} \ge 0\,. \tag{2.8}$$

The integration measure for the perpendicular components can then be written as

$$d^{d-M+i}k_{i\perp} = \frac{i\,\pi^{(d-M+i)/2}}{\Gamma((d-M+i)/2)} \left|\frac{G(q_i,\ldots,q_M)}{G(q_{i+1},\ldots,q_M)}\right|^{(d-M+i-2)/2} ds_{ii}\,. \tag{2.9}$$

[1] There is some subtlety in this decomposition with the Minkowski signature. We will assume that the parallel subspace contains space-like vectors (i.e., we work in the so-called "Euclidean" kinematic region), such that vectors in the perpendicular subspace are time-like. Results for physical kinematics can be obtained via analytic continuation.

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Using the above, we are able to change the integration variables from $\{k_i^\mu\}$ to $\{s_{ij}\}$. We can further change variable to the Baikov variables $\{z_n\}$ using eq. (2.5) and

$$\prod_{i=1}^{L}\prod_{j=i}^{M} ds_{ij} = |\det\left(\boldsymbol{A}^{-1}\right)| \prod_{n=1}^{N} dz_n \,. \tag{2.10}$$

Finally, we arrive at

$$F_{a_1,\ldots,a_N} = \frac{C_{L,E}\,|\det\left(\boldsymbol{A}^{-1}\right)|}{\left|G(p_1,\ldots,p_E)\right|^{(d-E-1)/2}} \int \prod_{n=1}^{N} dz_n \, \frac{u_{\text{std}}(z_1,\ldots,z_N)}{z_1^{a_1}\cdots z_N^{a_N}} \,, \tag{2.11}$$

where the prefactor is

$$C_{L,E} = \frac{e^{\epsilon\gamma_E L}\,\pi^{-L(L-1)/4-LE/2}}{\prod_{i=1}^{L}\Gamma\big((d-M+i)/2\big)} \,, \tag{2.12}$$

and the u_{std} function takes the form

$$u_{\text{std}}(z_1,\ldots,z_N) \equiv \left|P_{L,E}(z_1-f_1,\ldots,z_N-f_N)\right|^{(d-M-1)/2} \,, \tag{2.13}$$

with the Baikov polynomial ($x_n \equiv z_n - f_n$)

$$P_{L,E}(x_1,\ldots,x_N) = G(q_1,\ldots,q_M)\Big|_{s_{ij}=A_{ij}^n x_n} \,. \tag{2.14}$$

The integration domain for the Baikov variables can be deduce from eq. (2.8). We need to require $G(q_i,\ldots,q_M)/G(q_{i+1},\ldots,q_M) \leq 0$ for each $i = 1,\ldots,L$. The signs of individual Gram determinants can then be fixed according to the sign of the Gram determinant of the external momenta. These impose restrictions on the values of the Baikov variables. It is possible that the space of loop momenta is covered more than once when the variables are varied within this domain. In this case an extra normalization factor is required, which is however irrelevant to the purposes of this work. Later on we will regard the variables as complex, and the integration in the real domain can be deformed into the complex space. To do that we need to firstly rewrite the absolute value of the Gram determinants as $\pm G(q_i,\ldots,q_M)$ according to their signs. We will often suppress these $\pm$'s when they are not important, but they should be kept in mind when considering the integration domain.

2.2 An explicit example

Usually one would not directly use the Baikov representation to calculate Feynman integrals, since other parameterizations are often more convenient in this respect. In this subsection we use a simple example to explicitly demonstrate how the Baikov representation works and how to deal with the integration domain which will prove to be important later. The example is the one-loop bubble integral given by

$$\begin{aligned} I(Q^2,\epsilon) = e^{\epsilon\gamma_E} \int \frac{d^dk}{i\pi^{d/2}} \frac{1}{k^2} \frac{1}{(k+p)^2} &= e^{\epsilon\gamma_E}\Gamma(\epsilon) \int_0^1 dx \left[Q^2 x(1-x)\right]^{-\epsilon} \\ &= e^{\epsilon\gamma_E} \left(Q^2 - i0\right)^{-\epsilon} \frac{\Gamma^2(1-\epsilon)\,\Gamma(\epsilon)}{\Gamma(2-2\epsilon)} \,. \end{aligned} \tag{2.15}$$

where $d = 4-2\epsilon$ and $Q^2 \equiv -p^2 > 0$. We have suppressed the Feynman $+i0$ prescription until the last expression, which is important in the analytic continuation to the region $p^2 > 0$.

Now we follow the approach in the previous subsection to get the Baikov representation for the above integral. The Baikov variables are $z_1 = k^2$ and $z_2 = (k+p)^2$. The relevant Gram determinants are

$$G(p) = p^2 = -Q^2 \,, \qquad G(k,p) = k^2\,p^2 - (k\cdot p)^2 = -\frac{1}{4}\left[\left(z_1 - z_2 - Q^2\right)^2 + 4Q^2 z_1\right]. \tag{2.16}$$

Since $G(p) < 0$, the integration domain is determined by $G(k,p) \geq 0$. We change variable to $u = (z_1 - z_2)/Q^2$, $v = z_1/Q^2$, and define the polynomial

$$P(u,v) = \frac{4G(k,p)}{Q^4} = -(u-1)^2 - 4v \geq 0\,. \tag{2.17}$$

The integration domain for u and v is then

$$u \in (-\infty, +\infty)\,, \qquad v \in \left(-\infty, -\frac{(u-1)^2}{4}\right). \tag{2.18}$$

The Baikov representation can be written in the form

$$I\left(Q^2,\epsilon\right) = \mathcal{N}_\epsilon\left(Q^2\right)\, f(\epsilon)\,, \tag{2.19}$$

where

$$\mathcal{N}_\epsilon(Q^2) = \frac{e^{\epsilon\gamma_E}\,\Gamma(1-\epsilon)}{2\pi\Gamma(2-2\epsilon)}\left(Q^2\right)^{-\epsilon}\,,$$
$$f(\epsilon) = \int_{P\geq 0} du\,dv\,\frac{[P(u,v)]^{1/2-\epsilon}}{v\,(v-u)}\,. \tag{2.20}$$

The integration over v can be carried out using partial fraction, and we arrive at

$$f(\epsilon) = \frac{\pi}{\cos(\pi\epsilon)}\int_{-\infty}^{+\infty}\frac{du}{u}\left[\left((u+1)^2\right)^{1/2-\epsilon} - \left((u-1)^2\right)^{1/2-\epsilon}\right]. \tag{2.21}$$

Note that the integrand is not singular at $u=0$ due to the cancellation between the two terms. However in practice, it is more convenient to perform the integration for each term separately, which then requires some extra regularization. We employ the analytic regulator $u^{-1} \to (u^2)^\delta u^{-1}$, and take the limit $\delta \to 0$ in the end. This gives

$$\int_{-\infty}^{+\infty}\frac{du}{u}\left((u+1)^2\right)^{1/2-\epsilon} = -\int_{-\infty}^{+\infty}\frac{du}{u}\left((u-1)^2\right)^{1/2-\epsilon} = \cos(\pi\epsilon)\,\Gamma(\epsilon)\,\Gamma(1-\epsilon)\,. \tag{2.22}$$

Hence we have

$$f(\epsilon) = 2\pi\,\Gamma(\epsilon)\,\Gamma(1-\epsilon)\,. \tag{2.23}$$

Plugging the above back to eq. (2.19), we find a result in agreement with that from Feynman parameterization (2.15).

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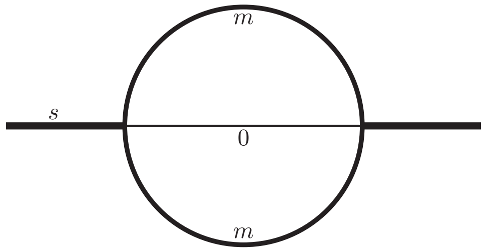

Figure 1. The sunrise diagram with two equal-mass propagators and one massless propagator.

2.3 The (generalized) loop-by-loop Baikov representation

The standard Baikov representation (2.11) works generically for multi-loop integrals. On the other hand, for $L > 1$ the number of positive a_i's in a given integral is often smaller than N. Those z_i's with zero or negative powers are called irreducible scalar products (ISPs). They may not directly appear in the corresponding Feynman integrals (or they may appear as numerators), but is necessary for a unique definition of the integral family, and is also necessary for the construction of the standard Baikov representation. We start with one of the ISPs, which, without loss of generality, is taken to be z_N. That is, we assume $a_N \leq 0$ in the following discussion. Starting from the standard representation, it is possible to integrate out z_N to arrive at a different, but equivalent representation of the same integral.

If $a_N = 0$, z_N only appears in the polynomial $P_{L,E}$, and hence it is often straightforward to integrate over it. The same practice may be carried out for other ISPs as well. The resulting representation is equivalent to eq. (2.11), but with fewer integration variables. This representation is the same as the so-called loop-by-loop (LBL) Baikov representation if the same set of Baikov variables are chosen in the latter. In the loop-by-loop approach, one performs the change of variables for a single loop momentum at a time, treating the others as external.

We take the sunrise integral family as an example. The diagram is shown in figure 1. The integral family is defined by the propagator denominators

$$\left\{ z_1 = k_1^2 - m^2,\, z_2 = (k_1 - k_2)^2,\, z_3 = (k_2 - p)^2 - m^2,\, z_4 = k_2^2 - m^2,\, z_5 = (k_1 - p)^2 - m^2 \right\}, \tag{2.24}$$

where $p^2 = s \neq 0$. Suppose that we are interested in integrals where only the first three propagators appear, namely, $F_{a_1,a_2,a_3,0,0}$. In the standard Baikov representation, we still need to include the last two denominators as ISPs. On the other hand, in the loop-by-loop approach, as the first step we perform the change of variables from k_1^μ to z_1 and z_2, treating k_2 as an external momentum. In the second step we perform the variable change from k_2^μ to z_3 and z_4. Here, the variable z_5 does not appear in the representation, and only one ISP,

z_4, is needed. The resulting representation reads

$$F_{a_1,a_2,a_3,0,0} \propto \int dz_1 dz_2 \int \frac{d^d k_2}{i\pi^{d/2}} \frac{[G(k_1,k_2)]^{(d-3)/2}}{[G(k_2)]^{(d-2)/2}} \frac{1}{z_1^{a_1}\, z_2^{a_2}\, z_3^{a_3}}$$
$$\propto \int dz_1 dz_2 dz_3 dz_4\, u_{\mathrm{LBL}}(z_1,z_2,z_3,z_4)\, \frac{1}{z_1^{a_1}\, z_2^{a_2}\, z_3^{a_3}}\,, \tag{2.25}$$

where we have omitted some constant prefactors, and the function

$$u_{\mathrm{LBL}}(z_1,z_2,z_3,z_4) = [G(k_2)]^{-1+\epsilon}\,[G(k_1,k_2)]^{1/2-\epsilon}\,[G(k_2,p)]^{1/2-\epsilon}\,. \tag{2.26}$$

Apparently, the LBL representation (2.25) can be straightforwardly applied to integrals with a non-zero a_4. On the other hand, it fails to capture those integrals with a non-zero a_5 (even if z_5 appears only in the numerator of the integrand, i.e., $a_5 < 0$).[2] The problem is that when we change variables from k_1^μ to the Baikov variables, we have to include z_5 since the integrand depends on $k_1 \cdot p$. As a result, we will end up with the standard Baikov representation following this approach. In this case, it is then useful to consider the LBL representation as the result of performing the integration over z_5 in the standard Baikov representation. From this viewpoint, it is possible to start from the standard Baikov representation with z_5 in the numerator, integrate out z_5, and arrive at a new representation without z_5. More generically, we consider a Feynman integral where z_N only appears in the numerator (i.e., $a_N \leq 0$). We construct its standard Baikov representation with the Baikov polynomial $P(z_1,\ldots,z_N)$. We consider $P(z_1,\ldots,z_N)$ as a quadratic polynomial of z_N while treating other variables as constants: $P(z_N) = -A_N z_N^2 + B_N z_N - C_N$, with A_N, B_N and C_N being polynomials of $\boldsymbol{z} \equiv \{z_1,\ldots,z_{N-1}\}$. The two roots of the polynomial are given by:

$$r_\pm = \frac{B_N \pm \sqrt{B_N^2 - 4A_N C_N}}{2A_N}\,. \tag{2.27}$$

The integration over z_N then gives

$$F_{a_1,\ldots,a_N} \propto \int d^{N-1}\boldsymbol{z}\, z_1^{-a_1} \cdots z_{N-1}^{-a_{N-1}} \int_{r_-}^{r_+} dz_N\, z_N^{-a_N}\, [P(z_N)]^\gamma$$
$$\propto \int d^{N-1}\boldsymbol{z}\, z_1^{-a_1} \cdots z_{N-1}^{-a_{N-1}}\, A_N^{-1-\gamma} \left(B_N^2 - 4A_N C_N\right)^{1/2+\gamma}$$
$$\times (r_-)^{-a_N}\, {}_2F_1\left(a_N, 1+\gamma, 2+2\gamma, 1 - \frac{r_+}{r_-}\right), \tag{2.28}$$

where γ is a parameter depending on ϵ. Since $a_N \leq 0$, the hypergeometric function in the above is in fact a polynomial of its argument:

$${}_2F_1\left(a_N, 1+\gamma, 2+2\gamma, 1 - \frac{r_+}{r_-}\right) = \sum_{n=0}^{-a_N} (-1)^n \binom{-a_N}{n} \frac{\Gamma(1+\gamma+n)\,\Gamma(2+2\gamma)}{\Gamma(1+\gamma)\,\Gamma(2+2\gamma+n)} \left(1 - \frac{r_+}{r_-}\right)^n. \tag{2.29}$$

[2] It should be noted that had we started from k_2 in the first step, we would end up with an alternative loop-by-loop representation in terms of the Baikov variables z_1, z_2, z_3 and z_5. This can be used to represent integrals with a non-zero a_5, but not those with a non-zero a_4. In any case, the conventional LBL approach cannot reduce the number of integration variables if both a_4 and a_5 are non-zero.

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If $a_N = 0$, eq. (2.28) simply reduces to the conventional loop-by-loop representation. The more interesting cases are those with $a_N < 0$. They describe integrals with z_N in the numerator, albeit z_N does not appear in the final integrand. For illustration purposes, we consider again the sunrise family with $a_5 = -1$. The standard Baikov polynomial is $P(z_5) = G(k_1, k_2, p)$. The coefficient of $-z_5^2$ in $P(z_5)$ can be easily seen to be $A_5 = G(k_2)/4$, while the discriminant of $P(z_5)$ can be shown to be[3]

$$B_5^2 - 4A_5C_5 = G(k_1, k_2)\, G(k_2, p)\,.$$

Eq. (2.28) in this case then gives

$$\begin{aligned}
&\int \frac{dz_1dz_2dz_3dz_4}{z_1^{a_1}\, z_2^{a_2}\, z_3^{a_3}\, z_4^{a_4}} \int_{r_-}^{r_+} dz_5\, z_5\, [P(z_5)]^{-\epsilon} \\
&\propto \int \frac{dz_1dz_2dz_3dz_4}{z_1^{a_1}\, z_2^{a_2}\, z_3^{a_3}\, z_4^{a_4}}\, A_5^{-1+\epsilon} \left(B_5^2 - 4A_5C_5\right)^{1/2-\epsilon} \frac{B_5}{A_5} \\
&\propto \int \frac{dz_1dz_2dz_3dz_4}{z_1^{a_1}\, z_2^{a_2}\, z_3^{a_3}\, z_4^{a_4}}\, u_{\mathrm{LBL}}(z_1, z_2, z_3, z_4)\, \frac{1}{G(k_2)} \left.\frac{\partial G(k_1, k_2, p)}{\partial z_5}\right|_{z_5=0},
\end{aligned}$$

where we have used the fact that B_5 is the coefficient of z_5 in $G(k_1, k_2, p)$, and $u_{\mathrm{LBL}}(z_1, z_2, z_3, z_4)$ is the same as eq. (2.26). For example, the integral $F_{1,1,1,0,-1}$ in the sunrise family can be represented by

$$F_{1,1,1,0,-1} \propto \int dz_1dz_2dz_3dz_4\, u_{\mathrm{LBL}}(z_1, z_2, z_3, z_4) \frac{1}{z_1z_2z_3\, G(k_2)} \left.\frac{\partial G(k_1, k_2, p)}{\partial z_5}\right|_{z_5=0}. \tag{2.30}$$

An important fact about the above representation is that certain polynomials of $\{z_i\}$ (e.g., $G(k_2) = z_4 + m^2$ in the above example) can appear in the denominators of the integrands. In generic situations where more than one ISPs are integrated out, more than one polynomials may appear in the denominators. These polynomials are factors of the u_{LBL} function. From the loop-by-loop approach described below eq. (2.24), one can see that at L loops there are $m = 2L - 1$ such polynomial factors. We denote them as $P_1, \ldots, P_m$. We will then refer to integrals of the form

$$\int \prod_{i=1}^{n} dz_i \frac{u_{\mathrm{LBL}}(z_1, \ldots, z_n)}{z_1^{a_1} \cdots z_n^{a_n}\, P_1^{b_1} \cdots P_m^{b_m}}\,, \tag{2.31}$$

as *generalized* loop-by-loop Baikov integrals, where the variables $z_1, \ldots, z_n$ are those not integrated out. A Feynman integral in this *generalized* loop-by-loop representation is written as a linear combination of integrals with the above form.

As we will see later, the introduction of polynomials in the denominators greatly broadens the possible forms of the integrands among which we will search for $d\log$ ones. That said, it is also clear that the polynomial denominators cannot appear arbitrarily, but must be accompanied by a suitable numerator. Otherwise it is possible that the expression does not correspond to (a combination of) Feynman integrals.[4] We will come back to this point later from the viewpoint of the intersection theory.

[3]In this simple case, these relations can be easily deduced by brute-force expansion of the Gram determinants. We will give more generalized relations of this kind in later sections.

[4]This fact has also been observed in [62], where suitable combinations of generalized LBL integrals are treated as Feynman integrals in shifted spacetime dimensions.

2.4 Cuts of integrals in the Baikov representation

It is often useful to consider cuts of integrals in the Baikov representation [61]. Cutting a propagator variable z_i amounts to localize its integration contour around the point $z_i = 0$. For example, consider cutting the variables $z_1, \dots, z_r$ in a (standard or generalized loop-by-loop) Baikov representation. The result is given by:

$$\int \prod_{j=r+1}^{n} dz_j \prod_{i=1}^{r} \oint_{z_i=0} dz_i \, \frac{u(z_1, \dots, z_n)}{z_1^{a_1} \cdots z_n^{a_n} \, P_1^{b_1} \cdots P_m^{b_m}} \,. \tag{2.32}$$

Apparently, cutting a variable z_i is equivalent to taking the residue of the integrand at $z_i = 0$.

Cut Baikov integrals are useful due to the fact that they satisfy the same IBP relations and the same differential equations as the uncut ones. Let's take again the sunrise integral family as an example. For simplicity we consider cases with $a_4 = a_5 = 0$, and omit them from the subscripts. Any integral in this family F_{a_1,a_2,a_3} can be expressed as a linear combination of three master integrals, chosen as $F_{1,1,1}$, $F_{1,1,2}$ and $F_{1,0,1}$:

$$F_{a_1,a_2,a_3} = c_1 F_{1,1,1} + c_2 F_{1,1,2} + c_3 F_{1,0,1} \,. \tag{2.33}$$

We can now take the maximal cut (i.e., cutting z_1, z_2 and z_3) on both sides of the above equality. Note that cutting z_2 on $F_{1,0,1}$ leads to a vanishing result. Therefore, we have the relation

$$F_{a_1,a_2,a_3}\big|_{\text{3-cut}} = c_1 F_{1,1,1}\big|_{\text{3-cut}} + c_2 F_{1,1,2}\big|_{\text{3-cut}} \,. \tag{2.34}$$

Determining the coefficients c_1 and c_2 from the cut-version of the IBP relations is simpler than solving the full IBP relations. The same is true when using the intersection theory to calculate the coefficients. The simplification is much more pronounced in more complicated situations. Note however, after taking the cuts, we lose the information about c_3 completely, which can be recovered in the next step by loosing the cuts.

From the definition of the cut, it is clear that if the power a_i is non-positive, cutting z_i will lead to a vanishing result. On the other hand, if $a_i > 0$, the z_i-cut integral is usually non-zero. This property is often used to select integrals belonging to a particular sector. However, one should be careful with some exceptions to the above rule, especially when cutting multiple variables. It is possible that when taking several variables to zero, the function $u(z_1, \dots, z_n)$ vanishes. Since the u function consists of polynomials raised to non-integer powers, this means that all its derivatives also vanish in this limit. In this case, even if all the a_i's are positive, the cut integral still vanishes. This does not necessarily mean that this sector is reducible, but is just an accidental fact of this particular representation. There exist other exceptional cases where a cut on variables in the denominator could lead to a vanishing result. It is possible that localizing the variables to zero may force the integration over the remaining variables to be scaleless, or the integrand may become a total derivative. In all the above situations, if one still wants to study this particular cut, an alternative representation has to be used. We will see examples in later sections.

3 The intersection theory of Baikov and Feynman integrals

From the discussions in the previous section, it is clear that we sometimes need to consider integrals in the generalized LBL representation, where polynomials of Baikov variables may appear in the denominator of the integrand. We will need to convert them to linear combinations of Feynman integrals appearing in scattering amplitudes. This can be achieved via generalized IBP relations [62] or via the method of intersection theory [65–68, 71, 72]. In this section, we briefly introduce the concept of intersection theory in the context of Baikov and Feynman integrals. For a more detailed explanation, we refer the readers to the original literature.

We will be dealing with Aomoto-Gelfand general hypergeometric functions [73] which can be defined via integrals of the form

$$I[\varphi] = \int_{\mathcal{C}} u(\boldsymbol{z})\varphi(\boldsymbol{z}) \,, \tag{3.1}$$

where $\varphi(\boldsymbol{z})$ is a single-valued differential n-form on an n-dimensional manifold, and $u(\boldsymbol{z})$ is a multi-valued function which vanishes on the boundary $\partial\mathcal{C}$ of the integration domain $\mathcal{C}$. It is required that $\varphi(\boldsymbol{z})$ can only be singular on the boundary $\partial\mathcal{C}$, where the singularity is regularized by the vanishing $u(\boldsymbol{z})$. We will often work with a particular coordinate system. In that case the point $\boldsymbol{z}$ is parametrized by n variables $\{z_1, z_2, \ldots, z_n\}$, and the n-form can be written as $\varphi(\boldsymbol{z}) = \hat{\varphi}(\boldsymbol{z})d^n\boldsymbol{z}$, where $\hat{\varphi}(\boldsymbol{z})$ is a single-valued function and $d^n\boldsymbol{z} = dz_1 \wedge \cdots \wedge dz_n$.

We are interested in the relations among integrals with a given $u(\boldsymbol{z})$ and a given $\mathcal{C}$. It is clear that different φ's may give rise to the same integral due to the IBP identity:

$$0 = \int_{\mathcal{C}} d\left(u(\boldsymbol{z})\xi(\boldsymbol{z})\right) = \int_{\mathcal{C}} u(\boldsymbol{z})\left(d + \omega\wedge\right)\xi(\boldsymbol{z}) \equiv \int_{\mathcal{C}} u(\boldsymbol{z})\nabla_\omega\xi(\boldsymbol{z}) \,, \tag{3.2}$$

where $\xi(\boldsymbol{z})$ is a differential $(n-1)$-form, $\omega \equiv d\log u(\boldsymbol{z})$ is a 1-form, and $\nabla_\omega \equiv d + \omega\wedge$ is a covariant derivative with ω as the connection. It follows that for a given φ and an arbitrary ξ, the following relation holds:

$$I[\varphi] = I[\varphi + \nabla_\omega\xi] \,. \tag{3.3}$$

The above identity can be understood as an equivalence relation between the two n-forms:

$$\varphi \sim \varphi + \nabla_\omega\xi \,. \tag{3.4}$$

We collect all n-forms equivalent to φ into an equivalence class denoted as a bra $\langle\varphi|$, which is also called a twisted cocycle. The set of all twisted cocycles forms a vector space called the nth twisted cohomology group H^n_ω with respect to the connection ω.

It is easy to see that the generalized LBL Baikov representation introduced in the last section is a special case of general hypergeometric functions. The $u(\boldsymbol{z})$ function corresponds to the u_{LBL} function consisting of Gram determinants raised to non-integer powers:

$$u(\boldsymbol{z}) = [P_1(\boldsymbol{z})]^{\gamma_1} \cdots [P_m(\boldsymbol{z})]^{\gamma_m} \,. \tag{3.5}$$

The n-forms $\varphi(\boldsymbol{z})$ are linear combinations of the building blocks

$$\frac{dz_1 \wedge \cdots \wedge dz_n}{z_1^{a_1} \cdots z_n^{a_n} P_1^{b_1} \cdots P_m^{b_m}}\,, \tag{3.6}$$

The non-integer power γ_i serves as a regulator for the possible singularity of $\varphi(\boldsymbol{z})$ when $P_i \to 0$. On the other hand, the singularity at $z_i \to 0$ is not regularized by $u(\boldsymbol{z})$. Therefore, it is necessary to multiply $u(\boldsymbol{z})$ by an extra factor $z_i^{\rho_i}$ for each $a_i > 0$ in order to satisfy the requirement of general hypergeometric functions. One takes the limit $\rho_i \to 0$ at the end of calculations.

The dimension ν of the twisted cohomology group H^n_ω counts the number of independent integrals of the form (3.1). It can be computed by counting the number of proper critical points [65, 70, 72, 74, 75].[5] A critical point is a solution to the set of equations[6]

$$\omega_i \equiv \partial_{z_i} \log u(\boldsymbol{z}) = 0\,, \quad (i = 1, \ldots, n)\,. \tag{3.7}$$

Given the form of the $u(\boldsymbol{z})$ function in eq. (3.5), the equations can be recasted to

$$\begin{aligned} \beta_i(\boldsymbol{z}) &= 0\,, \quad (i = 1, \ldots, n)\,, \\ P_j(\boldsymbol{z}) &\neq 0\,, \quad (j = 1, \ldots, m)\,, \end{aligned} \tag{3.8}$$

where

$$\beta_i(\boldsymbol{z}) \equiv \sum_{j=1}^{m} \partial_{z_i} P_j(\boldsymbol{z}) \prod_{k \neq j} P_k(\boldsymbol{z})\,. \tag{3.9}$$

We introduce an additional variable z_0 and define the polynomial

$$\beta_{n+1}(z_0, \boldsymbol{z}) \equiv z_0 \prod_{j=1}^{m} P_j(\boldsymbol{z}) - 1\,. \tag{3.10}$$

The conditions $P_j(\boldsymbol{z}) \neq 0$ can then be imposed by asking for a solution of z_0 to the equation $\beta_{n+1}(z_0, \boldsymbol{z}) = 0$. The number of solutions to the set of equations $\beta_i = 0, (i = 1, \ldots, n+1)$ is equal to the dimension of the quotient ring

$$\mathbb{C}[z_0, z_1, \ldots, z_n]/\mathcal{I}\,, \tag{3.11}$$

where $\mathcal{I}$ is the ideal generated by the polynomials $\{\beta_i\}$, i.e.,

$$\mathcal{I} = \langle \beta_1, \ldots, \beta_n, \beta_{n+1} \rangle\,. \tag{3.12}$$

The dimension of the quotient ring can be obtained using methods from computational algebraic geometry.

[5] We assume that all critical points are non-degenerate and isolated.

[6] The powers $\{\gamma_i\}$ in the $u(\boldsymbol{z})$ function are assumed to be generic non-integers, e.g., containing the dimensional regulator ϵ. Otherwise the number of solutions could be smaller than the actual number of independent integrals. In this case, it is necessary to add an extra regulator for these γ_i's, and take the regulators to zero in the last step.

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When working with generalized LBL representations, it is often the case where the dimension ν is different from the number of independent Feynman integrals found by reduction programs. The dimension ν can be larger than the number of independent integrals if there exist certain symmetry relations among the integrals which are not captured by the IBP relations (but are considered by reduction programs). This is apparently harmless since these symmetries can be easily incorporated later. After taking into account the symmetry relations, it is still possible that ν is larger than the number of independent Feynman integrals. This leads us to conclude that, certain integrals of the form (3.1) actually do not correspond to Feynman integrals, as we have already mentioned in the previous section. Therefore, the space of Feynman integrals can be regarded as a subspace of the vector space H^n_ω. We will refer to this subspace as the FI-subspace. It is our quest to identify the FI-subspace, and look for $d\log$-form integrands inside it.

Before considering the subspace, we briefly discuss how to work with H^n_ω using the intersection theory. Since H^n_ω is a vector space of dimension ν, one may choose a basis of it consisting of vectors $\langle e_1|, \langle e_2|, \dots, \langle e_\nu|$, such that any vector $\langle\varphi| \in H^n_\omega$ can be expressed as a linear combination of the basis vectors:

$$\langle\varphi| = c_1 \langle e_1| + c_2 \langle e_2| + \cdots + c_\nu \langle e_\nu| \,. \tag{3.13}$$

In the context of Feynman integrals, this gives the reduction of an integral as a linear combination of MIs. To calculate the coefficients c_i, one introduces the dual space of H^n_ω, denoted as $(H^n_\omega)^*$. It turns out that $(H^n_\omega)^*$ is isomorphic to $H^n_{-\omega}$, i.e., the twisted cohomology group with respect to the connection $-\omega$. We denote a vector in $(H^n_\omega)^*$ as a ket $|\varphi\rangle$, which is the equivalence class

$$|\varphi\rangle : \varphi \sim \varphi - \nabla_\omega \xi \,. \tag{3.14}$$

Between a bra $\langle\varphi_L|$ and a ket $|\varphi_R\rangle$ one can define a bilinear pairing $\langle\varphi_L|\varphi_R\rangle$ called an *intersection number* [73, 76–79]. This serves as an inner product between the vector space H^n_ω and its dual. With this, it is straightforward to compute the coefficients c_i by first choosing a basis $\{|h_1\rangle, |h_2\rangle, \dots, |h_\nu\rangle\}$ of the dual space $(H^n_\omega)^*$, and then use

$$c_i = \sum_{j=1}^{\nu} \langle\varphi|h_j\rangle \left(\boldsymbol{C}^{-1}\right)_{ji} \,, \tag{3.15}$$

where $\boldsymbol{C}$ is a $\nu\times\nu$ matrix with elements $\boldsymbol{C}_{ij} \equiv \langle e_i|h_j\rangle$. We will not discuss the computation of the intersection numbers in detail, but refer the interested readers to the original articles. It suffices to mention that, if both $e_i(\boldsymbol{z})$ and $h_j(\boldsymbol{z})$ are $d\log$-forms (which have only simple poles), the computation of $\langle e_i|h_j\rangle$ is greatly simplified. Therefore, having a $d\log$ basis not only simplifies the differential equations, but also helps the integral reduction using the intersection theory.

We now come back to the possible cases where *not* all linear combinations of $\{\langle e_i|\}$ correspond to Feynman integrals. In this case the dimension ν of H^n_ω is larger than the number ν_f of independent Feynman integrals. Equipped with the intersection theory, it is straightforward to identify the FI-subspace: one chooses a set of ν_f master Feynman integrals,[7] and projects them onto the basis $\{\langle e_i|\}$ using intersection theory. These ν_f linear combinations of $\{\langle e_i|\}$ span a ν_f-dimensional subspace, and we will look for $d\log$-form integrals inside this subspace.

Let's look at an example in the sunrise family introduced in the previous section. For simplicity, we consider cutting the two variables z_1 and z_3 (i.e., the two massive propagators) in the generalized LBL representation with z_4 as the ISP. We do not introduce the regulator for z_2, which means that a_2 can only be nonpositive. The integrals then take the form

$$\int u(\boldsymbol{z})\varphi(\boldsymbol{z}) = \int u_{\text{cut}}(z_2, z_4)\, \frac{z_2^{-a_2} z_4^{-a_4}\, dz_2 \wedge dz_4}{[P_1(z_4)]^{b_1}\,[P_2(z_2,z_4)]^{b_2}\,[P_3(z_4)]^{b_3}}\,, \tag{3.16}$$

where

$$\begin{aligned} u_{\text{cut}}(z_2, z_4) &= [P_1(z_4)]^{-1+\epsilon}\,[P_2(z_2,z_4)]^{1/2-\epsilon}\,[P_3(z_4)]^{1/2-\epsilon}\,, \\ P_1(z_4) &= G(k_2)\Big|_{z_1=z_3=0} = z_4 + m^2\,, \\ P_2(z_2,z_4) &= G(k_1,k_2)\Big|_{z_1=z_3=0} = z_2 m^2 - \frac{1}{4}(z_2 - z_4)^2\,, \\ P_3(z_4) &= G(k_2,p)\Big|_{z_1=z_3=0} = s m^2 - \frac{1}{4}(s - z_4)^2\,. \end{aligned} \tag{3.17}$$

It is easy to determine the dimension of the corresponding twisted cohomology group by computing the connection $\omega = d\log u_{\text{cut}}$ and counting the number of critical points which are solutions of $\omega = 0$. The result is $\nu = 2$, which means that there are two independent integrals of the form (3.16). Indeed, we can choose a basis $\{\langle e_1|\,, \langle e_2|\}$ where $e_i = \hat{e}_i(z_2,z_4)\, dz_2 \wedge dz_4$ with

$$\hat{e}_1(z_2,z_4) = \frac{m^2}{P_2 P_3}\,, \quad \hat{e}_2(z_2,z_4) = \frac{z_4}{P_2 P_3}\,, \tag{3.18}$$

which can be shown to be independent.[8] On the other hand, the topology under consideration is just the product of two massive tadpoles. It is easy to see that there is only one independent Feynman integral in this sector, i.e., $\nu_f = 1$. We can arbitrarily choose a Feynman integral, e.g., $F_{1,0,1,0,0}$, whose corresponding 2-form is simply $\varphi = dz_2 \wedge dz_4$. Computing the intersection numbers, we get

$$\langle\varphi| = \frac{(1-2\epsilon)^2 s m^4}{4(1-\epsilon)^2}\,(\langle e_1| + \langle e_2|)\,. \tag{3.19}$$

Therefore, we conclude that Feynman integrals live in the 1-dimensional subspace spanned by $\langle e_1| + \langle e_2|$.

[7] This task can be accomplished using any suitable reduction method, e.g., momentum-space IBP, Baikov IBP, or intersection theory.

[8] The basis is of course not unique. We have made this choice for the sake of simplicity (both in the computation of intersection numbers and in the final expression (3.19)).

4 Constructing $d\log$-form integrals

We now come to the construction of UT Feynman integrals satisfying canonical differential equations in a given integral family. The idea [55] is very simple: we conjecture that each UT Feynman integral should admit a representation of a generalized $d\log$-form, i.e., can be written as

$$I[\varphi] = \mathcal{N}_\epsilon \int_{\mathcal{C}} u(\boldsymbol{z})\varphi(\boldsymbol{z}) = \tilde{\mathcal{N}_\epsilon} \int_{\mathcal{C}} [G(\boldsymbol{z})]^\epsilon \bigwedge_{j=1}^{n} d\log f_j(\boldsymbol{z}) \,, \tag{4.1}$$

where $G(\boldsymbol{z})$ is a rational function and $f_j(\boldsymbol{z})$'s are algebraic functions of the Baikov variables. $\mathcal{N}_\epsilon$ is the prefactor arising from the Baikov representation, while $\tilde{\mathcal{N}_\epsilon}$ is a UT factor depending on ϵ and external variables (i.e., masses and scalar products of external momenta). In this section, we will ignore the factor $\tilde{\mathcal{N}_\epsilon}/\mathcal{N}_\epsilon$ which needs to be built into $\varphi(\boldsymbol{z})$. For applications in later sections, this factor can be easily deduced from the Gamma functions appearing in $\mathcal{N}_\epsilon$. With a slight abuse of notation, we will call $\varphi(\boldsymbol{z})$ a $d\log$ n-form, although it needs to be combined with some factors in $u(\boldsymbol{z})$ to be written as a $d\log$ integrand.

It should be noted that a UT integral can have many different representations, some of which are of the $d\log$-form while others are not. For example, a UT integral might be $d\log$ in the loop-by-loop Baikov representation, while the same is not true in the standard representation. For our purpose, it is sufficient to construct *one* $d\log$ representation for each candidate of a UT integral. To do that, it is sometimes necessary to try different representations until an appropriate one is found.

The n-form $\varphi(\boldsymbol{z})$ is a linear combination of the building blocks

$$\frac{dz_1 \wedge \cdots \wedge dz_n}{z_1^{a_1} \cdots z_n^{a_n} P_1^{b_1} \cdots P_m^{b_m}} \,, \tag{4.2}$$

where $P_1, \ldots, P_m$ are irreducible polynomial factors of $G(\boldsymbol{z})$ (and hence of $u(\boldsymbol{z})$). It should be emphasized that $\varphi(\boldsymbol{z})$ must be a single-valued differential n-form, whose denominator can only contain Baikov variables and the polynomial factors of $u(\boldsymbol{z})$. The $d\log$-form of eq. (4.1) puts further constraints on the properties of $\varphi(\boldsymbol{z})$. Most importantly, $u(\boldsymbol{z})\varphi(\boldsymbol{z})$ can only have simple poles in all the variables. This requirement puts upper and lowers bounds on the powers $\{a_i\}$ and $\{b_i\}$ (note the poles at infinity). In the following, we first show a systematic way to construct such $d\log$ n-forms both in the univariate and the multivariate cases, and then discuss how to convert them to UT Feynman integrals.

4.1 The univariate case

We start with the cases where only one Baikov variable is involved in the integrals. This can happen when we consider the maximal cuts of many integrals. We refer to this variable simply as z, and the $u(z)$ function can always be factorized into the form.[9]

$$u(z) = \frac{\mathcal{K}_1^\epsilon}{\mathcal{K}_0} \prod_{i=0}^{\nu} (z - c_i)^{-\gamma_i - \beta_i \epsilon} \,, \tag{4.3}$$

[9] In this expression, we have dropped some possible minus signs for the $(z - c_i)$ factors. In this section we'll not be worried about these signs and the integration domain. They will be recovered for the examples in later sections.

where β_i are integers and γ_i can be either integers or half-integers; $\mathcal{K}_0$ is an algebraic function and $\mathcal{K}_1$ is a rational function of external variables, respectively. In the above expression, we assume that the roots c_i are all distinct. The dimensionality of the cohomology group is then given by ν.

If the number of half-integer γ_i's in $u(z)$ is larger than two, this sector involves elliptic integrals or more complicated functional structures. In such case it is still possible to construct several $d\log$-form integrals, but one does not expect to have a complete canonical basis. If none of the γ_i's is a half-integer, we can choose

$$\varphi_i(z) = \frac{\mathcal{K}_0\, dz}{z - c_i} \prod_{j=0}^{\nu} (z - c_j)^{\gamma_j}\,, \quad (i = 0, \ldots, \nu)\,. \tag{4.4}$$

This gives

$$u(z)\varphi_i(z) = \left(\mathcal{K}_1 \prod_{j=0}^{\nu} (z - c_j)^{-\beta_j}\right)^{\epsilon} d\log(z - c_i)\,, \tag{4.5}$$

which takes the desired $d\log$-form (4.1). Note that there are $\nu + 1$ 1-forms in the above, but only ν of them are independent.

If there's one half-integer γ_i, without loss of generality, we take it to be γ_0. We can perform the construction using the identity

$$d\log \frac{1 + \sqrt{\frac{c_0 - z}{c_0 - c}}}{1 - \sqrt{\frac{c_0 - z}{c_0 - c}}} = -\frac{\sqrt{c - c_0}\, dz}{(z - c)\sqrt{z - c_0}}\,, \tag{4.6}$$

for arbitrary $c \neq c_0$. Evidently we can choose

$$\varphi_i(z) = -\mathcal{K}_0\, dz\, \frac{\sqrt{c_i - c_0}}{z - c_i}\, (z - c_0)^{\gamma_0 - 1/2} \prod_{j=1}^{\nu} (z - c_j)^{\gamma_j}\,, \quad (i = 1, \ldots, \nu)\,, \tag{4.7}$$

such that $u(z)\varphi_i(z)$ takes the $d\log$-form:

$$u(z)\varphi_i(z) = \left(\mathcal{K}_1 \prod_{j=0}^{\nu} (z - c_j)^{-\beta_j}\right)^{\epsilon} d\log \frac{1 + \sqrt{\frac{c_0 - z}{c_0 - c_i}}}{1 - \sqrt{\frac{c_0 - z}{c_0 - i}}}\,. \tag{4.8}$$

Things are quite similar in the case of two half-integer γ_i's. We take them to be γ_0 and γ_1. Here we employ the identities

$$\begin{aligned} d\log \frac{1 + \sqrt{\frac{(c_0 - z)}{(c_1 - z)}}}{1 - \sqrt{\frac{(c_0 - z)}{(c_1 - z)}}} &= \frac{dz}{\sqrt{(z - c_0)(z - c_1)}}\,, \\ d\log \frac{1 + \sqrt{\frac{(c_1 - c)(c_0 - z)}{(c_0 - c)(c_1 - z)}}}{1 - \sqrt{\frac{(c_1 - c)(c_0 - z)}{(c_0 - c)(c_1 - z)}}} &= -\frac{\sqrt{(c_0 - c)(c_1 - c)}\, dz}{(z - c)\sqrt{(z - c_0)(z - c_1)}}\,. \end{aligned} \tag{4.9}$$

We can then construct the following 1-forms:

$$\varphi_1(z) = \mathcal{K}_0\, dz\, (z-c_0)^{\gamma_0-1/2}(z-c_1)^{\gamma_1-1/2}\prod_{j=2}^{\nu}(z-c_j)^{\gamma_j}\,,$$
$$\varphi_i(z) = \mathcal{K}_0\, dz\, \frac{\sqrt{(c_0-c_i)(c_1-c_i)}}{z-c_i}\,(z-c_0)^{\gamma_0-1/2}(z-c_1)^{\gamma_1-1/2}\prod_{j=2}^{\nu}(z-c_j)^{\gamma_j}\,, \tag{4.10}$$

where $i = 2, \ldots, \nu$. The corresponding $u(z)\varphi_i(z)$ again have forms similar to eqs. (4.5) and (4.8).

4.2 The multivariate cases

We now want generalize the above procedure to multivariate cases. Our approach is to perform the construction one-by-one for each variable. In the first step, we select a variable which allows us to apply the methodology of the previous subsection, while treating the other variables as "external" at the moment. We call this variable z_1.[10] Using the univariate constructions, we can construct functions $\hat{\varphi}_i^{(1)}(\boldsymbol{z})$ such that

$$u(\boldsymbol{z})\,\hat{\varphi}_i^{(1)}(\boldsymbol{z}) = [G(\boldsymbol{z})]^{\epsilon}\,\frac{\partial}{\partial z_1}\log f_i^{(1)}(\boldsymbol{z})\,. \tag{4.11}$$

We call the combination $u(\boldsymbol{z})\hat{\varphi}_i^{(1)}(\boldsymbol{z})dz_1$ as a partial-$d\log$-form integrand in z_1. Here it should be noted that z_1 could be a propagator denominator instead of an ISP. In that case there is a regularization factor z_1^{ρ} in $u(\boldsymbol{z})$, and z_1 itself should be regarded as one of the "polynomial factors" of $u(\boldsymbol{z})$. This means that one of the c_i's in eq. (4.3) is zero. Note that $\hat{\varphi}_i^{(1)}(\boldsymbol{z})$ is in general not a rational function, which is a problem to be dealt with later.

Given the above partial results, the next step is to pick a variable z_2 and repeat the procedure. Namely, we try to construct functions $\hat{\varphi}_{i,j}^{(2)}(\boldsymbol{z}')/\Lambda_i(\boldsymbol{z}')$ such that $u(\boldsymbol{z})\hat{\varphi}_i^{(1)}(\boldsymbol{z})dz_1 \wedge \hat{\varphi}_{i,j}^{(2)}(\boldsymbol{z}')dz_2/\Lambda_i(\boldsymbol{z}')$ are partial-$d\log$-form integrands in the two variables z_1 and z_2, where $\boldsymbol{z}' = \{z_2, \ldots, z_n\}$. The algebraic functions $\Lambda_i(\boldsymbol{z}')$ are meant to cancel certain factors in $\hat{\varphi}_i^{(1)}(\boldsymbol{z})$, such that $\hat{\varphi}_i^{(1)}(\boldsymbol{z})/\Lambda_i(\boldsymbol{z}')$ become rational functions. Such a recursive procedure, if succeeded, leads to $d\log$-form integrals we want, with the full $\hat{\varphi}(\boldsymbol{z})$ a rational function. There is, however, a few complications in the second step (and further steps). We will address them in the following.

4.2.1 Square roots from the previous step

First of all, in eqs. (4.4), (4.7) and (4.10), denominators of the form $(z-c_i)$ appear. These are allowed in the univariate case according to the generic form (4.2), since $(z-c_i)$ *is* a polynomial factor of $u(z)$.[11] However, this is problematic in the multivariate case, since c_i is in general an algebraic function of the remaining Baikov variables $\boldsymbol{z}'$. Hence $(z_1 - c_i(\boldsymbol{z}))$ may *not* be a polynomial factor of the full $u(\boldsymbol{z})$, and cannot appear in the denominator alone.

[10] The order of variables is sometimes important, and one may need to try different orders to arrive at a successful construction.

[11] Here "polynomial" regards the Baikov variables only. The coefficients can be algebraic functions of external variables.

In this case it is necessary to make a linear combination of several terms, such that their common denominator becomes one of the polynomial factors $P_i(\boldsymbol{z})$ of $u(\boldsymbol{z})$. Fortunately, such linear combinations can be worked out rather generically, which depend on which of the formulas (4.4), (4.7) and (4.10) was used in the previous step.

The simplest case is eq. (4.4), where no square roots are involved. One can simply make linear combinations of the form

$$u(\boldsymbol{z})\,\hat{\varphi}^{(1)}(\boldsymbol{z}) = [G(\boldsymbol{z})]^{\epsilon} \sum_i \frac{r_i}{z_1 - c_i(\boldsymbol{z}')}\,, \tag{4.12}$$

with rational coefficients r_i. The sum in the above expression is over a subset of $\{0, \ldots, n\}$ such that $\mathcal{K}(\boldsymbol{z}')\prod_i(z_1 - c_i(\boldsymbol{z}'))$ is an irreducible polynomial factor (say, $P(\boldsymbol{z})$) of $u(\boldsymbol{z})$, where $\mathcal{K}(\boldsymbol{z}')$ is a polynomial in $\boldsymbol{z}'$ (which is the coefficient of the highest power of z_1 in $P(\boldsymbol{z})$). The coefficients r_i need to be chosen such that the numerator (after combining the denominators into $P(\boldsymbol{z})$) is either a rational function of $\boldsymbol{z}$, or the square root of a rational function of $\boldsymbol{z}'$. In the former case $\hat{\varphi}^{(1)}(\boldsymbol{z})$ is already single-valued with the correct denominator, and one can continue the construction for the remaining variables. In the latter case one needs to incorporate the square root in the next step to make the whole $\varphi(\boldsymbol{z})$ a single-valued differential form. In practice we most often encounter cases where $P(\boldsymbol{z})$ is quadratic in z_1, and it is straightforward to choose $r_i = \pm 1$. The two candidates are then simply given by the symmetric and anti-symmetric combinations:

$$\begin{aligned} u(\boldsymbol{z})\,\varphi_+(\boldsymbol{z}) &= [G(\boldsymbol{z})]^{\epsilon}\,\frac{1}{P(\boldsymbol{z})}\frac{\partial P(\boldsymbol{z})}{\partial z_1} dz_1 \wedge \hat{\varphi}'(\boldsymbol{z}')d^{n-1}\boldsymbol{z}'\,, \\ u(\boldsymbol{z})\,\varphi_-(\boldsymbol{z}) &= [G(\boldsymbol{z})]^{\epsilon}\,\frac{\mathcal{K}(\boldsymbol{z}')\,[c_i(\boldsymbol{z}') - c_j(\boldsymbol{z}')]\,dz_1}{P(\boldsymbol{z})} \wedge \frac{\hat{\varphi}'(\boldsymbol{z}')d^{n-1}\boldsymbol{z}'}{c_i(\boldsymbol{z}') - c_j(\boldsymbol{z}')}\,, \end{aligned} \tag{4.13}$$

where c_i and c_j are the two roots of $P(\boldsymbol{z})$ in z_1, and $\hat{\varphi}'(\boldsymbol{z}')$ is a rational function of $\boldsymbol{z}'$ which remains to be constructed.

We now turn to the case where the second line of eq. (4.10) is used in the previous step of construction (if the first line is used, the situation is very simple). Here we have

$$u(\boldsymbol{z})\,\hat{\varphi}_i^{(1)}(\boldsymbol{z}) = [G(\boldsymbol{z})]^{\epsilon}\,\frac{\sqrt{(c_0(\boldsymbol{z}') - c_i(\boldsymbol{z}'))\,(c_1(\boldsymbol{z}') - c_i(\boldsymbol{z}'))}}{(z_1 - c_i(\boldsymbol{z}'))\,\sqrt{(z_1 - c_0(\boldsymbol{z}'))\,(z_1 - c_1(\boldsymbol{z}'))}}\,, \tag{4.14}$$

where $i = 2, \ldots, \nu$. We again need to make linear combinations of the above to proceed with the next variable. We first note that c_0 and c_1 are the two roots of a quadratic polynomial $Q(z_1, \boldsymbol{z}')$ with respect to z_1, and therefore the square roots in the numerator and the denominator can be rescaled to $Q(c_i, \boldsymbol{z}')$ and $Q(z_1, \boldsymbol{z}')$, respectively. As before, we also assume that c_i is a root of the irreducible polynomial factor $P(z_1, \boldsymbol{z}')$. If c_i itself is a polynomial of $\boldsymbol{z}'$ (including the case where c_i is a constant), $P(z_1, \boldsymbol{z}')$ is simply $z_1 - c_i$. Hence we can readily write down the candidate

$$u(\boldsymbol{z})\,\varphi(\boldsymbol{z}) = [G(\boldsymbol{z})]^{\epsilon}\,\frac{dz_1\sqrt{Q(c_i, \boldsymbol{z}')}}{(z_1 - c_i)\sqrt{Q(z_1, \boldsymbol{z}')}} \wedge \frac{\hat{\varphi}'(\boldsymbol{z}')d^{n-1}\boldsymbol{z}'}{\sqrt{Q(c_i, \boldsymbol{z}')}}\,. \tag{4.15}$$

The construction can then be continued recursively.

On the other hand, more generally c_i is an algebraic function of $\boldsymbol{z}'$, and $P(z_1, \boldsymbol{z}')$ is a non-linear polynomial of z_1. There seems to be no way to continue the construction with $\sqrt{Q(c_i, \boldsymbol{z}')}$ (square root inside a square root) in the denominator. Fortunately, very often it can be expressed in a simpler form due to relations among Gram determinants.[12] To understand that, we first define a generalized Gram determinant with two sets of momenta:

$$G(\{q_1, \ldots, q_n\}, \{l_1, \ldots, l_n\}) \equiv \det(q_i \cdot l_j) \equiv \det \begin{pmatrix} q_1 \cdot l_1 & q_1 \cdot l_2 & \cdots & q_1 \cdot l_n \\ q_2 \cdot l_1 & q_2 \cdot l_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ q_n \cdot l_1 & \cdots & \cdots & q_n \cdot l_n \end{pmatrix} . \tag{4.16}$$

It then follows from Sylvester's determinant identity that

$$\begin{aligned} & [G(\{q_1, \ldots, q_n, k\}, \{q_1, \ldots, q_n, q_{n+1}\})]^2 \\ & = G(k, q_1, \ldots, q_n)\, G(q_1, \ldots, q_{n+1}) - G(k, q_1, \ldots, q_{n+1})\, G(q_1, \ldots, q_n) . \end{aligned} \tag{4.17}$$

The above identity can be used in various ways. As an example (which often appears in practice), consider $Q(z_1, \boldsymbol{z}') = G(k, q_1, \ldots, q_{n+1})$ and $P(z_1, \boldsymbol{z}') = G(k, q_1, \ldots, q_n)$, where z_1 is one of the Baikov variables associated with the loop momentum k (in the loop-by-loop sense). We introduce a short-hand notation for the polynomial

$$R(z_1, \boldsymbol{z}') \equiv G(\{q_1, \ldots, q_n, k\}, \{q_1, \ldots, q_n, q_{n+1}\}) = -\frac{1}{2} \frac{\partial}{\partial(k \cdot q_{n+1})} G(k, q_1, \ldots, q_{n+1}) , \tag{4.18}$$

where the last equal sign follows from Jacobi's formula. Because $z_1 = c_i$ is a zero point of $P(z_1, \boldsymbol{z}')$, we immediately find that

$$\sqrt{Q(c_i, \boldsymbol{z}')} = \frac{\pm R(c_i, \boldsymbol{z}')}{\sqrt{-G(q_1, \ldots, q_n)}} . \tag{4.19}$$

Since $G(q_1, \ldots, q_n)$ is independent of z_1, the above expression is actually a polynomial of c_i. It is then possible to build linear combinations of the form

$$u(\boldsymbol{z})\, \hat{\varphi}^{(1)}(\boldsymbol{z}) = [G(\boldsymbol{z})]^{\epsilon} \frac{1}{\sqrt{-G(q_1, \ldots, q_n)}\, \sqrt{Q(z_1, \boldsymbol{z}')}} \sum_i \frac{r_i\, R(c_i, \boldsymbol{z}')}{(z_1 - c_i)} , \tag{4.20}$$

where the rational coefficients r_i are chosen to satisfy conditions similar to the discussions below eq. (4.12). To see how such linear combinations can be found generically in the above situation, it is enough to consider $P(z_1, \boldsymbol{z})$ to be quadratic in z_1 (since Q is quadratic and the degree of P cannot exceed Q in terms of z_1). We can then use the fact that $R(z_1, \boldsymbol{z}')$ is a linear function of z_1 to write

$$R(c_i, \boldsymbol{z}') = \left(1 - (z_1 - c_i) \frac{\partial}{\partial z_1}\right) R(z_1, \boldsymbol{z}') . \tag{4.21}$$

[12] These relations have also been presented in the appendix of [62].

Taking $r_i = \pm 1$, the two linear combinations are then given by

$$
\begin{aligned}
u(\boldsymbol{z})\,\hat{\varphi}_{+}^{(1)}(\boldsymbol{z}) &= [G(\boldsymbol{z})]^{\epsilon}\,\frac{1}{\sqrt{-G(q_1,\ldots,q_n)}\,\sqrt{Q(z_1,\boldsymbol{z}')}}\left(\frac{R(z_1,\boldsymbol{z}')}{P(\boldsymbol{z})}\frac{\partial P(\boldsymbol{z})}{\partial z_1}-2\frac{\partial R(z_1,\boldsymbol{z}')}{\partial z_1}\right),\\
u(\boldsymbol{z})\,\hat{\varphi}_{-}^{(1)}(\boldsymbol{z}) &= [G(\boldsymbol{z})]^{\epsilon}\,\frac{1}{\sqrt{-G(q_1,\ldots,q_n)}\,\sqrt{Q(z_1,\boldsymbol{z}')}}\,\frac{\mathcal{K}(\boldsymbol{z}')\,[c_i(\boldsymbol{z}')-c_j(\boldsymbol{z}')]}{P(\boldsymbol{z})}\,R(z_1,\boldsymbol{z}')\,,
\end{aligned}
\tag{4.22}
$$

where the meaning of $\mathcal{K}$, c_i and c_j are the same as in the discussions below eq. (4.12). Similar constructions can be applied for the case $P = G(k, q_1, \ldots, q_{n+1})$ and $Q = G(k, q_1, \ldots, q_n)$, which also appears in practice quite often. The considerations outlined above is also valid if eq. (4.7) is used for the construction of z_1, and we will not go into details about that.

4.2.2 Higher-degree polynomial with a half-integer power

Unlike the univariate case, the appearance of higher-degree polynomials with half-integer powers does not necessarily mean that the result is elliptic. It is possible that a canonical basis is known to exist (from the maximal cuts of various sectors), but at some point in the construction procedure one encounters square root of a polynomial with degree higher than two for each remaining variable. Hence the construction cannot proceed straightforwardly. There are a couple of ways to circumvent this issue, and we will briefly discuss them in the following.

An apparent possibility is to perform a variable transformation such that the polynomial becomes quadratic in one of the new variables (which is similar in spirit to [53]). This can be illustrated by a simple example:

$$
\int [G(x,y)]^{\epsilon}\,\frac{\hat{\varphi}(x,y)\;dx\wedge dy}{\sqrt{(x^2+y^2-6xy)\,(x^2+y^2+2xy+2x+2y-2)}}\,. \tag{4.23}
$$

It is not easy to see how to construct the function $\hat{\varphi}(x, y)$ such that the above integrand becomes $d\log$. However, it is easy to find a change of variables $u = x + y$ and $v = x - y$, and the square root becomes

$$
\frac{du\wedge dv}{2\sqrt{(2v^2-u^2)((u+1)^2-3)}}\,. \tag{4.24}
$$

It is now straightforward to perform the construction first in v and then in u.

A different type of variable transformation is to "rationalize" part of the square root. Suppose that we have a square root of the polynomial $P(\boldsymbol{z})$. It is possible to find a rational change of variable $\boldsymbol{z} \to \tilde{\boldsymbol{z}}$, such that

$$
P(\boldsymbol{z}) = R^2(\tilde{\boldsymbol{z}})\,Q(\tilde{\boldsymbol{z}})\,, \tag{4.25}
$$

where R is a rational function of the new variables $\tilde{\boldsymbol{z}}$, and Q is a polynomial which is quadratic in some of the new variables. This possibility has also been used in [62]. Note that for this single square-root, the rationalizing transformation can be found algorithmically when it exists [80, 81].

Another possibility to avoid higher-degree polynomials is to perform the construction in reducible super-sectors. Given a sector with some propagator denominators and several ISPs, a super-sector is a sector where some of the ISPs are allowed to appear in the denominator. Sometimes a super-sector can be reducible. This means that all integrals in that super-sector can be expressed as linear combinations of integrals in lower sectors. We find that a canonical integral may have a very complicated form in its own sector (involving variable transformations as mentioned above), but is much simpler when expressed in the Baikov representation of a reducible super-sector. For this reason we usually look into the super-sectors first before attempting variable transformations.

In the beginning of this section, we have emphasized that a UT integral can have many equivalent but different representations, and it is enough for us to find one representation that is $d\log$. In the above, we have searched only in the generalized LBL representations. It is sometimes useful to extend these representations by introducing an additional fold of integration over an extra variable in the intermediate steps of the construction. We will call them "extended Baikov representations". As will be demonstrated in a practical example later, this extra variable is not randomly chosen, but is often motivated by (but not the same as) the variables in the standard Baikov representations. In this way it is easy to show that such an extended integral is indeed equivalent to integrals (we will refer to their integrands as "equivalently-$d\log$ integrands") in the original representation. A benefit of such an extension is that higher-degree polynomials might disappear, leading to a successful construction. We will see an example of this method later in the top sector of outer-massive double box family.

4.3 From $d\log$-forms to canonical Feynman integrals

The generic procedure outlined in the previous subsections allows us to construct $d\log$ Baikov integrals for a given $u(\boldsymbol{z})$. On the other hand, the main goal of this section is to construct UT Feynman integrals satisfying a canonical set of differential equations. In this subsection we show how to convert between these two in a systematic way.

As extensively discussed in section 2 and 3, Feynman integrals live in a subspace of the space of generalized Baikov integrals. It is hence easy to understand that the $d\log$ Baikov integrals are not necessarily expressible as linear combinations of Feynman integrals. As a result, we usually need to construct more $d\log$ Baikov integrals than the number of independent Feynman integrals. We call these extra ones *auxiliary* $d\log$ *forms*. With them, we can make linear combinations belonging to the FI-subspace using the method outlined at the end of section 3. We note that these combinations, after being put into a common denominator, generically take the form

$$\frac{N(\boldsymbol{z})\, dz_1 \wedge \cdots \wedge dz_n}{z_1 \cdots z_n\, P_1^{b_1} \cdots P_m^{b_m}}\,, \tag{4.26}$$

where b_i is either 0 or 1, and $N(\boldsymbol{z})$ is a polynomial in the numerator. This provides hint on which auxiliary $d\log$ forms can be combined together, and is very helpful in many cases.

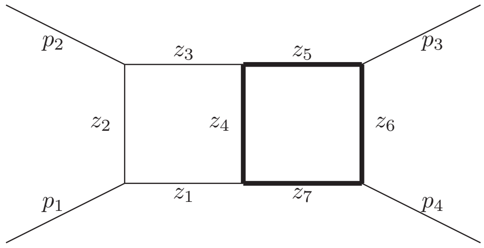

Figure 2. Inner-massive double box family. Thick lines represent propagators with a mass m, while thin lines represent massless propagators.

To summarize this section, we list below the procedure for constructing UT Feynman integrals for a given integral family:

1. Starting from a top sector, use IBP programs such as `FIRE`, `LiteRed`, `Reduze` or `Kira` to find all irreducible sectors containing master Feynman integrals (unique sectors in the notation of `LiteRed`), as well as the number of master Feynman integrals in each sector. Also compute the number of critical points to identify the dimension of the cohomology group for each sector, which corresponds to the number of independent Baikov integrals (here we also consider reducible sectors if necessary, which may provide simpler super-sector constructions as well as auxiliary $d\log$ forms).

2. Write down the generalized LBL Baikov representation for each sector, and apply the construction method to find enough $d\log$ Baikov integrals.

3. Identify linear combinations of $d\log$ Baikov integrals that belong to the FI-subspace, and transform them to Feynman integrals using either Baikov IBP, dimensional recurrence relations, or intersection theory.

In the following sections we demonstrate this procedure in several non-trivial examples. More examples can be found in the appendices.

5 Inner-massive double box

As the first example, we consider the double box integral family where the propagators in an "inner" loop have the same mass m, while the other propagators as well as external legs are massless. We take all external momenta to be incoming. The propagator denominators and the relevant scalar products are given by

$$\begin{aligned}
&\Big\{k_1^2,\, (k_1-p_1)^2\,,\, (k_1-p_1-p_2)^2\,,\, (k_1-k_2)^2-m^2,\, (k_2-p_1-p_2)^2-m^2, \\
&\quad (k_2-p_1-p_2-p_3)^2-m^2,\, k_2^2-m^2,\, (k_2-p_1)^2-m^2,\, (k_1-p_1-p_2-p_3)^2\Big\}\,, \\
&p_i^2=0\,, \qquad (p_1+p_2)^2=s\,, \qquad (p_2+p_3)^2=t\,, \qquad\qquad (5.1)
\end{aligned}$$

where the last two propagator denominators appear as ISPs. The corresponding diagram is depicted in figure 2.

This integral family has been considered in [82]. There are 20 unique sectors with 32 master integrals in total. The construction of $d\log$ forms in most sectors is straightforward. In the following, we discuss three representative sectors where special treatments are required. The complete results will be given in appendix A.2.

5.1 Sector {1,1,0,1,1,1,0,0,0}

We construct the LBL Baikov representation for this sector with z_7 and z_8 as ISPs. The relevant ingredients are given by:

$$\mathcal{N}_\epsilon = -\frac{e^{2\epsilon\gamma_E}}{4\pi^3\Gamma(1-2\epsilon)}\left[-st(s+t)\right]^\epsilon ,$$
$$u(\boldsymbol{z}) = P_1^{-1/2+\epsilon}\, P_2^{-\epsilon}\, P_3^{-1/2-\epsilon} , \tag{5.2}$$

where the three polynomials are

$$P_1(z_7,z_8) = 4\,G(k_2,p_1) = -(z_7-z_8)^2 ,$$
$$P_2(z_1,z_2,z_4,z_7,z_8) = 4\,G(k_1,k_2,p_1) ,$$
$$P_3(z_5,z_6,z_7,z_8) = 16\,G(k_2,p_1,p_2,p_3) . \tag{5.3}$$

It is attempting to apply maximal cut to this representation, to count the dimension of the corresponding cohomology group. However, in this case the maximal cut does not work for this specific LBL representation. The reason is that P_2 equals to 0 when z_1 and z_2 are set to zero. Since P_2 comes with a power of $-\epsilon$, the cut integral is identically zero in dimensional regularization. This situation has been discussed in section 2.4, and we need to search for other representations to perform the counting.

We can try a different LBL Baikov representation with z_3, z_7 and z_8 as ISPs. Applying maximal cut, we have

$$u_{\text{cut}}(\boldsymbol{z}) = P_{1,\text{cut}}^{\epsilon}\, P_{2,\text{cut}}^{-1/2-\epsilon}\, P_{3,\text{cut}}^{-1/2-\epsilon} , \tag{5.4}$$

where

$$P_{1,\text{cut}}(z_7,z_8) = z_8(s-z_7+z_8)+sm^2 ,$$
$$P_{2,\text{cut}}(z_3,z_7,z_8) = \left[sz_8+z_3\,(z_7-z_8)\right]^2 ,$$
$$P_{3,\text{cut}}(z_7,z_8) = 4m^2st(s+t)-(sz_8+tz_7-st)^2 . \tag{5.5}$$

Note that the integration domain is determined by $P_{3,\text{cut}} \geq 0$ and $P_{2,\text{cut}}/P_{1,\text{cut}} \geq 0$. These conditions do not constrain z_3, which means that the integration range of z_3 is $(-\infty,+\infty)$. In dimensional regularization such kind of integrals vanish. Hence we still cannot study the maximal cut using this representation.

In the end, we find that we need to employ the LBL Baikov representation which keeps z_3, z_7 and z_9 as ISPs in order to apply the maximal cut. The result is

$$u_{\text{cut}}(\boldsymbol{z}) = P_{1,\text{cut}}^{\epsilon}\, P_{2,\text{cut}}^{-1/2-\epsilon}\, P_{3,\text{cut}}^{-1/2-\epsilon} , \tag{5.6}$$

where

$$
\begin{aligned}
P_{1,\mathrm{cut}}(z_3,z_9) &= z_9(s-z_3+z_9)\,,\\
P_{2,\mathrm{cut}}(z_3,z_7,z_9) &= (z_7z_9-z_3z_7-sz_9)^2-4m^2sz_9(s-z_3+z_9)\,,\\
P_{3,\mathrm{cut}}(z_3,z_9) &= [s(z_9-t)+tz_3]^2\,.
\end{aligned}
\tag{5.7}
$$

Counting the number of critical points we get the result $\nu=2$, which coincides with the number of master integrals in this sector found by `Kira` and `Reduze`.

Now we need to construct two $d\log$-forms corresponding to Feynman integrals in this sector. This can be done in any representation that is convenient for the purpose, and we choose to work in the representation (5.2). Since this is the first practical example in this paper, we will demonstrate the construction procedure step-by-step. We first note that, for this sector we require all of z_1, z_2, z_4, z_5 and z_6 to appear in the denominator, otherwise the integrand likely belongs to a sub-sector. We also need to add regulators for these five variables in the context of intersection theory, such that the $u(\boldsymbol{z})$ function now becomes

$$
u(\boldsymbol{z}) = z_1^\rho z_2^\rho z_4^\rho z_5^\rho z_6^\rho\,P_1^{-1/2+\epsilon}\,P_2^{-\epsilon}\,P_3^{-1/2-\epsilon}\,,
\tag{5.8}
$$

where the polynomials P_1, P_2 and P_3 are given in eq. (5.3), and the regulator ρ will be taken to zero in the end.

Observing that z_1, z_2 and z_4 do not appear in the polynomials P_1 and P_3,[13] the construction for them is straightforward according to eq. (4.4). And we immediately know that the desired 7-form takes the form

$$
\frac{dz_1}{z_1}\wedge\frac{dz_2}{z_2}\wedge\frac{dz_4}{z_4}\wedge\varphi(z_5,z_6,z_7,z_8)\,.
\tag{5.9}
$$

We now need to find a 4-form $\varphi(z_5,z_6,z_7,z_8)$ such that $\varphi(z_5,z_6,z_7,z_8)/\sqrt{P_1P_3}$ is a $d\log$ form. Since z_5 and z_6 only appear in P_3 but not P_1, we choose to work with them first. For the readers' convenience, we recall that

$$
\begin{aligned}
P_1 &= -(z_7-z_8)^2\,,\\
P_3 &= (st-sz_8-tz_7+sz_6+tz_5)^2-4st\Big[tz_5+sz_6+(z_5-z_6)(z_7-z_8)+m^2(s+t)\Big]\,.
\end{aligned}
\tag{5.10}
$$

P_3 is a quadratic polynomial of z_5, and we can apply the second equation in (4.9) where $c=0$. This give rise to the $d\log$ factor

$$
\frac{\varphi(z_5,z_6,z_7,z_8)}{\sqrt{P_1}\sqrt{P_3}} = \frac{\sqrt{P_3(z_5=0)}}{z_5\sqrt{P_3}}\,dz_5\wedge\cdots\,,
\tag{5.11}
$$

where the ellipsis denotes the part yet to be constructed. We now need to take care of a factor of $1/\sqrt{P_3(z_5=0)}$ in the construction for the remaining variables. Note that

[13] The polynomial P_2 is irrelevant here since it comes with a power of $-\epsilon$ in the $u(\boldsymbol{z})$ function.

$P_3(z_5 = 0)$ is again a quadratic polynomial of z_6, and hence we can apply the second equation in (4.9) here. This leads to

$$\frac{\varphi(z_5, z_6, z_7, z_8)}{\sqrt{P_1}\sqrt{P_3}} = \frac{\sqrt{P_3(z_5 = 0)}}{z_5\sqrt{P_3}}\, dz_5 \wedge \frac{\sqrt{P_3(z_5 = 0, z_6 = 0)}}{z_6\sqrt{P_3(z_5 = 0)}}\, dz_6 \wedge \cdots . \tag{5.12}$$

We can now continue with the construction for z_7 and z_8. $P_3(z_5 = 0, z_6 = 0)$ can now be regarded as a quadratic polynomial of z_7, and $\sqrt{P_1}$ can be written as a constant factor multiplying $(z_7 - z_8)$.[14] We apply again the second equation of (4.9) with $c = z_8$, and obtain

$$\begin{aligned}\frac{\varphi(z_5, z_6, z_7, z_8)}{\sqrt{P_1}\sqrt{P_3}} = \frac{\sqrt{P_3(z_5 = 0)}}{z_5\sqrt{P_3}}\, dz_5 \wedge \frac{\sqrt{P_3(z_5 = 0, z_6 = 0)}}{z_6\sqrt{P_3(z_5 = 0)}}\, dz_6 \\ \wedge \frac{\sqrt{P_3(z_5 = 0, z_6 = 0, z_7 = z_8)}}{(z_7 - z_8)\sqrt{P_3(z_5 = 0, z_6 = 0)}}\, dz_7 \wedge \cdots .\end{aligned} \tag{5.13}$$

We are left with the last variable z_8, with

$$P_3(z_5 = 0, z_6 = 0, z_7 = z_8) = [st - (s+t)z_8]^2 - 4m^2 st(s+t)\,. \tag{5.14}$$

We now apply the first equation in (4.9) to arrive at

$$\frac{\varphi(z_5, z_6, z_7, z_8)}{\sqrt{P_1}\sqrt{P_3}} = \cdots \wedge \frac{s+t}{\sqrt{P_3(z_5 = 0, z_6 = 0, z_7 = z_8)}}\, dz_8\,, \tag{5.15}$$

which is the final answer for a $d\log$-form integrand in this sector. The corresponding $\hat{\varphi}$ function is simply given by

$$\hat{\varphi}_9 = \frac{s+t}{z_1 z_2 z_4 z_5 z_6}\,, \tag{5.16}$$

where the numbering follows the list in appendix A.2.

It is possible to construct the second $d\log$-form within this sector, albeit a bit tricky. It is much simpler to employ the reducible super-sector $\{1, 1, 0, 1, 1, 1, 1, 0, 0\}$. Namely, we allow z_7 to appear in the denominator of the generalized LBL Baikov representation, and add the necessary z_7^ρ factor in the $u(\boldsymbol{z})$ function. All integrals in this super-sector can be reduced to the sector under consideration and its sub-sectors. It is now straightforward to construct the second $d\log$-form:

$$\hat{\varphi}_{10} = \frac{\sqrt{st(st - 4m^2(s+t))}}{z_1 z_2 z_4 z_5 z_6 z_7}\,. \tag{5.17}$$

It is clear that both $d\log$-forms correspond to Feynman integrals, and hence the construction for this sector completes.

As a final remark here, we note that the prefactor $\mathcal{N}_\epsilon$ in eq. (5.2) is already a UT function with weight -3. Hence we can directly take $\tilde{\mathcal{N}}_\epsilon = \mathcal{N}_\epsilon$ in eq. (4.1). We also note that the 7-fold integrations over $d\log$-forms leads to UT functions with weight $+7$, and the final results $\langle\varphi_9|$ and $\langle\varphi_{10}|$ are weight $+4$ functions. This should be kept in mind when constructing other sectors, since we would like to have a canonical basis with the same transcendental weight.

[14] A constant factor does not affect the construction of $d\log$ integrands, and we will simply drop it.

5.2 Sector {0,1,0,1,1,1,1,0,0}

We construct the LBL Baikov representation for this sector with z_8 as the ISP. To do that we first perform the momenta shifts $k_1 \to k_1 + p_1$ and $k_2 \to k_2 + p_1$ in the propagator denominators in eq. (5.1). The resulting ingredients are:

$$\begin{aligned}\mathcal{N}_\epsilon &= \frac{1}{1-2\epsilon} \frac{e^{2\epsilon\gamma_E}\Gamma^2(-\epsilon)}{16\pi^3\Gamma^2(-2\epsilon)} \left[st(s+t)\right]^\epsilon , \\ u(\boldsymbol{z}) &= (z_8+m^2)^{-1+\epsilon} P_2^{1/2-\epsilon} P_3^{-1/2-\epsilon} , \end{aligned} \tag{5.18}$$

where the two polynomials are

$$\begin{aligned} P_2(z_2, z_4, z_8) &= 4\, G(k_1, k_2) , \\ P_3(z_5, z_6, z_7, z_8) &= 16\, G(k_2, p_1, p_2, p_3) . \end{aligned} \tag{5.19}$$

Under maximal cut there are three critical points, corresponding to three MIs in this sector. Here $\mathcal{N}_\epsilon$ is not a UT function due to the factor of $(1-2\epsilon)$. Taking that into account and performing the construction, we arrive at three $d\log$-forms:

$$\begin{aligned} \hat{\varphi}'_{14} &= \frac{(1-2\epsilon)\left(m^2+z_8\right)\sqrt{st\left(st-4m^2(s+t)\right)}}{\epsilon z_2 z_4 z_5 z_6 z_7 P_2} , \\ \hat{\varphi}'_{15} &= \frac{(1-2\epsilon) z_8 \sqrt{s^2\left(t-m^2\right)^2 - 4m^2 st^2}}{\epsilon z_2 z_4 z_5 z_6 z_7 P_2} , \\ \hat{\varphi}'_{16} &= \frac{s z_8 (1-2\epsilon)\left(m^2+z_8\right)}{\epsilon z_2 z_4 z_5 z_6 z_7 P_2} . \end{aligned} \tag{5.20}$$

Note that the polynomial P_2 appears in the denominators, and hence it is not clear at first sight whether the above three correspond to Feynman integrals.

Since we require that the constructed $d\log$-forms are Feynman integrals without any cuts, it is necessary to consider the equivalence classes of the full 6-forms without any cuts

$$\frac{z_8^{-a_8}}{z_2^{a_2} z_4^{a_4} z_5^{a_5} z_6^{a_6} z_7^{a_7} (z_8+m^2)^{b_1} P_2^{b_2} P_3^{b_3}} . \tag{5.21}$$

Computing the number of critical points we get $\nu = 20$ (and after taking into account a symmetry between z_5 and z_7, there are 18 independent integrals), while from an IBP reduction we know that there are only $\nu_f = 12$ independent Feynman integrals in this sector including sub-sectors. It is therefore not surprising that some 6-forms like eq. (5.21) do not correspond to Feynman integrals. Following the strategy outlined in section 3, we could use the intersection theory to find the FI-subspace of the twisted cohomology group. In practice, the 6-fold intersection numbers are computationally heavy. However, observing that P_2 only depends on z_2, z_4 and z_8, we find it sufficient to consider integrals with cut on z_5, z_6 and z_7, such that only 3-fold intersection numbers are involved. The number of critical points is $\nu = 5$ in this situation, and the number of master Feynman integrals is $\nu_f = 4$ (three in this sector and one in a sub-sector).

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We choose the following basis for this 5-dimensional cohomology group:

$$\hat{e}_1 = \frac{1}{z_2 z_4}\,, \qquad \hat{e}_2 = \frac{1}{z_2 z_4^2}\,, \qquad \hat{e}_3 = \frac{1}{z_2 z_4^3}\,, \qquad \hat{e}_4 = \frac{1}{z_4^3}\,, \qquad \hat{e}_5 = \frac{1}{z_2 z_4 P_{2,\text{cut}}}\,. \tag{5.22}$$

It is clear that the first four vectors correspond to Feynman integrals (under the cut), while the last one does not (since it is linearly independent from the first four). To compute the intersection numbers, we need to multiply $u(\boldsymbol{z})$ by $z_2^\rho z_4^\rho$ as a regulator, and take the limit $\rho \to 0$ by the end of the calculation. Performing the decomposition, we have

$$\begin{aligned}
\langle \varphi'_{14,\text{cut}}| &= \frac{\sqrt{st\,(st - 4m^2(s+t))}}{\epsilon} \left(\langle e_2| + \frac{m^2}{\epsilon} \langle e_3| \right), \\
\langle \varphi'_{15,\text{cut}}| &= \frac{\sqrt{s^2\,(t-m^2)^2 - 4m^2 st^2}}{\epsilon} \left(\langle e_2| + \frac{m^2}{\epsilon} \langle e_3| - (1-2\epsilon) m^2 \langle e_5| \right), \\
\langle \varphi'_{16,\text{cut}}| &= -\frac{s}{\epsilon} \left((1-2\epsilon) \langle e_1| - m^2 \langle e_2| + \frac{m^2}{\epsilon} \langle e_4| \right).
\end{aligned} \tag{5.23}$$

We can see that $\langle \varphi'_{14,\text{cut}}|$ and $\langle \varphi'_{16,\text{cut}}|$ have no components in $\langle e_5|$, and hence they are candidates for canonical Feynman integrals. On the other hand, $\langle \varphi'_{15,\text{cut}}|$ is not a Feynman integral since the coefficient in front of $\langle e_5|$ is non-zero. We discuss how to transform it into a canonical Feynman integral in the following.

According to the discussion in section 3, the quest is to find a $d\log$-form $\tilde{\varphi}_{15}$ such that projection from $\langle \varphi'_{15}| + \langle \tilde{\varphi}_{15}|$ to $\langle e_5|$ vanishes. We call the $d\log$-forms such as $\tilde{\varphi}_{15}$ as *auxiliary* $d\log$-forms. We know that $\langle \tilde{\varphi}_{15}|$'s decomposition coefficient in front of $\langle e_5|$ must be negative to that of $\langle \varphi'_{15}|$. This requirement greatly constrains the possible forms of $\tilde{\varphi}_{15}$, and it's easy to construct two candidates:

$$\begin{aligned}
\hat{\phi}_1 &= \frac{(1-2\epsilon)\sqrt{s^2\,(t-m^2)^2 - 4m^2 st^2}}{\epsilon z_4 z_5 z_6 z_7 P_2}\,, \\
\hat{\phi}_2 &= \frac{(1-2\epsilon)\sqrt{s^2\,(t-m^2)^2 - 4m^2 st^2}}{\epsilon z_2 z_5 z_6 z_7 P_2}\,.
\end{aligned} \tag{5.24}$$

Cutting on z_5, z_6, z_7, and projecting them onto our basis we find

$$\begin{aligned}
\langle \phi_{1,\text{cut}}| &= \frac{\sqrt{s^2\,(t-m^2)^2 - 4m^2 st^2}}{\epsilon} \left(-\frac{m^2}{\epsilon} \langle e_3| + (1-2\epsilon) m^2 \frac{\epsilon - \rho}{\epsilon - 2\rho} \langle e_5| \right), \\
\langle \phi_{2,\text{cut}}| &= \frac{\sqrt{s^2\,(t-m^2)^2 - 4m^2 st^2}}{\epsilon} \left((1-2\epsilon) m^2 \frac{\rho}{\epsilon - 2\rho} \langle e_5| \right),
\end{aligned} \tag{5.25}$$

where we have taken the limit $\rho \to 0$ except for the coefficient in front of $\langle e_5|$. Note that $\langle \phi_{2,\text{cut}}|$ actually vanishes in that limit, which means that it is zero to begin with. In fact, the uncut version $\langle \phi_2|$ is also zero. This is something similar to the usual rule that scaleless integrals in dimensional regularization is zero. While it is not necessary, we will keep it in order to subtract exactly the $\langle e_5|$ term from $\langle \varphi'_{15}|$.

From the above results, one can identify the auxiliary $d\log$-form as $\tilde{\varphi}_{15} = \phi_1 - \phi_2$. It turns out that this is enough even in the uncut case, and one doesn't need to introduce more auxiliary $d\log$-forms in the sub-sectors. Hence the correct $d\log$-form and its relation to Feynman integrals are given by

$$\langle \varphi_{15}| = \langle \varphi'_{15}| + \langle \phi_1| - \langle \phi_2| = \frac{\sqrt{s^2\left(t-m^2\right)^2 - 4m^2 s t^2}}{\epsilon} \langle F_{010211100}| \,. \tag{5.26}$$

Note that the above combination actually takes the form

$$\begin{aligned}
\hat{\varphi}_{15} &= -\frac{(1-2\epsilon)\sqrt{s^2\left(t-m^2\right)^2 - 4m^2 s t^2}(z_4 - z_2 - z_8)}{\epsilon z_2 z_4 z_5 z_6 z_7 P_2} \\
&= -\frac{(1-2\epsilon)\sqrt{s^2\left(t-m^2\right)^2 - 4m^2 s t^2}}{\epsilon z_2 z_4 z_5 z_6 z_7} \frac{1}{2P_2} \frac{\partial P_2}{\partial z_4} \,,
\end{aligned} \tag{5.27}$$

which is manifestly a Feynman integral after performing an IBP with respect to z_4. This motivates another way to look for linear combinations of auxiliary $d\log$-forms, similar to the idea of syzygies [83].

5.3 Top-sector {1,1,1,1,1,1,1,0,0}

We now turn to the top sector, and construct the LBL Baikov representation with z_8 as the ISP. The relevant ingredients are:

$$\begin{aligned}
\mathcal{N}_\epsilon &= \frac{e^{2\epsilon\gamma_E}\Gamma^2(-\epsilon)}{16\pi^4\Gamma^2(-2\epsilon)} \left[s^2 t(s+t)\right]^\epsilon \,, \\
u(\boldsymbol{z}) &= P_1^\epsilon P_2^{-1/2-\epsilon} P_3^{-1/2-\epsilon} \,,
\end{aligned} \tag{5.28}$$

where the three polynomials are

$$\begin{aligned}
P_1(z_5, z_7, z_8) &= -\frac{4}{s} G(k_2, p_1, p_2) \,, \\
P_2(z_5, z_6, z_7, z_8) &= 16\, G(k_2, p_1, p_2, p_3) \,, \\
P_3(z_1, z_2, z_3, z_4, z_5, z_7, z_8) &= 16\, G(k_1, k_2, p_1, p_2) \,.
\end{aligned} \tag{5.29}$$

There are four master integrals in this sector. Performing the construction, we arrive at the following four $d\log$-forms:

$$\begin{aligned}
\hat{\varphi}_1 &= \frac{s\sqrt{st\left(st - 4m^2(s+t)\right)}}{z_1 z_2 z_3 z_4 z_5 z_6 z_7} \,, \\
\hat{\varphi}_2 &= \frac{s^2 z_8}{z_1 z_2 z_3 z_4 z_5 z_6 z_7} \,, \\
\hat{\varphi}'_3 &= \frac{z_8}{z_1 z_2 z_3 z_4 z_5 z_6 z_7} \left(\frac{\partial^2 P_2}{\partial z_6 \partial z_8} - \frac{1}{2P_1} \frac{\partial P_2}{\partial z_6} \frac{\partial P_1}{\partial z_8} \right) , \\
\hat{\varphi}'_4 &= \frac{\sqrt{s(s-4m^2)}\, z_8}{z_1 z_2 z_3 z_4 z_5 z_6 z_7} \frac{1}{2P_1} \frac{\partial P_2}{\partial z_6} \,.
\end{aligned} \tag{5.30}$$

While $\hat{\varphi}_1$ and $\hat{\varphi}_2$ can be straightforwardly identified as Feynman integrals, $\hat{\varphi}'_3$ and $\hat{\varphi}'_4$ are not. We again need to add linear combinations of auxiliary $d\log$-forms to bring them into the FI-subspace. The suitable auxiliary $d\log$-forms can be constructed systematically, and we leave the details to appendix A. The final combinations are given by

$$\begin{aligned}\hat{\varphi}_3 &= \frac{1}{z_1 z_2 z_3 z_4 z_5 z_6 z_7}\Big[-2s^2 t + 2s^2(z_8+z_9) + 4s z_8 z_9 \\ &\qquad + 2tz_1(s+2z_5) + 2tz_3(s+2z_7) - 2sz_2(s-z_5+2z_6-z_7) - 4stz_4\Big], \\ \hat{\varphi}_4 &= \frac{-2st + 2sz_9 + 2t(z_1+z_3)\sqrt{s(s-4m^2)}}{z_1 z_2 z_3 z_4 z_5 z_6 z_7}. \end{aligned} \tag{5.31}$$

5.4 The complete canonical basis as Feynman integrals

We now list the complete canonical basis of the inner-massive double box family, written as linear combinations of Feynman integrals:

$$\begin{aligned}
\langle\varphi_1| &= s\sqrt{st(st-4m^2(s+t))}\,\langle F_{111111100}|\,, \\
\langle\varphi_2| &= s^2\,\langle F_{1111111-10}|\,, \\
\langle\varphi_3| &= -2s^2t\,\langle F_{111111100}| + 2s^2\,\langle F_{1111111-10}| + 2s^2\,\langle F_{11111110-1}| + 4s\,\langle F_{1111111-1-1}| \\
&\quad - 2s^2\,\langle F_{101111100}| + 4st\,\langle F_{110111100}| - 4s\,\langle F_{101110100}| + 4s\,\langle F_{101111000}| \\
&\quad - \frac{4t(1-2\epsilon)}{\epsilon}\,\langle F_{101011100}| + 8t\,\langle F_{110111000}|\,, \\
\langle\varphi_4| &= \sqrt{s(s-4m^2)}\left(-2st\,\langle F_{111111100}| + 2s\,\langle F_{11111110-1}| + 4t\,\langle F_{110111100}|\right), \\
\langle\varphi_5| &= st\,\langle F_{111111000}|\,, \\
\langle\varphi_6| &= s\sqrt{s(s-4m^2)}\,\langle F_{101111100}|\,, \\
\langle\varphi_7| &= \frac{s\sqrt{t(t-4m^2)}}{\epsilon}\,\langle F_{111102000}|\,, \\
\langle\varphi_8| &= \frac{1-2\epsilon}{\epsilon^2}\,\langle F_{101200000}| - \frac{s(1-2\epsilon)}{\epsilon}\,\langle F_{111101000}| - \frac{s(t-4m^2)}{\epsilon}\,\langle F_{111102000}|\,, \\
\langle\varphi_9| &= (s+t)\,\langle F_{110111000}|\,, \\
\langle\varphi_{10}| &= \sqrt{st(st-4m^2(s+t))}\,\langle F_{110111100}|\,, \\
\langle\varphi_{11}| &= s\,\langle F_{101111000}|\,, \\
\langle\varphi_{12}| &= \frac{s(1-2\epsilon)}{\epsilon}\,\langle F_{101110100}|\,, \\
\langle\varphi_{13}| &= \frac{s(1-2\epsilon)}{\epsilon}\,\langle F_{101011100}|\,, \\
\langle\varphi_{14}| &= \frac{\sqrt{st(st-4m^2(s+t))}}{\epsilon}\,\langle F_{010211100}| + \frac{m^2\sqrt{st(st-4m^2(s+t))}}{\epsilon^2}\,\langle F_{010311100}|\,,
\end{aligned}$$

$$
\begin{aligned}
\langle\varphi_{15}| &= \frac{\sqrt{s(s(t-m^2)^2-4t^2m^2)}}{\epsilon}\,\langle F_{010211100}|\ ,\\
\langle\varphi_{16}| &= -\frac{sm^2}{\epsilon^2}\,\langle F_{000311100}| - \frac{s(1-2\epsilon)}{\epsilon}\,\langle F_{010111100}| + \frac{sm^2}{\epsilon}\,\langle F_{010211100}|\ ,\\
\langle\varphi_{17}| &= \frac{s}{\epsilon}\,\langle F_{101102000}|\ ,\\
\langle\varphi_{18}| &= \frac{s\sqrt{s(s+4m^2)}}{\epsilon^2}\,\langle F_{102102000}| - \frac{\sqrt{s(s+4m^2)}}{2m^2\epsilon(1+2\epsilon)}\,\langle F_{000220000}|\ ,\\
\langle\varphi_{19}| &= -\frac{\sqrt{s(s-4m^2)}(1-2\epsilon)}{\epsilon^2}\,\langle F_{101010200}|\ ,\\
\langle\varphi_{20}| &= \frac{s}{2\epsilon}\,\langle F_{100211000}| - \frac{s}{4\epsilon^2}\,\langle F_{100220000}|\ ,\\
\langle\varphi_{21}| &= -\frac{t}{2\epsilon}\,\langle F_{010211000}| + \frac{t}{4\epsilon^2}\,\langle F_{010202000}|\ ,\\
\langle\varphi_{22}| &= -\frac{s}{\epsilon}\,\langle F_{010210100}| - \frac{sm^2}{\epsilon^2}\,\langle F_{010310100}|\ ,\\
\langle\varphi_{23}| &= -\frac{\sqrt{s(s-4m^2)}(1-2\epsilon)}{\epsilon^2}\,\langle F_{010110200}| - \frac{\sqrt{s(s-4m^2)}}{\epsilon}\,\langle F_{010210100}|\ ,\\
\langle\varphi_{24}| &= \frac{s}{\epsilon}\,\langle F_{010210100}| + \frac{2sm^2}{\epsilon^2}\,\langle F_{010310100}|\ ,\\
\langle\varphi_{25}| &= \frac{sm^2}{\epsilon^2}\,\langle F_{000311100}|\ ,\\
\langle\varphi_{26}| &= \frac{1-2\epsilon}{\epsilon}\,\langle F_{101200000}|\ ,\\
\langle\varphi_{27}| &= \frac{\sqrt{s(s-4m^2)}}{4\epsilon^2}\,\langle F_{100220000}| + \frac{\sqrt{s(s-4m^2)}}{2\epsilon^2}\,\langle F_{200120000}|\ ,\\
\langle\varphi_{28}| &= \frac{1}{4\epsilon^2}\,\langle F_{000220000}| - \frac{s}{2\epsilon^2}\,\langle F_{100220000}|\ ,\\
\langle\varphi_{29}| &= \frac{\sqrt{t(t-4m^2)}}{4\epsilon^2}\,\langle F_{010202000}| + \frac{\sqrt{t(t-4m^2)}}{2\epsilon^2}\,\langle F_{020102000}|\ ,\\
\langle\varphi_{30}| &= \frac{1}{4\epsilon^2}\,\langle F_{000220000}| - \frac{t}{2\epsilon^2}\,\langle F_{010202000}|\ ,\\
\langle\varphi_{31}| &= \frac{\sqrt{s(s-4m^2)}}{2\epsilon^2}\,\langle F_{000210200}|\ ,\\
\langle\varphi_{32}| &= \frac{1}{\epsilon^2}\,\langle F_{000220000}|\ .
\end{aligned}
\tag{5.32}
$$

We have worked out the differential equations of the above basis, and verified that they indeed take the ϵ-form.

6 Outer-massive double box

Now we tackle a different double-box family with one mass, where the propagators in the "outer" loop are taken to have the mass m. The propagator denominators and kinematic

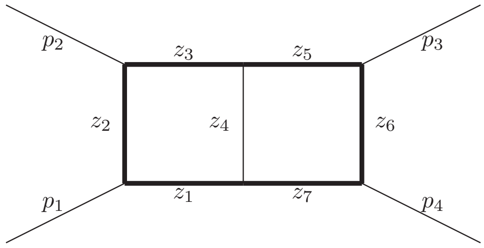

Figure 3. The outer-massive double-box integral family. Thick lines represent propagators with mass m, while thin lines represent massless propagators.

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invariants are

$$\begin{aligned}&\Big\{k_1^2-m^2,\,(k_1-p_1)^2-m^2,\,(k_1-p_1-p_2)^2-m^2,\,(k_1-k_2)^2,\,(k_2-p_1-p_2)^2-m^2,\\&(k_2-p_1-p_2-p_3)^2-m^2,\,k_2^2-m^2,\,(k_2-p_1)^2-m^2,\,(k_1-p_1-p_2-p_3)^2-m^2\Big\}\,,\\&p_i^2=0\,,\qquad (p_1+p_2)^2=s\,,\qquad (p_2+p_3)^2=t\,.\end{aligned}\tag{6.1}$$

The integral family with z_8 and z_9 as ISPs corresponds to the diagram in figure 3.

This integral family has already been considered in [84]. There are 17 unique sectors with 29 master integrals found by `LiteRed` [16] and `Kira` [20]. The construction of canonical integrals in most sectors is straightforward following the procedure in the last section. The only non-trivial sector is the top sector with 7 propagators.

6.1 Top sector {1,1,1,1,1,1,1,0,0}

We first construct the loop-by-loop Baikov representation with z_8 as ISP. The results are:

$$\begin{aligned}\mathcal{N}_\epsilon&=\frac{e^{2\epsilon\gamma_E}\Gamma^2(-\epsilon)}{16\pi^4\Gamma^2(-2\epsilon)}\left[s^2t(s+t)\right]^\epsilon\,,\\ u(\boldsymbol{z})&=P_1^\epsilon P_2^{-\epsilon-1/2}P_3^{-\epsilon-1/2}\,,\end{aligned}\tag{6.2}$$

where

$$\begin{aligned}P_1(z_5,z_7,z_8)&=-\frac{4}{s}\,G(k_2,p_1,p_2)\,,\\ P_2(z_1,z_2,z_3,z_4,z_5,z_7,z_8)&=16\,G(k_1,k_2,p_1,p_2)\,,\\ P_3(z_6,z_5,z_7,z_8)&=16\,G(k_2,p_1,p_2,p_3)\,.\end{aligned}\tag{6.3}$$

The construction can be performed easily for the variables z_1, z_2, z_3, z_4 and z_6. However, after that we are left with the expression

$$u_\epsilon(\boldsymbol{z})\,d\log f_1\wedge d\log f_2\wedge d\log f_3\wedge d\log f_4\wedge d\log f_6\wedge\frac{\hat{\varphi}(z_5,z_7,z_8)dz_5\wedge dz_7\wedge dz_8}{\sqrt{\bar{P}_2\bar{P}_3}}\,,\tag{6.4}$$

where $\hat{\varphi}(z_5, z_7, z_8)$ is the remaining function we need to construct, and

$$\begin{aligned}
u_\epsilon(\boldsymbol{z}) &= P_1^\epsilon P_2^{-\epsilon} P_3^{-\epsilon} \,, \\
\bar{P}_2(z_5, z_7, z_8) &= P_2(0,0,0,0,z_5,z_7,z_8) = s\left(4m^2\left(z_7 - z_8\right)\left(z_8 - z_5\right) + s z_8^2\right), \\
\bar{P}_3(z_5, z_7, z_8) &= P_3(0, z_5, z_7, z_8) = \left(s z_8 - st + t(z_5 + z_7)\right)^2 - 4t(s+t)\left(z_5 z_7 + s m^2\right) .
\end{aligned} \tag{6.5}$$

Since $\bar{P}_2\bar{P}_3$ is a quartic polynomial of z_8 and is also cubic in z_5 and z_7, the usual construction strategy breaks down here. On the other hand, from the maximal cut we know that there are no elliptic integrals involved in this sector. Hence it is expected that a $d\log$ representation must exist somehow.

We may continue the construction by recalling that the LBL Baikov representation can be obtained by integrating out z_9 from the standard Baikov representation, as shown in section 2.3. In particular, the polynomials P_2 and P_3 are related to the two roots of the polynomial $P_0(\boldsymbol{z}, z_9) = 16G(k_1, k_2, p_1, p_2, p_3)/s$ with respect to z_9 ($P_2 P_3 \propto B_N^2 - 4A_N C_N$ in eq. (2.28)). Let $r_\pm(\boldsymbol{z})$ denote the two roots, we can write

$$P_0^{-\delta} \frac{d^8\boldsymbol{z}\, dz_9}{P_0(\boldsymbol{z}, z_9)} = P_0^{-\delta} \frac{d^8\boldsymbol{z}}{\sqrt{P_2(\boldsymbol{z}) P_3(\boldsymbol{z})}}\, d\log\frac{z_9 - r_+}{z_9 - r_-} \,, \tag{6.6}$$

where δ serves as a regulator. Note that the above relation still holds if we set z_1, z_2, z_3, z_4 and z_6 to zero in all of $P_i(\boldsymbol{z})$ and $r_\pm(\boldsymbol{z})$. Hence we have

$$\bar{P}_0^{-\delta} \frac{f(z_5, z_7, z_8, z_9) dz_8 dz_9}{\bar{P}_0(z_5, z_7, z_8, z_9)} = \bar{P}_0^{-\delta} \frac{f(z_5, z_7, z_8, z_9) dz_8}{\sqrt{\bar{P}_2 \bar{P}_3}}\, d\log\frac{z_9 - \bar{r}_+}{z_9 - \bar{r}_-} \,, \tag{6.7}$$

where the notations $\bar{P}_0$ and $\bar{r}_\pm$ should be clear to the readers, and we have suppressed the other factors in eq. (6.4). Here, $f(z_5, z_7, z_8, z_9)$ is an arbitrary rational function whose singularities are properly regularized. The left-hand side of the above equation can also be written as

$$\bar{P}_0^{-\delta} \frac{f(z_5, z_7, z_8, z_9) dz_8 dz_9}{\bar{P}_0(z_5, z_7, z_8, z_9)} = d\log\frac{z_8 - \bar{t}_+}{z_8 - \bar{t}_-}\, \bar{P}_0^{-\delta} \frac{f(z_5, z_7, z_8, z_9) dz_9}{\sqrt{\bar{Q}_2(z_9)\bar{Q}_3(z_5, z_7, z_9)}} \,, \tag{6.8}$$

where $\bar{t}_\pm$ are the two roots of $\bar{P}_0$ with respect to z_8, and the two polynomials $\bar{Q}_2$ and $\bar{Q}_3$ are given by

$$\begin{aligned}
\bar{Q}_2(z_9) &= 16\, G(k_1, p_1, p_1 + p_2, p_3)\Big|_{z_1, z_2, z_3 \to 0} \\
&= s\left(4m^2 st + 4m^2 t^2 - st^2 + 2st z_9 - s z_9^2\right), \\
\bar{Q}_3(z_5, z_7, z_9) &= 16\, G(k_2, k_1, p_1 + p_2, p_3)\Big|_{z_1, z_3, z_4, z_6 \to 0} \\
&= 4m^2 s z_9^2 + 4m^2 s z_5 z_7 + 4m^2 s z_5 z_9 + 4m^2 s z_7 z_9 - s^2 z_9^2 + 2s z_5 z_9^2 \\
&\quad + 2s z_7 z_9^2 + 4s z_5 z_7 z_9 - z_5^2 z_9^2 - z_7^2 z_9^2 + 2 z_5 z_7 z_9^2 \,.
\end{aligned} \tag{6.9}$$

Now comes the crucial observation: the function $f(z_5, z_7, z_8, z_9)$ can be seen as a representative of an equivalence class under IBP relations. It is possible that the following equivalence relation holds:

$$f(z_5, z_7, z_8, z_9) \sim \hat{\varphi}(z_5, z_7, z_8) \sim \hat{\phi}(z_5, z_7, z_9)\,, \tag{6.10}$$

where $\hat{\varphi}$ does not depend on z_9, and $\hat{\phi}$ does not depend on z_8. In this case, we can construct the function $\hat{\phi}(z_5, z_7, z_9)$ such that the right-hand size of eq. (6.8) becomes a $d\log$-form (with $\hat{\phi}$ in place of f). Then by inserting the corresponding $\hat{\varphi}(z_5, z_7, z_8)$ into eq. (6.4), we find a candidate for canonical integrals in the original LBL Baikov representation. A dictionary of equivalent $\hat{\varphi}$'s and $\hat{\phi}$'s can be generated by putting different forms of $f(z_5, z_7, z_8, z_9)$ into eq. (6.8), and integrating out z_9 (which gives a $\hat{\varphi}$) or z_8 (which gives a $\hat{\phi}$). Since we are interested in integrands with simple poles, it is enough to consider four possible kinds of the function f: $f = c$, $f = cz_8$, $f = cz_9$ and $f = cz_8z_9$, where c denotes a "constant" polynomial independent of z_8 and z_9. The integration over z_8 or z_9 can be performed using eq. (2.28) and the relation

$$\bar{P}_0(z_5, z_7, z_8, z_9) = P_1(z_5, z_7, z_8)\,(r_+ - z_9)(z_9 - r_-) = Q_1(z_9)\,(t_+ - z_8)(z_8 - t_-)\,, \tag{6.11}$$

where P_1 was given in eq. (6.3), and

$$Q_1(z_9) = -\frac{4}{s}\,G(k_1, p_1 + p_2, p_3)\Big|_{z_1, z_3 \to 0} = z_9^2 + s z_9 + s m^2\,. \tag{6.12}$$

We then find the following correspondences between $\hat{\varphi}(z_5, z_7, z_8)$ and $\hat{\phi}(z_5, z_7, z_9)$:

$$\begin{aligned} 1 &\longleftrightarrow 1\,, \\ -\frac{R_1(z_5, z_7, z_8)}{2s\,P_1(z_5, z_7, z_8)} &\longleftrightarrow z_9\,, \\ z_8 &\longleftrightarrow -\frac{R_2(z_5, z_7, z_9)}{2s\,Q_1(z_9)}\,, \\ \frac{z_8\,R_1(z_5, z_7, z_8)}{P_1(z_5, z_7, z_8)} &\longleftrightarrow \frac{z_9\,R_2(z_5, z_7, z_9)}{Q_1(z_9)}\,, \end{aligned} \tag{6.13}$$

with the polynomials

$$\begin{aligned} R_1 &= -2st\left[m^2(z_5 + z_7) + z_5 z_7\right] + s z_8\left[2m^2(s + 2t) - st + t(z_5 + z_7)\right] + s^2 z_8^2\,, \\ R_2 &= z_9 N_1 + (z_5 + z_7) z_9 N_2 + 2t(z_5 + z_7) Q_1\,, \\ N_1 &= s(2m^2(s + 2t) - st + s z_9)\,, \\ N_2 &= -st - (s + 2t) z_9\,. \end{aligned} \tag{6.14}$$

Now, since $\bar{Q}_2$ doesn't depend on z_5 and z_7 and $\bar{Q}_3$ is quadratic in all variables, it is straightforward to construct the following candidates for $\hat{\phi}(z_5, z_7, z_9)$ such that $\hat{\phi}/\sqrt{\bar{Q}_2 \bar{Q}_3}$

is $d\log$:

$$\begin{aligned}
&\frac{\sqrt{s(s-4m^2)}\sqrt{st(st-4m^2(s+t))}}{z_5z_7}\,,\\
&\frac{\sqrt{s(s-4m^2)}}{z_5z_7}\frac{z_9N_1}{Q_1}\,, \qquad \frac{s(s-4m^2)}{z_5z_7}\frac{z_9N_2}{Q_1}\,,\\
&\frac{\sqrt{s(s-4m^2)}(z_5+z_7)}{z_5z_7}\frac{z_9N_2}{Q_1}\,, \frac{(z_5+z_7)}{z_5z_7}\frac{z_9N_1}{Q_1}\,, \qquad \frac{s(z_5+z_7)z_9}{z_5z_7}\,.
\end{aligned} \tag{6.15}$$

We need to identify three linear combinations of the above candidates which can be converted to $\hat{\varphi}(z_5,z_7,z_8)$ using the dictionary eq. (6.13). These are

$$\begin{aligned}
\hat{\phi}_1 &= \frac{\sqrt{s(s-4m^2)}\sqrt{st(st-4m^2(s+t))}}{z_5z_7}\,,\\
\hat{\phi}_2 &= -\frac{\sqrt{s(s-4m^2)}}{z_5z_7}\frac{z_9N_1+(z_5+z_7)z_9N_2}{2Q_1}\\
&= -\frac{\sqrt{s(s-4m^2)}}{z_5z_7}\left[\frac{R_2}{2Q_1}-t(z_5+z_7)\right],\\
\hat{\phi}_3 &= \frac{(z_5+z_7)}{z_5z_7}\frac{z_9N_1}{2Q_1}-\frac{s(z_5+z_7)z_9}{z_5z_7}+\frac{s\left(s-4m^2\right)}{z_5z_7}\frac{z_9N_2}{2Q_1}\\
&= \frac{s}{z_5z_7}\left[\frac{z_9R_2}{sQ_1}-sz_9+\frac{R_2}{2Q_1}-t(z_5+z_7)\right],
\end{aligned} \tag{6.16}$$

where in the last equation we have used the following identity

$$z_9R_2 = s(s-z_5-z_7)z_9Q_1+\frac{1}{2}z_9(z_5+z_7)(N_1-sN_2)-\frac{1}{2}sz_9\left(N_1-\left(s-4m^2\right)N_2\right)\,. \tag{6.17}$$

Their corresponding $\hat{\varphi}(z_5,z_7,z_8)$'s can be easily found. Multiplying them by the other factors from the construction of z_1, z_2, z_3, z_4 and z_6, we obtain

$$\begin{aligned}
\hat{\varphi}_1 &= \frac{\sqrt{s(s-4m^2)}\sqrt{st(st-4m^2(s+t))}}{z_1z_2z_3z_4z_5z_6z_7}\,,\\
\hat{\varphi}_2 &= \frac{\sqrt{s(s-4m^2)}(sz_8+t(z_5+z_7))}{z_1z_2z_3z_4z_5z_6z_7}\,,\\
\hat{\varphi}_3' &= \frac{(s+2z_8)R_1}{2z_1z_2z_3z_4z_5z_6z_7P_1}-\frac{s(sz_8+t(z_5+z_7))}{z_1z_2z_3z_4z_5z_6z_7}\,.
\end{aligned} \tag{6.18}$$

One can see that φ_1 and φ_2 are Feynman integrals while φ_3 contains a polynomial denominator. We needs to again introduce auxiliary $d\log$s from sub-sectors to convert it to

Feynman integrals. Further details can be found in appendix B. The final results are:[15]

$$\begin{aligned}
\langle\varphi_1| &= \sqrt{s(s-4m^2)}\sqrt{st(st-4m^2(s+t))}\,F_{111111100}\,,\\
\langle\varphi_2| &= \sqrt{s(s-4m^2)}\,(sF_{1111111-10}+2tF_{111101100})\,,\\
\langle\varphi_3| &= -2sF_{1111111-1-1}-2s^2F_{1111111-10}-4stF_{111111000}+2stF_{111011100}\\
&\quad +2s^2F_{101111100}+2sF_{101110100}-4tF_{110111000}\,.
\end{aligned}\tag{6.19}$$

Note that in eqs. (6.14) and (6.17), we have used the fact that both R_2 and z_9R_2 can be written as linear combinations of z_9N_1, z_9N_2 and Q_1, where the coefficients of z_9N_1 and z_9N_2 are independent of z_9 and the coefficient of Q_1 is at most linear in z_9. This is not a coincidence as is shown in appendix B.

6.2 The canonical basis

The $d\log$-forms in the other sectors can be constructed and converted to Feynman integrals similarly. We list the canonical basis from our construction in the following:

$$\begin{aligned}
\langle\varphi_1| &= \sqrt{s(s-4m^2)}\sqrt{st(st-4m^2(s+t))}\,\langle F_{111111100}|\\
\langle\varphi_2| &= \sqrt{s(s-4m^2)}\,(s\,\langle F_{1111111-10}|+2t\,\langle F_{111111000}|)\\
\langle\varphi_3| &= -2s\,\langle F_{1111111-1-1}|-2s^2\,\langle F_{1111111-10}|-4st\,\langle F_{111111000}|+2st\,\langle F_{111011100}|\\
&\quad +2s^2\,\langle F_{101111100}|+2s\,\langle F_{101110100}|-4t\,\langle F_{110111000}|\\
\langle\varphi_4| &= \sqrt{st\,(st-4m^2(s+t))}\,\langle F_{111111000}|\\
\langle\varphi_5| &= s\,\langle F_{110111000}|-s\,\langle F_{11111100-1}|\\
\langle\varphi_6| &= s^2\,\langle F_{111011100}|\\
\langle\varphi_7| &= \frac{\sqrt{st\,(st-4m^2(s+t))}}{2\epsilon}\,\langle F_{111102000}|+\frac{\sqrt{st\,(st-4m^2(s+t))}}{2\epsilon}\,\langle F_{111201000}|\\
\langle\varphi_8| &= \frac{\sqrt{s\,(s(t-m^2)^2-4m^2t^2)}}{\epsilon}\,\langle F_{111102000}|\\
\langle\varphi_9| &= \frac{sm^2}{\epsilon}\,\langle F_{111102000}|-\frac{s(1-2\epsilon)}{\epsilon}\,\langle F_{111101000}|+\frac{s}{2\epsilon}\,\langle F_{111020000}|\\
\langle\varphi_{10}| &= \frac{s\sqrt{s\,(s-4m^2)}}{\epsilon}\,\langle F_{111010200}|\\
\langle\varphi_{11}| &= (s+t)\,\langle F_{110111000}|\\
\langle\varphi_{12}| &= \frac{\sqrt{st\,(st-4m^2(s+t))}}{\epsilon}\,\langle F_{110211000}|\\
\langle\varphi_{13}| &= \frac{s}{\epsilon}\,\langle F_{11021100-1}|+\frac{t}{\epsilon}\,\langle F_{11-1211000}|
\end{aligned}$$

[15]Here we have utilized some symmetry relations such as $F_{111101100}=F_{111111000}$ and $F_{011101100}=F_{110111000}$. These relations can be automatically detected by `Kira`.

$$
\begin{aligned}
\langle\varphi_{14}| &= s\,\langle F_{101111000}| \\
\langle\varphi_{15}| &= s\sqrt{s\,(s-4m^2)}\,\langle F_{101111100}| \\
\langle\varphi_{16}| &= \frac{s}{\epsilon}\,\langle F_{101102000}| \\
\langle\varphi_{17}| &= -\frac{s}{2\epsilon}\,\langle F_{101102000}| - \frac{s}{2\epsilon}\,\langle F_{101201000}| \\
\langle\varphi_{18}| &= \frac{\sqrt{s\,(s-4m^2)}}{\epsilon}\,\langle F_{101102000}| + \frac{(1-2\epsilon)\sqrt{s\,(s-4m^2)}}{\epsilon^2}\,\langle F_{102101000}| \\
\langle\varphi_{19}| &= \frac{s}{\epsilon}\,\langle F_{111020000}| \\
\langle\varphi_{20}| &= t\,\langle F_{011111000}| \\
\langle\varphi_{21}| &= \frac{s}{2\epsilon}\,\langle F_{110120000}| - \frac{s}{4\epsilon^2}\,\langle F_{200120000}| \\
\langle\varphi_{22}| &= \frac{t}{2\epsilon}\,\langle F_{110102000}| - \frac{t}{4\epsilon^2}\,\langle F_{020102000}| \\
\langle\varphi_{23}| &= \frac{s\,(s-4m^2)}{\epsilon^2}\,\langle F_{102010200}| \\
\langle\varphi_{24}| &= \frac{\sqrt{s\,(s-4m^2)}}{2\epsilon^2}\,\langle F_{100220000}| + \frac{\sqrt{s\,(s-4m^2)}}{4\epsilon^2}\,\langle F_{200120000}| \\
\langle\varphi_{25}| &= \frac{s}{4\epsilon^2}\,\langle F_{200120000}| \\
\langle\varphi_{26}| &= \frac{\sqrt{t\,(t-4m^2)}}{2\epsilon^2}\,\langle F_{010202000}| + \frac{\sqrt{t\,(t-4m^2)}}{4\epsilon^2}\,\langle F_{020102000}| \\
\langle\varphi_{27}| &= \frac{t}{4\epsilon^2}\,\langle F_{020102000}| \\
\langle\varphi_{28}| &= \frac{\sqrt{s\,(s-4m^2)}}{\epsilon^2}\,\langle F_{102020000}| \\
\langle\varphi_{29}| &= \frac{1}{\epsilon^2}\,\langle F_{200020000}|
\end{aligned}
\tag{6.20}
$$

We have checked that their differential equations with respect to s, t and m^2 all take the ϵ-form.

7 Summary and outlook

In this paper, we have explored the properties of the generalized Baikov representation, which allows additional polynomials of the Baikov variables to appear in the denominator. We have investigated its difference and relation with the usual Baikov representation of Feynman integrals using the language of intersection theory. We find that Feynman integrals span a subspace of the vector space of generalized Baikov integrals. This explains why the dimension-counting by computing the number of critical points in the loop-by-loop Baikov representation often gives a number larger than that of independent Feynman integrals. We have further discussed how to identify this so-called FI-subspace using intersection theory, optionally supplemented with IBP relations.

Utilizing the generalized Baikov integrals, we have proposed a novel method to construct canonical Feynman integrals satisfying ϵ-form differential equations. The method start with constructing $d\log$-form integrands in the generalized Baikov representation. The construction is performed variable-by-variable, and we show in detail how to deal with the square roots appearing in the intermediate steps using properties of Gram determinants. The $d\log$ Baikov integrals are then converted to Feynman integrals by looking for linear combinations belonging to the FI-subspace. In this way a complete canonical basis is obtained. We emphasize that the constructed $d\log$ Baikov integrals are fully d-dimensional without any cuts. The resulting Feynman integrals therefore automatically have uniform transcendentality without further manipulations.

We have demonstrated our method using several examples including two kinds of one-mass double box families, and further examples are given in the appendices. In all cases we have verified the differential equations for the constructed canonical bases, which indeed take the ϵ-form. Such equations allow solutions in terms of iterated integrals which satisfy nice algebraic properties and can be easily evaluated numerically.

At one-loop, the $d\log$-form of the UT integrals helps to determine the letters appearing in the differential equations, as well as the symbols of the solutions [85–88]. It is interesting to investigate whether similar results can be obtained for higher loops using the construction in this work. It is also interesting to extend our framework to study integral families involving elliptic integrals. We leave these investigations to future works.

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A Further details for the inner-massive double box family

In this appendix, we give further details for the inner-massive double box family. We first elaborate on the construction of auxiliary $d\log$-forms which are necessary to find the UT Feynman integrals in the top sector. We then list all the $d\log$-forms for sectors corresponding to the canonical basis (5.32) in the main text. We also provide an alternative way to convert $d\log$ Baikov integrals to Feynman integrals.

A.1 The auxiliary $d\log$-forms for the top sector

In this subsection, we discuss the construction of auxiliary $d\log$-forms that allow us to arrive at $\hat{\varphi}_3$ and $\hat{\varphi}_4$ in eq. (5.31). These auxiliary $d\log$s must belong to sub-sectors, since the number of maximally-cut Baikov integrals is the same as the number of master Feynman integrals in the top sector. We therefore choose to subtract the top-sector components from φ_3' and φ_4' as the first step. Note that both φ_3' and φ_4' have the polynomial P_1 in the denominator. This fact leads us to choose the following four master Feynman integrals in the top sector:

$$F_{111111100}\,,\; F_{1111111-10}\,,\; F_{11111110-1}\,,\; F_{1111111-1-1}\,, \tag{A.1}$$

where the last two have z_9 in the numerator (which gives P_1 denominator in the generalized loop-by-loop representation without z_9 ISP).

Performing the projections using intersection theory we have

$$\begin{aligned}
\langle\varphi_3'| &= -2s^2t\,\langle F_{111111100}| + 2s^2\,\langle F_{1111111-10}| + 2s^2\,\langle F_{11111110-1}| \\
&\quad + 4s\,\langle F_{1111111-1-1}| + \text{sub-sector integrals}\,, \\
\langle\varphi_4'| &= \sqrt{s(s-4m^2)}\,(-2st\,\langle F_{111111100}| + 2s\,\langle F_{11111110-1}| + \text{sub-sector integrals})\,.
\end{aligned} \tag{A.2}$$

Subtracting the top-sector components, we arrive at the remainders

$$\begin{aligned}
R_3 &= \frac{N_3}{z_1z_2z_3z_4z_5z_6z_7P_1}\,, \\
R_4 &= \frac{\sqrt{s(s-4m^2)}\,N_4}{z_1z_2z_3z_4z_5z_6z_7P_1}\,,
\end{aligned} \tag{A.3}$$

where the numerators are given by

$$\begin{aligned}
N_3 &= z_1[-(z_5+z_7)(z_5-z_8)Q_1 - (z_5-z_8)Q_2 + 2(st+sz_5-sz_8+2tz_5)P_1] \\
&\quad + z_3[-(z_5+z_7)(z_7-z_8)Q_1 - (z_7-z_8)Q_2 + 2(st+sz_7-sz_8+2tz_7)P_1] \\
&\quad + z_2[-TQ_1 - (z_5+z_7)Q_2 - 2s(s-z_5+2z_6-z_7)P_1] \\
&\quad + z_4[-s(z_5+z_7)Q_1 + s(Q_2 - 2(s+2t)P_1)] \\
&\quad - sz_5z_8Q_1 - sz_7z_8Q_1\,, \\
N_4 &= z_1[2tP_1 - (z_5-z_8)Q_1] + z_3[2tP_1 - (z_7-z_8)Q_1] + z_4sQ_1 - z_2Q_2\,, \\
T &= \lambda(s,z_5,z_7) - 4sm^2\,,
\end{aligned} \tag{A.4}$$

with the Källén function $\lambda(x,y,z) \equiv x^2+y^2+z^2-2xy-2xz-2yz$, while Q_1 and Q_2 are the two polynomials already present in φ_3' and φ_4':

$$\begin{aligned}
Q_1 &= \frac{1}{2s}\frac{\partial P_2}{\partial z_6} \\
&= -st + sz_6 - sz_8 + tz_5 + tz_7 - 2tz_8\,, \\
Q_2 &= \frac{1}{2s}\left(2P_1\frac{\partial^2 P_2}{\partial z_6\partial z_8} - \frac{\partial P_2}{\partial z_6}\frac{\partial P_1}{\partial z_8}\right) \\
&= 2m^2s^2 + 4m^2st - s^2t + s^2z_8 + 2stz_7 + z_5\left(2z_7(s+t) + s\left(2t-z_8\right) - sz_6\right) \\
&\quad - sz_7z_8 + sz_6\left(s - z_7 + 2z_8\right) - tz_5^2 - tz_7^2\,.
\end{aligned} \tag{A.5}$$

Sector	Auxiliary $d\log$s
011111100	$\frac{\sqrt{s(s-4m^2)}(z_5-z_8)Q_1}{z_2z_3z_4z_5z_6z_7P_1}, \frac{(z_5-z_8)Q_2}{z_2z_3z_4z_5z_6z_7P_1}, \frac{s(z_5-z_8)}{z_2z_3z_4z_5z_6z_7}$
011101100	$\frac{(z_5-z_8)Q_1}{z_2z_3z_4z_6z_7P_1}$
011111000	$\frac{(z_5-z_8)Q_1}{z_2z_3z_4z_5z_6P_1}$
110111100	$\frac{\sqrt{s(s-4m^2)}(z_7-z_8)Q_1}{z_1z_2z_4z_5z_6z_7P_1}, \frac{\sqrt{s(s-4m^2)}(z_7-z_8)Q_2}{z_1z_2z_4z_5z_6z_7P_1}, \frac{s(z_7-z_8)}{z_1z_2z_4z_5z_6z_7}$
110101100	$\frac{(z_7-z_8)Q_1}{z_1z_2z_4z_6z_7P_1}$
110111000	$\frac{(z_7-z_8)Q_1}{z_1z_2z_4z_5z_6P_1}$
101111100	$\frac{TQ_1}{z_1z_3z_4z_5z_6z_7P_1}, \frac{\sqrt{s(s-4m^2)}Q_2}{z_1z_3z_4z_5z_6z_7P_1}, \frac{\sqrt{s(s-4m^2)}s}{z_1z_3z_4z_5z_6z_7}$
101101100	$\frac{Q_2}{z_1z_3z_4z_6z_7P_1}$
101111000	$\frac{Q_2}{z_1z_3z_4z_5z_6P_1}$
111011100	$\frac{s\sqrt{s(s-4m^2)}Q_1}{z_1z_2z_3z_5z_6z_7P_1}, \frac{sQ_2}{z_1z_2z_3z_5z_6z_7P_1}, \frac{s^2}{z_1z_2z_3z_5z_6z_7}$
111001100	$\frac{sQ_1}{z_1z_2z_3z_6z_7P_1}$
111011000	$\frac{sQ_1}{z_1z_2z_3z_5z_6P_1}$
111101100	$\frac{sz_8Q_1}{z_1z_2z_3z_4z_6z_7P_1}, \frac{st}{z_1z_2z_3z_4z_6z_7}$
111111000	$\frac{sz_8Q_1}{z_1z_2z_3z_4z_6z_7P_1}, \frac{st}{z_1z_2z_3z_4z_6z_7}$

Table 1. Auxiliary $d\log$s in the sub-sectors of the inner-massive double box family.

The expressions of R_3 and R_4 seem to be rather complicated. However, since we know that they are composed of $d\log$-forms in the sub-sectors, we can systematically construct these $d\log$-forms and use them to subtract all terms with P_1 in the denominator. After such a subtraction, the results must belong to the FI-subspace and it is then straightforward to convert them to Feynman integrals. In table 1 we list all relevant $d\log$s in the sub-sectors. From these it is easy to deduce the required combinations:

$$\begin{aligned}\hat{\phi}_3 &= \frac{z_1(z_5-z_8)\left(-Q_2-(z_5+z_7)Q_1+2sP_1\right)+z_3(z_7-z_8)\left(-Q_2-(z_5+z_7)Q_1+2sP_1\right)}{z_1z_2z_3z_4z_5z_6z_7P_1}\\ &\quad+\frac{-z_2\left(TQ_1+(z_5+z_7)Q_2\right)+z_4s\left(Q_2-(z_5+z_7)Q_1-2sP_1\right)-z_5sz_8Q_1-z_7sz_8Q_1}{z_1z_2z_3z_4z_5z_6z_7P_1},\\ \hat{\phi}_4 &= \frac{\sqrt{s^2-4m^2s}\left(-z_1(z_5-z_8)Q_1-z_3(z_7-z_8)Q_1-z_2Q_2+z_4sQ_1\right)}{z_1z_2z_3z_4z_5z_6z_7P_1}\,. \end{aligned} \tag{A.6}$$

Our final results for the last two UT Feynman integrals are hence

$$\hat{\varphi}_3 = \hat{\varphi}_3' - \hat{\phi}_3\,, \quad \hat{\varphi}_4 = \hat{\varphi}_4' - \hat{\phi}_4\,. \tag{A.7}$$

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A.2 List of $d\log$-forms for all sectors

Here we list all $d\log$-forms and hints for their construction. The Gram determinants in the following should be rewritten as functions of the propagator denominators z_i. These denominators are given by

$$\Big\{k_1^2,\ (k_1-p_1)^2,\ (k_1-p_1-p_2)^2,\ (k_1-k_2)^2-m^2,\ (k_2-p_1-p_2)^2-m^2,$$
$$(k_2-p_1-p_2-p_3)^2-m^2,\ k_2^2-m^2,\ (k_2-p_1)^2-m^2,\ (k_1-p_1-p_2-p_3)^2\Big\}\,. \tag{A.8}$$

- Sector $\{111111100\}$: z_8 as ISP.

$$\begin{aligned}
\hat{\varphi}_1 &= \frac{s\sqrt{st\,(st-4m^2(s+t))}}{z_1z_2z_3z_4z_5z_6z_7}\,,\\
\hat{\varphi}_2 &= \frac{s^2z_8}{z_1z_2z_3z_4z_5z_6z_7}\,,\\
\hat{\varphi}_3 &= \frac{1}{z_1z_2z_3z_4z_5z_6z_7}\Big[-2s^2t+2s^2(z_8+z_9)+4sz_8z_9\\
&\quad +2tz_1(s+2z_5)+2tz_3(s+2z_7)-2sz_2(s-z_5+2z_6-z_7)-4stz_4\Big]\,,\\
\hat{\varphi}_4 &= \frac{-2st+2sz_9+2t(z_1+z_3)\sqrt{s(s-4m^2)}}{z_1z_2z_3z_4z_5z_6z_7}\,.
\end{aligned} \tag{A.9}$$

- Sector $\{111111000\}$: z_9 as ISP.

$$\hat{\varphi}_5 = \frac{st}{z_1z_2z_3z_4z_5z_6}\,. \tag{A.10}$$

- Sector $\{101111100\}$: no ISP.

$$\hat{\varphi}_6 = \frac{s\sqrt{s\,(s-4m^2)}}{z_1z_3z_4z_5z_6z_7}\,. \tag{A.11}$$

- Sector $\{111101000\}$: z_9 as ISP.

$$\begin{aligned}
\hat{\varphi}_7 &= \frac{sz_9\,(1-2\epsilon)\,\sqrt{t\,(t-4m^2)}}{4\epsilon z_1z_2z_3z_4z_6\,G\left(\tilde{k}_1,\tilde{k}_2\right)}\,,\\
\hat{\varphi}_8 &= \frac{sz_9(1-2\epsilon)\,(z_9-t)}{4\epsilon z_1z_2z_3z_4z_6\,G\left(\tilde{k}_1,\tilde{k}_2\right)}\,,
\end{aligned} \tag{A.12}$$

  where $\tilde{k}_i = k_i-p_1-p_2-p_3$.

- Sector $\{110111000\}$: z_7 and z_8 as ISP.

$$\begin{aligned}
\hat{\varphi}_9 &= \frac{s+t}{z_1z_2z_4z_5z_6}\,,\\
\hat{\varphi}_{10} &= \frac{\sqrt{st(st-4m^2(s+t))}}{z_1z_2z_4z_5z_6z_7}\,.
\end{aligned} \tag{A.13}$$

- Sector $\{101111000\}$: z_9 as ISP.

$$\hat{\varphi}_{11} = \frac{s}{z_1 z_3 z_4 z_5 z_6}\,. \tag{A.14}$$

- Sector $\{101110100\}$: no ISP.

$$\hat{\varphi}_{12} = \frac{s(1-2\epsilon)}{\epsilon z_1 z_3 z_4 z_5 z_7}\,. \tag{A.15}$$

- Sector $\{101011100\}$: no ISP.

$$\hat{\varphi}_{13} = -\frac{s^3(1-2\epsilon)}{4\epsilon z_1 z_3 z_5 z_6 z_7\, G(k_1, p_1+p_2)} \tag{A.16}$$

- Sector $\{010111100\}$: z_8 as ISP.

$$\begin{aligned}
\hat{\varphi}_{14} &= \frac{(1-2\epsilon)\left(m^2+z_8\right)\sqrt{st\left(st-4m^2(s+t)\right)}}{4\epsilon z_2 z_4 z_5 z_6 z_7\, G(k_1-p_1, k_2-p_1)}\,, \\
\hat{\varphi}_{15} &= \frac{(1-2\epsilon)\sqrt{s^2(t-m^2)^2-4m^2st^2}}{2\epsilon z_2 z_4 z_5 z_6 z_7\, G(k_1-p_1, k_2-p_1)} \frac{\partial G(k_1-p_1, k_2-p_1)}{\partial z_4}\,, \\
\hat{\varphi}_{16} &= \frac{s z_8(1-2\epsilon)\left(m^2+z_8\right)}{4\epsilon z_2 z_4 z_5 z_6 z_7\, G(k_1-p_1, k_2-p_1)}\,.
\end{aligned} \tag{A.17}$$

- Sector $\{101101000\}$: z_9 as ISP.

$$\begin{aligned}
\hat{\varphi}_{17} &= \frac{(1-2\epsilon)s z_9}{4\epsilon z_1 z_3 z_4 z_6\, G(\tilde{k}_1, \tilde{k}_2)}\,, \\
\hat{\varphi}_{18} &= -\frac{(1-2\epsilon)s^2 z_9^2\sqrt{s(s+4m^2)}}{16\epsilon z_1 z_3 z_4 z_6\, G(\tilde{k}_1, \tilde{k}_2)\, G(\tilde{k}_1, p_1+p_2, p_3)}\,,
\end{aligned} \tag{A.18}$$

where $\tilde{k}_i = k_i - p_1 - p_2 - p_3$.

- Sector $\{101010100\}$: no ISP.

$$\hat{\varphi}_{19} = \frac{s^3(1-2\epsilon)^2\sqrt{s\,(s-4m^2)}}{16\epsilon^2 z_1 z_3 z_5 z_7\, G(k_1, p_1+p_2)\, G(k_2, p_1+p_2)}\,. \tag{A.19}$$

- Sector $\{100111000\}$: z_7 as ISP.

$$\hat{\varphi}_{20} = \frac{(1-2\epsilon)s\, G(k_2)}{4\epsilon z_1 z_4 z_5 z_6\, G(k_1, k_2)}\,. \tag{A.20}$$

- Sector $\{010111000\}$: z_8 as ISP.

$$\hat{\varphi}_{21} = \frac{(1-2\epsilon)t\, G(k_2-p_1)}{\epsilon z_2 z_4 z_5 z_6\, G(k_1-p_1, k_2-p_1)} \tag{A.21}$$

- Sector $\{010110100\}$: z_8 as ISP.

$$\begin{aligned}
\hat{\varphi}_{22} &= -\frac{s(1-2\epsilon)\left(m^2+z_8\right)}{4\epsilon z_2 z_4 z_5 z_7\, G(k_1-p_1,k_2-p_1)}\,,\\
\hat{\varphi}_{23} &= \frac{s^2 z_8(1-2\epsilon)\sqrt{s\,(s-4m^2)}\left(m^2+z_8\right)}{16\epsilon z_2 z_4 z_5 z_7\, G(k_1-p_1,k_2-p_1)\, G(k_2-p_1,p_1,p_2)}\,,\\
\hat{\varphi}_{24} &= -\frac{s z_8(1-2\epsilon)\left(m^2+z_8\right)}{4\epsilon z_2 z_4 z_5 z_7\, G(k_1-p_1,k_2-p_1)\, G(k_2-p_1,p_1,p_2)}\frac{\partial G(k_2-p_1,p_1,p_2)}{\partial z_8}\,.
\end{aligned} \tag{A.22}$$

- Sector $\{000111100\}$: z_1 as ISP.

$$\hat{\varphi}_{25} = -\frac{(1-2\epsilon)s\, G(k_2)}{4\epsilon z_4 z_5 z_6 z_7\, G(k_1,k_2)}\,. \tag{A.23}$$

- Sector $\{101100000\}$: no ISP.

$$\hat{\varphi}_{26} = \frac{s^2(1-2\epsilon)(1-\epsilon)}{4\epsilon^2 z_1 z_3 z_4\, G(k_2)\, G(k_1,p_1+p_2)}\,. \tag{A.24}$$

- Sector $\{100110000\}$: z_3 as ISP.

$$\begin{aligned}
\hat{\varphi}_{27} &= \frac{s z_3\,(1-2\epsilon)^2\,\sqrt{s\,(s-4m^2)}}{16\epsilon^2 z_1 z_4 z_5\, G\left(\tilde{k}_1,\tilde{k}_2\right)\, G\left(\tilde{k}_1,p_1+p_2\right)}\,,\\
\hat{\varphi}_{28} &= \frac{s z_3\,(1-2\epsilon)^2\,(z_3-s)}{16\epsilon^2 z_1 z_4 z_5\, G\left(\tilde{k}_1,\tilde{k}_2\right)\, G\left(\tilde{k}_1,p_1+p_2\right)}\,,
\end{aligned} \tag{A.25}$$

where $\tilde{k}_i = k_i - p_1 - p_2$.

- Sector $\{010101000\}$: z_9 as ISP.

$$\begin{aligned}
\hat{\varphi}_{29} &= \frac{t z_9\,(1-2\epsilon)^2\,\sqrt{t\,(t-4m^2)}}{16\epsilon^2 z_2 z_4 z_6\, G\left(\tilde{k}_1,\tilde{k}_2\right)\, G\left(\tilde{k}_1,p_2+p_3\right)}\,,\\
\hat{\varphi}_{30} &= \frac{t z_9\,(1-2\epsilon)^2\,(z_9-t)}{16\epsilon^2 z_2 z_4 z_6\, G\left(\tilde{k}_1,\tilde{k}_2\right)\, G\left(\tilde{k}_1,p_2+p_3\right)}\,,
\end{aligned} \tag{A.26}$$

where $\tilde{k}_i = k_i - p_1 - p_2 - p_3$.

- Sector $\{000110100\}$: z_1 as ISP.

$$\hat{\varphi}_{31} = \frac{s(1-2\epsilon)^2\sqrt{s\,(s-4m^2)}\, G(k_2)}{16\epsilon^2 z_4 z_5 z_7\, G(k_1,k_2)\, G(k_2,p_1+p_2)}\,. \tag{A.27}$$

- Sector $\{000110000\}$: no ISP.

$$\hat{\varphi}_{32} = \frac{(1-\epsilon)^2}{\epsilon^2 z_4 z_5\, G(k_1-k_2)\, G(k_2-p_1-p_2)}\,. \tag{A.28}$$

Sector	Auxiliary $d\log$s
011111100	$\frac{s(z_5-z_8)}{z_2z_3z_4z_5z_6z_7}$, $\frac{(z_5-z_8)}{z_2z_3z_4z_5z_6z_7}\frac{E_1}{P_1}$, $\frac{(z_5-z_8)(z_5+z_7)}{z_2z_3z_4z_5z_6z_7}\frac{E_2}{P_1}$
110111100	$\frac{s(z_7-z_8)}{z_1z_2z_4z_5z_6z_7}$, $\frac{(z_7-z_8)}{z_1z_2z_4z_5z_6z_7}\frac{E_1}{P_1}$, $\frac{(z_7-z_8)(z_5+z_7)}{z_1z_2z_4z_5z_6z_7}\frac{E_2}{P_1}$
101111100	$\frac{s(z_5+z_7)}{z_1z_3z_4z_5z_6z_7}$, $\frac{(z_5+z_7)}{z_1z_3z_4z_5z_6z_7}\frac{E_1}{P_1}$, $\frac{1}{z_1z_3z_4z_5z_6z_7}\frac{TE_2}{P_1}$
111011100	$\frac{s^2}{z_1z_2z_3z_5z_6z_7}$, $\frac{s}{z_1z_2z_3z_5z_6z_7}\frac{E_1}{P_1}$, $\frac{s(z_5+z_7)}{z_1z_2z_3z_5z_6z_7}\frac{E_2}{P_1}$
111110100	$\frac{s}{z_1z_2z_3z_4z_5z_7}\frac{sF_1}{P_1}$, $\frac{s(z_5+z_7)}{z_1z_2z_3z_4z_5z_7}\frac{sF_2}{P_1}$

Table 2. Auxiliary $d\log$s needed for the outer-massive double box family.

B Further details for the outer-massive double box family

B.1 Auxiliary $d\log$s for the top sector

The only non-trivial conversion from $d\log$-forms to Feynman integrals in the outer-massive double box family is that of $\hat{\varphi}_3'$ in (6.18). We apply the same method as in the inner-massive double box family. We construct the auxiliary $d\log$s in the sub-sectors and list them in table 2. The polynomials appearing in the numerators are given by

$$\begin{aligned} E_1 &= st - tz_5 - tz_7 - sz_6 + (s+2t)\,z_8\,, \\ E_2 &= (z_5+z_7)\,E_1 + 2\,(s+2t)\,P_1 - (s+2z_8)\,E_1\,, \\ T &= \lambda\,(s,z_5,z_7) - 4sm^2\,, \\ F_1 &= \left(z_8+2m^2\right)(z_5+z_7) + 2P_1 - (s+2z_8)\left(z_8+2m^2\right)\,, \\ F_2 &= z_8 + 2m^2\,, \end{aligned} \tag{B.1}$$

and P_1 is defined in eq. (6.3).

We subtract the top-sector components of

$$\langle\varphi_3'| = -2s^2\,\langle F_{1111111-10}| - 2s\,\langle F_{1111111-1-1}| + \text{sub-sector integrals}\,, \tag{B.2}$$

to arrive at the remainder

$$\mathcal{R}_3 = \frac{\mathcal{N}}{2z_1z_2z_3z_4z_5z_6z_7P_1} - \frac{st(z_5+z_7)}{z_1z_2z_3z_4z_5z_6z_7}\,. \tag{B.3}$$

The numerator $\mathcal{N}$ is given by

$$\begin{aligned} \mathcal{N} &= z_1\mathcal{N}_1 + z_2\mathcal{N}_2 + z_3\mathcal{N}_3 + z_4\mathcal{N}_4 + z_6\mathcal{N}_6\,, \\ \mathcal{N}_1 &= (z_5-z_8)E_1 - (z_5-z_8)(z_5+z_7)E_2 - 2(st+sz_5-sz_8+2tz_5)P_1\,, \\ \mathcal{N}_2 &= (z_5+z_7)E_1 - TE_2 + 2s(s-z_5-z_7+2z_6)P_1\,, \\ \mathcal{N}_3 &= (z_7-z_8)E_1 - (z_7-z_8)(z_5+z_7)E_2 - 2(st+sz_7-sz_8+2tz_7)P_1\,, \\ \mathcal{N}_4 &= -sE_1 + s(z_5+z_7)E_2 + 2s(s+2t)P_1\,, \\ \mathcal{N}_6 &= s^2(s+2z_8)(z_8+2m^2) = -s^2F_1 + s^2(z_5+z_7)F_2 + 2s^2P_1\,. \end{aligned} \tag{B.4}$$

We can now use the auxiliary $d\log$s listed in table 2 to cancel the terms with a P_1 denominator. We then have

$$\hat{\varphi}_3 = \frac{1}{z_1 z_2 z_3 z_4 z_5 z_6 z_7} \Big[-2s^2 z_8 - 2s z_8 z_9 - (st + 2t z_5) z_1 - (st + 2t z_7) z_3 \\ + (s^2 + 2s z_6) z_2 + 2st z_4 - st(z_5 + z_7) + s^2 z_6 \Big] . \tag{B.5}$$

This is apparently a combination of Feynman integrals and we can arrive at the final result in eq. (6.19).

B.2 Some relations used in the construction for the top sector

In this appendix, we discuss the relations among the polynomials N_1, N_2, Q_1, R_2 and $z_9 R_2$ appearing in and below eq. (6.14). We'd like to show that both R_2 and $z_9 R_2$ can be expressed in the form

$$c_1 z_9 N_1 + c_2 z_9 N_2 + c_3(z_9) Q_1 , \tag{B.6}$$

where c_1 and c_2 are independent of z_9, and $c_3(z_9)$ is at most linear in z_9.

We first note that R_2 and Q_1 appear in the polynomial $\bar{P}_0(z_8, z_9)$ in eq. (6.11):

$$\bar{P}_0(z_8, z_9) = Q_1(z_9) z_8^2 + R_2(z_9) z_8 + C(z_9) , \tag{B.7}$$

where we have suppressed the dependence on z_5 and z_7. In the construction for the variable z_9, we need to employ Sylvester's determinant identity eq. (4.17). In the current case it reads

$$R_2^2 - 4 Q_1 C = \bar{Q}_2(z_9)\, \bar{Q}_3(z_9) . \tag{B.8}$$

The polynomials N_1 and N_2 comes from the square roots of $\bar{Q}_2$ (see section 4.2.1). Hence one can imagine that R_2, N_1 and N_2 are related through Q_1 and $\bar{Q}_2$. In fact, the relations are not restricted to the special case here, but are universally applicable to quadratic polynomials satisfying Sylvester's determinant identity. So hereafter we'll take Q_1, Q_2, Q_3, R_2 and C to be generic quadratic polynomials of the variable z, where Q_2 and Q_3 do not share common factors. These polynomials satisfy

$$[R_2(z)]^2 - 4 Q_1(z) C(z) = Q_2(z) Q_3(z) . \tag{B.9}$$

Writing Q_1 as $(z - c_+)(z - c_-)$, we have

$$\begin{aligned} \frac{N_1(z)}{Q_1 \sqrt{Q_2}} &= \frac{\sqrt{Q_2(z = c_+)}}{(z - c_+)\sqrt{Q_2}} + \frac{\sqrt{Q_2(z = c_-)}}{(z - c_-)\sqrt{Q_2}} , \\ \frac{(c_+ - c_-) N_2(z)}{Q_1 \sqrt{Q_2}} &= \frac{\sqrt{Q_2(z = c_+)}}{(z - c_+)\sqrt{Q_2}} - \frac{\sqrt{Q_2(z = c_-)}}{(z - c_-)\sqrt{Q_2}} . \end{aligned} \tag{B.10}$$

According to eq. (B.9), we know that

$$\sqrt{Q_2(z = c_\pm)\, Q_3(z = c_\pm)} = R_2(z = c_\pm) , \tag{B.11}$$

is a polynomial. Hence we can define the linear functions $S_2(z)$ and $S_3(z)$ (similar to those in eqs. (4.19) and (4.21)), such that

$$\sqrt{Q_2(z=c_\pm)} = S_2(z=c_\pm)\,, \quad \sqrt{Q_3(z=c_\pm)} = S_3(z=c_\pm)\,. \tag{B.12}$$

Plugging the above equations back to eq. (B.10), we find

$$\begin{aligned} \frac{N_1(z)}{Q_1\sqrt{Q_2}} &= \frac{(z-c_-)\,S_2(z=c_+)+(z-c_+)\,S_2(z=c_-)}{Q_1\sqrt{Q_2}}\,, \\ \frac{(c_+-c_-)N_2(z)}{Q_1\sqrt{Q_2}} &= \frac{(z-c_-)\,S_2(z=c_+)-(z-c_+)\,S_2(z=c_-)}{Q_1\sqrt{Q_2}}\,. \end{aligned} \tag{B.13}$$

Using that $S_2(z)$ is a linear function of z, we can rewrite the above in a more instructive form:

$$N_2(z) = S_2(z) \equiv A_2 z + B_2\,, \quad 2z\,N_2(z) = N_1(z) + (c_+ + c_-)N_2(z) + 2A_2\,Q_1(z)\,. \tag{B.14}$$

Now, using $N_2(z) = S_2(z)$ together with eqs. (B.9), (B.11) and (B.12), we can deduce that

$$R_2(z) = a\,Q_1(z) + A_3\,zN_2(z) + B_3 N_2(z)\,, \tag{B.15}$$

where we have written $S_3(z) \equiv A_3 z + B_3$. We can get rid of the $B_3 N_2$ term using the following identity

$$2c_+c_-N_2(z) = -zN_1(z) + (c_+ + c_-)zN_2(z) + 2B_2 Q_1(z)\,, \tag{B.16}$$

which follows from eq. (B.14). This shows that $R_2(z)$ can be written in the form of eq. (B.6), as presented in eq. (6.14). To show that $zR_2(z)$ can also be written in this way, we just need to note that we can use the second equation in eq. (B.14) to reduce the power of z in front of $N_2(z)$. Our purpose is then achieved.

We finally note that Q_2 and Q_3 are symmetric, and the above arguments also apply in case $1/\sqrt{Q_3}$ appears in the construction of $d\log$-forms.

B.3 List of $d\log$-forms for all sectors

Here we list all $d\log$-forms in the outer-massive double box family. The denominators z_i are given by

$$\begin{aligned} &\Big\{k_1^2-m^2,\ (k_1-p_1)^2-m^2,\ (k_1-p_1-p_2)^2-m^2,\ (k_1-k_2)^2,\ (k_2-p_1-p_2)^2-m^2, \\ &(k_2-p_1-p_2-p_3)^2-m^2,\ k_2^2-m^2,\ (k_2-p_1)^2-m^2,\ (k_1-p_1-p_2-p_3)^2-m^2\Big\}\,. \end{aligned} \tag{B.17}$$

- Sector $\{111111100\}$: z_8 as ISP.

$$\begin{aligned} \hat{\varphi}_1 &= \frac{\sqrt{s(s-4m^2)}\sqrt{st(st-4m^2(s+t))}}{z_1z_2z_3z_4z_5z_6z_7}\,, \\ \hat{\varphi}_2 &= \frac{\sqrt{s(s-4m^2)}(sz_8+t(z_5+z_7))}{z_1z_2z_3z_4z_5z_6z_7}\,, \\ \hat{\varphi}_3 &= \frac{1}{z_1z_2z_3z_4z_5z_6z_7}\Big[-2s^2z_8-2sz_8z_9-(st+2tz_5)z_1-(st+2tz_7)z_3 \\ &\qquad +(s^2+2sz_6)z_2+2stz_4-st(z_5+z_7)+s^2z_6\Big]\,. \end{aligned} \tag{B.18}$$

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- Sector $\{111111000\}$: z_9 as ISP.

$$\begin{aligned}\hat{\varphi}_4 &= \frac{\sqrt{st(st-4m^2(s+t))}}{z_1z_2z_3z_4z_5z_6}\,,\\ \hat{\varphi}_5 &= \frac{s(z_3-z_9)}{z_1z_2z_3z_4z_5z_6}\,.\end{aligned}\tag{B.19}$$

- Sector $\{111011100\}$: no ISP.

$$\hat{\varphi}_6 = \frac{s^2}{z_1z_2z_3z_5z_6z_7}\,.\tag{B.20}$$

- Sector $\{111101000\}$: z_9 as ISP.

$$\begin{aligned}\hat{\varphi}_7 &= \frac{1-2\epsilon}{\epsilon}\frac{(z_9+m^2)\sqrt{st(st-4m^2(s+t))}}{4z_1z_2z_3z_4z_6\,G(k_1-p_1-p_2-p_3,k_2-p_1-p_2-p_3)}\,,\\ \hat{\varphi}_8 &= \frac{1-2\epsilon}{\epsilon}\frac{(z_9+z_4-z_6)\sqrt{s(s(t-m^2)^2-4m^2t^2)}}{4z_1z_2z_3z_4z_6\,G(k_1-p_1-p_2-p_3,k_2-p_1-p_2-p_3)}\,,\\ \hat{\varphi}_9 &= \frac{1-2\epsilon}{\epsilon}\frac{sz_9\,(z_9+m^2)}{4z_1z_2z_3z_4z_6\,G(k_1-p_1-p_2-p_3,k_2-p_1-p_2-p_3)}\,.\end{aligned}\tag{B.21}$$

- Sector $\{111010100\}$: no ISP.

$$\hat{\varphi}_{10} = \frac{1-2\epsilon}{\epsilon}\frac{s^2\sqrt{s(s-4m^2)}}{4z_1z_2z_3z_5z_7\,G(k_2,p_1+p_2)}\,.\tag{B.22}$$

- Sector $\{110111000\}$: z_3 and z_9 as ISPs.

$$\begin{aligned}\hat{\varphi}_{11} &= \frac{s+t}{z_1z_2z_4z_5z_6}\,,\\ \hat{\varphi}_{12} &= \frac{(z_5-z_6)(z_3-z_9)\sqrt{st(st-4m^2(s+t))}}{4z_1z_2z_4z_5z_6\,G(k_2-p_1-p_2,k_1-p_1-p_2,p_3)}\,,\\ \hat{\varphi}_{13} &= \frac{(z_5-z_6)(z_3-z_9)(sz_9+tz_3)}{4z_1z_2z_4z_5z_6\,G(k_2-p_1-p_2,k_1-p_1-p_2,p_3)}\,.\end{aligned}\tag{B.23}$$

- Sector $\{101111000\}$: z_9 as ISP (super-sector with z_7 for $\hat{\varphi}_{15}$).

$$\begin{aligned}\hat{\varphi}_{14} &= \frac{s}{z_1z_3z_4z_5z_6}\,,\\ \hat{\varphi}_{15} &= \frac{s\sqrt{s(s-4m^2)}}{z_1z_3z_4z_5z_6z_7}\,.\end{aligned}\tag{B.24}$$

- Sector $\{101101000\}$: z_9 as ISP.

$$\begin{aligned}\hat{\varphi}_{16} &= -\frac{1-2\epsilon}{\epsilon}\frac{4s\,(z_9+z_4-z_6)}{z_1z_3z_4z_6\,G\left(\tilde{k}_1,\tilde{k}_2\right)}\,,\\ \hat{\varphi}_{17} &= \frac{1-2\epsilon}{\epsilon}\frac{s\,(z_9+m^2)}{4z_1z_3z_4z_6\,G\left(\tilde{k}_1,\tilde{k}_2\right)}\,,\\ \hat{\varphi}_{18} &= -\frac{1-2\epsilon}{\epsilon}\frac{s^2z_9\,(z_9+m^2)\,\sqrt{s\,(s-4m^2)}}{16z_1z_3z_4z_6\,G\left(\tilde{k}_1,\tilde{k}_2\right)\,G\left(\tilde{k}_1,p_1+p_2,p_3\right)}\,,\end{aligned}\tag{B.25}$$

where $\tilde{k}_1 = k_1-p_1-p_2-p_3$ and $\tilde{k}_2 = k_2-p_1-p_2-p_3$.

- Sector $\{111010000\}$: no ISP.

$$\hat{\varphi}_{19} = \frac{1-\epsilon}{\epsilon} \frac{s}{z_1 z_2 z_3 z_5 \, G(k_2 - p_1 - p_2)} \,. \tag{B.26}$$

- Sector $\{011111000\}$: z_9 as ISP.

$$\hat{\varphi}_{20} = \frac{t}{z_2 z_3 z_4 z_5 z_6} \,. \tag{B.27}$$

- Sector $\{110110000\}$: z_3 as ISP.

$$\hat{\varphi}_{21} = \frac{1-2\epsilon}{\epsilon} \frac{s\, G(k_1 - p_1 - p_2)}{4 z_1 z_2 z_4 z_5 \, G(k_1 - p_1 - p_2, k_2 - p_1 - p_2)} \,. \tag{B.28}$$

- Sector $\{110101000\}$: z_9 as ISP.

$$\hat{\varphi}_{22} = \frac{1-2\epsilon}{\epsilon} \frac{t\, G(k_1 - p_1 - p_2)}{4 z_1 z_2 z_4 z_6 \, G(k_1 - p_1 - p_2, k_2 - p_1 - p_2)} \,. \tag{B.29}$$

- Sector $\{101010100\}$: no ISP.

$$\hat{\varphi}_{23} = \frac{(1-2\epsilon)^2}{\epsilon^2} \frac{s^3 (s - 4m^2)}{16 z_1 z_3 z_5 z_7 \, G(k_1, p_1 + p_2)\, G(k_2, p_1 + p_2)} \,. \tag{B.30}$$

- Sector $\{100110000\}$: z_3 as ISP.

$$\begin{aligned} \hat{\varphi}_{24} &= \frac{(1-2\epsilon)^2}{\epsilon^2} \frac{s\sqrt{s\,(s-4m^2)}\, G\left(\tilde{k}_1\right)}{15 z_1 z_4 z_5 \, G\left(\tilde{k}_1, \tilde{k}_2\right)\, G\left(\tilde{k}_1, p_1 + p_2\right)} \,, \\ \hat{\varphi}_{25} &= \frac{(1-2\epsilon)^2}{\epsilon^2} \frac{s z_3 \, G\left(\tilde{k}_1\right)}{16 z_1 z_4 z_5 \, G\left(\tilde{k}_1, \tilde{k}_2\right)\, G\left(\tilde{k}_1, p_1 + p_2\right)} \,, \end{aligned} \tag{B.31}$$

where $\tilde{k}_1 = k_1 - p_1 - p_2$ and $\tilde{k}_2 = k_2 - p_1 - p_2$.

- Sector $\{010101000\}$: z_9 as ISP.

$$\begin{aligned} \hat{\varphi}_{26} &= \frac{(1-2\epsilon)^2}{\epsilon^2} \frac{t\sqrt{t\,(t-4m^2)}\, G\left(\tilde{k}_1\right)}{16 z_2 z_4 z_6 \, G\left(\tilde{k}_1, \tilde{k}_2\right)\, G\left(\tilde{k}_1, p_2 + p_3\right)} \,, \\ \hat{\varphi}_{27} &= \frac{(1-2\epsilon)^2}{\epsilon^2} \frac{t z_9 \, G\left(\tilde{k}_1\right)}{16 z_2 z_4 z_6 \, G\left(\tilde{k}_1, \tilde{k}_2\right)\, G\left(\tilde{k}_1, p_2 + p_3\right)} \,, \end{aligned} \tag{B.32}$$

where $\tilde{k}_1 = k_1 - p_1 - p_2 - p_3$ and $\tilde{k}_2 = k_2 - p_1 - p_2 - p_3$.

- Sector $\{101010000\}$: no ISP.

$$\hat{\varphi}_{28} = \frac{(1-2\epsilon)(1-\epsilon)}{\epsilon^2} \frac{s\sqrt{s(s-4m^2)}}{4 z_1 z_3 z_5 \, G(k_2 - p_1 - p_2)\, G(k_1, p_1 + p_2)} \,. \tag{B.33}$$

- Sector $\{100010000\}$: no ISP.

$$\hat{\varphi}_{29} = \frac{(1-\epsilon)^2}{\epsilon^2} \frac{1}{z_1 z_5 \, G(k_1)\, G(k_2 - p_1 - p_2)} \,. \tag{B.34}$$

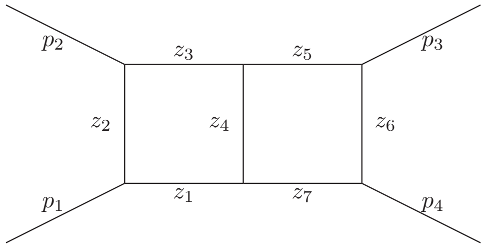

Figure 4. Massless double box. All propagators and external legs are massless.

C Massless double box

This is a simpler example since we have fewer mass scales in the problem. The diagram is depicted in figure 4, where all external momenta are outgoing. The propagator denominators $\{z_i\}$ $(i = 1, \dots, 9)$ are given by

$$\Big\{k_1^2,\, (k_1-p_1)^2,\, (k_1-p_1-p_2)^2,\, (k_1-k_2)^2,\\ (k_2-p_1-p_2)^2,\, (k_2-p_1-p_2-p_3)^2,\, k_2^2,\, (k_2-p_1)^2,\, (k_1-p_1-p_2-p_3)^2\Big\}\,. \tag{C.1}$$

The momentum invariants are $p_i^2 = 0$ for $i = 1, \dots, 4$, and

$$(p_1+p_2)^2 = s\,, \qquad (p_2+p_3)^2 = t\,, \qquad (p_1+p_3)^2 = -s-t\,. \tag{C.2}$$

We consider the top sector {1,1,1,1,1,1,1,0,0} and its sub-sectors. Using `Kira` we find 8 MIs in total, spanning the top sector and 6 sub-sectors. We list the $d\,$log-forms in the following:

- Sector {1,1,1,1,1,1,1,0,0}: z_9 as ISP.

$$\hat\varphi_1 = \frac{s^2 t}{z_1 z_2 z_3 z_4 z_5 z_6 z_7}\,, \quad \hat\varphi_2 = \frac{s^2 z_9}{z_1 z_2 z_3 z_4 z_5 z_6 z_7}\,. \tag{C.3}$$

- Sector {1,1,1,1,0,1,0,0,0}: z_9 as ISP.

$$\hat\varphi_3 = -\frac{1-2\epsilon}{\epsilon}\frac{s t z_9}{4 z_1 z_2 z_3 z_4 z_6\, G(k_1-k_2, k_1-p_1-p_2-p_3)}\,. \tag{C.4}$$

- Sector {1,1,0,1,1,1,0,0,0}: z_7 and z_8 as ISPs.

$$\hat\varphi_4 = \frac{s+t}{z_1 z_2 z_4 z_5 z_6}\,. \tag{C.5}$$

- Sector {101101000}: z_9 as ISP.

$$\hat\varphi_5 = -\frac{1-2\epsilon}{\epsilon}\frac{s z_9}{4 z_1 z_3 z_4 z_6\, G(k_2-p_1-p_2-p_3, k_1-p_1-p_2-p_3)}\,. \tag{C.6}$$

- Sector $\{101010100\}$: no ISP.

$$\hat{\varphi}_6 = \frac{(1-2\epsilon)^2}{\epsilon^2} \frac{s^4}{16 z_1 z_3 z_5 z_7\, G(k_1, p_1+p_2)\, G(k_2, p_1+p_2)}\,. \tag{C.7}$$

- Sector $\{010101000\}$: z_8 as ISP.

$$\hat{\varphi}_7 = \frac{(1-2\epsilon)^2}{\epsilon^2} \frac{t^2 z_8}{16 z_2 z_4 z_6\, G(k_2-p_1, k_1-p_1)\, G(k_2-p_1, p_2+p_3)}\,. \tag{C.8}$$

- Sector $\{001100100\}$: z_5 as ISP.

$$\hat{\varphi}_8 = \frac{(1-2\epsilon)^2}{\epsilon^2} \frac{s^2 z_5}{16 z_3 z_4 z_7\, G(k_2-p_1-p_2, k_1-p_1-p_2)\, G(k_2-p_1-p_2, p_1+p_2)}\,. \tag{C.9}$$

  Note that this sector is symmetric with respect to the previous one under the replacements

$$z_3 \leftrightarrow z_2\,, \qquad z_5 \leftrightarrow z_8\,, \qquad z_7 \leftrightarrow z_6\,, \qquad s \leftrightarrow t\,. \tag{C.10}$$

Some of the above $d\log$-forms already appear to be Feynman integrals, and the others can be converted using dimensional recurrence relations. Hence we don't bother to invoke intersection theory here. The results are given by

$$\begin{aligned}
\langle\varphi_1| &= s^2 t\, \langle F_{1111111}|\,, \\
\langle\varphi_2| &= s^2\, \langle F_{11111110-1}|\,, \\
\langle\varphi_3| &= \frac{3(2\epsilon-1)(3\epsilon-1)}{2\epsilon^2} \langle F_{1011010}| + \frac{3s(2\epsilon-1)}{\epsilon} \langle F_{1111010}|\,, \\
\langle\varphi_4| &= (s+t)\, \langle F_{1101110}|\,, \\
\langle\varphi_5| &= -\frac{(2\epsilon-1)(3\epsilon-1)}{2\epsilon^2} \langle F_{1011010}|\,, \\
\langle\varphi_6| &= \frac{(2\epsilon-1)^2}{\epsilon^2} \langle F_{1010101}|\,, \\
\langle\varphi_7| &= \frac{3(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{2t\epsilon^3} \langle F_{0101010}|\,, \\
\langle\varphi_8| &= \frac{3(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{2s\epsilon^3} \langle F_{0011001}|\,.
\end{aligned} \tag{C.11}$$

It is straightforward to derive the differential equations of the above basis with respect to s and t. We may multiply the basis by a factor of $(-s)^{2\epsilon}$ to make it dimensionless, and introduce the dimensionless variable $x = t/s$. Denoting the basis as $\vec{\phi}$, we can write the differential equations as

$$\partial_x \vec{\phi} = \epsilon \left(\frac{A_1}{x} + \frac{A_2}{x+1} \right) \vec{\phi}\,, \tag{C.12}$$

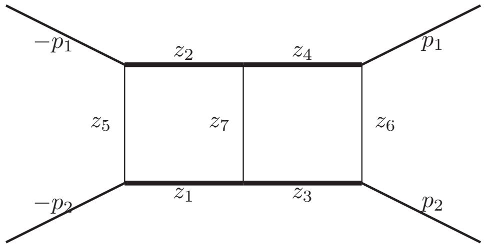

Figure 5. The integral family relevant to the HW^+W^- vertex. Note that we only consider its various sub-sectors obtained by pinching some propagators. Internal thick lines represent propagators with mass m_t, while internal thin lines represent massless propagators. The external momenta are outgoing with $p_1^2 = m_1^2$ and $p_2^2 = m_2^2$.

where the two matrices are given by

$$A_1 = \begin{pmatrix} -2 & 0 & -4 & 12 & 0 & 0 & -4 & -4 \\ -1 & 1 & -4 & 18 & 3 & -1 & -6 & -4 \\ 0 & 0 & -2 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & -2/3 & 2/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 2 & -2 & 4 & -12 & 6 & -2 & 4 & 8 \\ 1 & -1 & 4 & -18 & -3 & -1 & 6 & 4 \\ 0 & 0 & 1 & 0 & -3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{C.13}$$

We see that the equations are of the ϵ-form, and the solutions can be easily expressed as HPLs.

D The two-loop triangle family relevant to the HW^+W^- vertex

This two-loop triangle family is defined by four massive and three massless propagators. The diagram is depicted in figure 5, where all external momenta are outgoing. The propagator denominators are given by

$$\Big\{k_1^2 - m_t^2, (k_1 - p_1 - p_2)^2 - m_t^2, k_2^2 - m_t^2, (k_2 - p_1 - p_2)^2 - m_t^2, (k_1 - p_2)^2, \\ (k_2 - p_2)^2, (k_1 - k_2)^2\Big\}, \tag{D.1}$$

where the external momenta p_1 and p_2 satisfy

$$p_1^2 = m_1^2\,, \qquad p_2^2 = m_2^2\,, \qquad (p_1 + p_2)^2 = s\,. \tag{D.2}$$

For convenience we define the abbreviation $\lambda \equiv \lambda(s, m_1^2, m_2^2)$ where the Källén function is defined as

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz\,. \tag{D.3}$$

This family is relevant to the HW^+W^- vertex in the standard model [89, 90]. There are 38 MIs in 24 unique sectors. We perform the construction as follows:

- Sector {1, 1, 1, 1, 1, 1, 0}, z_7 as ISP.

$$\hat{\varphi}_1 = \frac{\lambda}{z_1 z_2 z_3 z_4 z_5 z_6}\,. \tag{D.4}$$

- Sector {1, 1, 1, 1, 0, 1, 0}: no ISP.

$$\hat{\varphi}_2 = -\frac{1-2\epsilon}{\epsilon}\frac{s\sqrt{\lambda}\sqrt{s(s-4m_t^2)}}{4z_1 z_2 z_3 z_4 z_6\, G(k_1, p_1 + p_2)}\,. \tag{D.5}$$

- Sector {1, 1, 1, 1, 0, 0, 0}: no ISP.

$$\hat{\varphi}_3 = \frac{(1-2\epsilon)^2}{\epsilon^2}\frac{s^3(s-4m_t^2)}{16 z_1 z_2 z_3 z_4\, G(k_1, p_1 + p_2)\, G(k_2, p_1 + p_2)}. \tag{D.6}$$

- Sector {1, 0, 1, 1, 1, 0, 1}: z_6 as ISP.

$$\begin{aligned} P_2(z_1, z_3, z_5, z_6, z_7) &= -4G(k_1, k_2, p_2)\,, \\ P_3(z_3, z_4, z_6) &= -4G(k_2, p_1, p_2)\,, \\ \hat{\varphi}_4 &= \frac{\sqrt{\lambda}}{z_1 z_3 z_4 z_5 z_7}\,, \\ \hat{\varphi}_5 &= \frac{\sqrt{\lambda}\sqrt{s(s-4m_t^2)}}{z_1 z_3 z_4 z_5 z_7}\frac{1}{P_3}\frac{\partial P_3}{\partial z_4}\,, \\ \hat{\varphi}_6 &= \frac{1}{z_1 z_3 z_4 z_5 z_7}\left[\frac{1}{P_3}\frac{\partial P_3}{\partial z_4}\frac{\partial P_3}{\partial z_6} - 2\frac{\partial^2 P_3}{\partial z_4 \partial z_6}\right], \\ \hat{\varphi}_7 &= \frac{\sqrt{\lambda}(m_2^2 - m_t^2)}{z_1 z_3 z_4 z_5 z_7}\frac{1}{P_2}\frac{\partial P_2}{\partial z_5}\,. \end{aligned} \tag{D.7}$$

- Sector $\{1, 0, 1, 1, 1, 1, 0\}$: no ISP.

$$\hat{\varphi}_8 = -\frac{1-2\epsilon}{\epsilon}\frac{\sqrt{\lambda}m_2^2(m_2^2-m_t^2)}{4z_1z_3z_4z_5z_6G(k_1,p_2)}\,. \tag{D.8}$$

- Sector $\{1, 0, 1, 1, 1, 0, 0\}$: no ISP.

$$\hat{\varphi}_9 = \frac{(1-2\epsilon)^2}{\epsilon^2}\frac{sm_2^2(m_2^2-m_t^2)\sqrt{s(s-4m_t^2)}}{16z_1z_3z_4z_5\,G(k_2,p_1+p_2)\,G(k_1,p_2)}\,. \tag{D.9}$$

- Sector $\{1, 0, 1, 0, 1, 1, 0\}$: no ISP.

$$\hat{\varphi}_{10} = \frac{(1-2\epsilon)^2}{\epsilon^2}\frac{m_2^4(m_2^2-m_t^2)^2}{16z_1z_3z_5z_6\,G(k_1,p_2)\,G(k_2,p_2)}\,. \tag{D.10}$$

- Sector $\{0, 1, 1, 1, 1, 0, 1\}$: z_6 as ISP.

  The $d\log$-forms $\hat{\varphi}_{11}$, $\hat{\varphi}_{12}$, $\hat{\varphi}_{13}$ and $\hat{\varphi}_{14}$ in this sector can be obtained from sector $\{1, 0, 1, 1, 1, 0, 1\}$ by the replacements:

$$z_3 \leftrightarrow z_4\,, \quad m_1 \leftrightarrow m_2\,, \quad z_1 \leftrightarrow z_2\,.$$

- Sector $\{0, 1, 1, 1, 1, 1, 0\}$: no ISP.

$$\hat{\varphi}_{15} = -\frac{1-2\epsilon}{\epsilon}\frac{\sqrt{\lambda}m_1^2(m_1^2-m_t^2)}{4z_2z_3z_4z_5z_6\,G(k_1,p_1)}\,. \tag{D.11}$$

- Sector $\{0, 1, 1, 1, 1, 0, 0\}$: no ISP.

$$\hat{\varphi}_{16} = \frac{(1-2\epsilon)^2}{\epsilon^2}\frac{sm_1^2(m_1^2-m_t^2)\sqrt{s(s-4m_t^2)}}{16z_2z_3z_4z_5\,G(k_2,p_1+p_2)\,G(k_1,p_1)}\,. \tag{D.12}$$

- Sector $\{0, 1, 1, 1, 0, 1, 0\}$: no ISP.

$$\hat{\varphi}_{17} = \frac{1-\epsilon}{\epsilon}\frac{\sqrt{\lambda}}{z_2z_3z_4z_6\,G(k_1-p_1-p_2)}\,. \tag{D.13}$$

- Sector $\{0, 1, 1, 1, 0, 0, 0\}$: no ISP.

$$\hat{\varphi}_{18} = -\frac{(1-\epsilon)(1-2\epsilon)}{\epsilon^2}\frac{s\sqrt{s(s-4m_t^2)}}{4z_2z_3z_4\,G(k_1-p_1-p_2)\,G(k_2,p_1+p_2)}\,. \tag{D.14}$$

- Sector $\{0, 1, 1, 0, 1, 1, 1\}$: z_1 as ISP.

$$\hat{\varphi}_{19} = \frac{\sqrt{\lambda}}{z_2z_3z_5z_6z_7}. \tag{D.15}$$

- Sector $\{0, 1, 1, 0, 1, 0, 1\}$: z_1 as ISP.

$$\begin{aligned}
P_1(z_1) &= G(k_1)\,,\\
P_2(z_1,z_2,z_5) &= -4G(k_1,p_1,p_2)\,,\\
P_3(z_1,z_3,z_7) &= -4G(k_1,k_2)\,,\\
\hat{\varphi}_{20} &= \frac{1-2\epsilon}{\epsilon}\frac{\sqrt{\lambda}P_1}{z_2z_3z_5z_7P_3}\,,\\
\hat{\varphi}_{21} &= \frac{1-2\epsilon}{\epsilon}\frac{\lambda(m_1^2-m_t^2)z_1P_1}{z_2z_3z_5z_7P_2P_3}\,,\\
\hat{\varphi}_{22} &= \frac{1-2\epsilon}{\epsilon}\frac{\sqrt{\lambda}z_1P_1}{z_2z_3z_5z_7P_2P_3}\frac{\partial P_2}{\partial z_1}\,.
\end{aligned} \tag{D.16}$$

- Sector $\{0, 1, 1, 0, 1, 1, 0\}$: no ISP.

$$\hat{\varphi}_{23} = \frac{(1-2\epsilon)^2}{\epsilon^2}\frac{m_1^2m_2^2(m_1^2-m_t^2)(m_2^2-m_t^2)}{16z_2z_3z_5z_6\,G(k_1,p_1)\,G(k_2,p_2)}. \tag{D.17}$$

- Sector $\{0, 1, 1, 0, 0, 1, 1\}$: z_4 ISP.

  The $d\log$-forms $\hat{\varphi}_{24}$, $\hat{\varphi}_{25}$ and $\hat{\varphi}_{26}$ in this sector can be obtained from sector $\{0, 1, 1, 0, 1, 0, 1\}$ by the replacements:

$$z_1 \leftrightarrow z_4\,, \qquad z_2 \leftrightarrow z_3\,, \qquad z_5 \leftrightarrow z_7\,, \qquad m_1 \leftrightarrow m_2\,.$$

- Sector $\{0, 1, 1, 0, 0, 0, 1\}$: z_1 as ISP.

$$\begin{aligned}
\hat{\varphi}_{27} &= \frac{(1-2\epsilon)^2}{16\epsilon^2}\frac{s\sqrt{s(s-4m_t^2)}}{z_2z_3z_7}\frac{G(k_1)}{G(k_1,p_1+p_2)\,G(k_1,k_2)},\\
\hat{\varphi}_{28} &= \frac{(1-2\epsilon)^2}{16\epsilon^2}\frac{sz_1\,G(k_1)}{z_2z_3z_7\,G(k_1,p_1+p_2)\,G(k_1,k_2)}.
\end{aligned} \tag{D.18}$$

- Sector $\{0, 1, 1, 0, 0, 1, 0\}$: no ISP.

$$\hat{\varphi}_{29} = -\frac{(1-\epsilon)(1-2\epsilon)}{\epsilon^2}\frac{m_2^2(m_2^2-m_t^2)}{4z_2z_3z_6\,G(k_1-p_1-p_2)\,G(k_2,p_2)}\,. \tag{D.19}$$

- Sector $\{0, 1, 0, 1, 1, 1, 0\}$: no ISP.

$$\hat{\varphi}_{30} = \frac{(1-2\epsilon)^2}{16\epsilon^2}\frac{m_1^4(m_1^2-m_t^2)^2}{z_2z_4z_5z_6\,G(k_1,p_1)\,G(k_2,p_1)}. \tag{D.20}$$

- Sector $\{0, 1, 0, 1, 0, 1, 0\}$: z_5 as ISP.

$$\hat{\varphi}_{31} = \frac{(1-2\epsilon)^2}{16\epsilon^2}\frac{m_1^4(m_1^2-m_t^2)}{z_2z_4z_6\,G(k_1,p_1)\,G(k_2,p_1)}\,. \tag{D.21}$$

- Sector $\{0, 1, 0, 1, 0, 0, 0\}$: no ISP.

$$\hat{\varphi}_{32} = \frac{(1-\epsilon)^2}{\epsilon^2} \frac{1}{z_2 z_4\, G(k_1-p_1-p_2)\, G(k_2-p_1-p_2)} \,. \tag{D.22}$$

- Sector $\{0, 0, 1, 1, 1, 0, 1\}$: z_6 as ISP.

$$\begin{aligned} \hat{\varphi}_{33} &= -\frac{1-2\epsilon}{4\epsilon} \frac{\sqrt{\lambda}\, z_6}{z_3 z_4 z_5 z_7\, G(k_1,k_2)} \,, \\ \hat{\varphi}_{34} &= \frac{1-2\epsilon}{16\epsilon} \frac{\lambda \sqrt{s(s-4m_t^2)}\, z_6^2}{z_3 z_4 z_5 z_7\, G(k_1,k_2)\, G(k_2,p_1,p_2)} \,. \end{aligned} \tag{D.23}$$

- Sector $\{0, 0, 1, 0, 1, 0, 1\}$: z_1 as ISP.

$$\begin{aligned} \hat{\varphi}_{35} &= \frac{(1-2\epsilon)^2}{16\epsilon^2} \frac{m_2^2(m_2^2-m_t^2)}{z_3 z_5 z_7} \frac{G(k_1)}{G(k_1,p_2)\, G(k_1,k_2)} \,, \\ \hat{\varphi}_{36} &= \frac{(1-2\epsilon)^2}{16\epsilon^2} \frac{m_2^2 z_1}{z_3 z_5 z_7} \frac{G(k_1)}{G(k_1,p_2)\, G(k_1,k_2)} \,. \end{aligned} \tag{D.24}$$

- Sector $\{0, 0, 0, 1, 1, 0, 1\}$: z_6 as ISP.

$$\begin{aligned} \hat{\varphi}_{37} &= \frac{(1-2\epsilon)^2}{16\epsilon^2} \frac{m_1^2 z_6^2}{z_4 z_5 z_7\, G(k_2,p_1)\, G(k_1,k_2)} \,, \\ \hat{\varphi}_{38} &= \frac{(1-2\epsilon)^2}{16\epsilon^2} \frac{m_1^2(m_1^2-m_t^2) z_6}{z_4 z_5 z_7\, G(k_2,p_1)\, G(k_1,k_2)} \,. \end{aligned} \tag{D.25}$$

We now list the canonical basis in terms of Feynman integrals. For convenience, we introduce the following dimensionless variables:

$$\begin{aligned} &u = -\frac{s}{4m_t^2} \,, \qquad v = -\frac{m_1^2}{4m_t^2} \,, \qquad w = -\frac{m_2^2}{4m_t^2} \,, \\ &R_1 = \sqrt{u(u+1)} \,, \qquad R_2 = \sqrt{\lambda(u,v,w)} \,. \end{aligned} \tag{D.26}$$

$$\begin{aligned} \langle \varphi_1 | &= 16 R_2^2 m_t^4 \, \langle F_{1111110} | \,, \\ \langle \varphi_2 | &= -\frac{16 R_1 R_2 m_t^4}{\epsilon} \, \langle F_{2111010} | \,, \\ \langle \varphi_3 | &= \frac{16 R_1^2 m_t^4}{\epsilon^2} \, \langle F_{2121000} | \,, \\ \langle \varphi_4 | &= 4 R_2 m_t^2 \, \langle F_{1011101} | \,, \\ \langle \varphi_5 | &= -\frac{16 R_1 R_2 m_t^4}{\epsilon} \, \langle F_{1012101} | \,, \end{aligned}$$

$$
\begin{aligned}
\langle\varphi_6| &= -\frac{4m_t^2(u+v-w)}{\epsilon}\langle F_{0210011}| - 8m_t^2(u-v+w)\langle F_{1011101}| \\
&\quad - \frac{8um_t^4(2u-2v-2w+1)}{\epsilon}\langle F_{1012101}| + \frac{8um_t^2}{\epsilon}\langle F_{10121-11}|\,, \\
\langle\varphi_7| &= \frac{4R_2(4w+1)m_t^4}{\epsilon}\langle F_{1011201}|\,, \\
\langle\varphi_8| &= -\frac{16R_2wm_t^4}{\epsilon}\langle F_{2011110}| - \frac{2R_2m_t^2}{\epsilon}\langle F_{0211010}|\,, \\
\langle\varphi_9| &= \frac{16R_1wm_t^4}{\epsilon^2}\langle F_{2021100}| + \frac{2R_1m_t^2}{\epsilon^2}\langle F_{0212000}|\,, \\
\langle\varphi_{10}| &= \frac{16w^2m_t^4}{\epsilon^2}\langle F_{2020110}| + \frac{4wm_t^2}{\epsilon^2}\langle F_{0220010}| + \frac{1}{4\epsilon^2}\langle F_{0202000}|\,, \\
\langle\varphi_{11}| &= 4R_2m_t^2\langle F_{0111101}|\,, \\
\langle\varphi_{12}| &= -\frac{16R_1R_2m_t^4}{\epsilon}\langle F_{0121101}|\,, \\
\langle\varphi_{13}| &= -\frac{4m_t^2(u-v+w)}{\epsilon}\langle F_{0120101}| - 8m_t^2(u+v-w)\langle F_{0111101}| \\
&\quad - \frac{8um_t^4(2u-2v-2w+1)}{\epsilon}\langle F_{0121101}| + \frac{8um_t^2}{\epsilon}\langle F_{01211-11}|\,, \\
\langle\varphi_{14}| &= \frac{4R_2(4v+1)m_t^4}{\epsilon}\langle F_{0111201}|\,, \\
\langle\varphi_{15}| &= -\frac{16R_2vm_t^4}{\epsilon}\langle F_{0211110}| - \frac{2R_2m_t^2}{\epsilon}\langle F_{0211010}|\,, \\
\langle\varphi_{16}| &= \frac{16R_1vm_t^4}{\epsilon^2}\langle F_{0221100}| + \frac{2R_1m_t^2}{\epsilon^2}\langle F_{0212000}|\,, \\
\langle\varphi_{17}| &= \frac{4R_2m_t^2}{\epsilon}\langle F_{0211010}|\,, \\
\langle\varphi_{18}| &= \frac{4R_1m_t^2}{\epsilon^2}\langle F_{0212000}|\,, \\
\langle\varphi_{19}| &= 4R_2m_t^2\langle F_{0110111}|\,, \\
\langle\varphi_{20}| &= \frac{2R_2m_t^2}{\epsilon}\langle F_{0110102}| + \frac{2R_2m_t^2}{\epsilon}\langle F_{0120101}|\,, \\
\langle\varphi_{21}| &= -\frac{2m_t^2f(u,v,w)}{\epsilon}\langle F_{0110102}| - \frac{(4w+1)m_t^2}{2\epsilon^2}\langle F_{0010202}| + \frac{(8w-1)m_t^2}{\epsilon^2}\langle F_{0020201}| \\
&\quad - \frac{4m_t^2f(u,v,w)}{\epsilon}\langle F_{0120101}| - \frac{4vm_t^2}{\epsilon^2}\langle F_{0202010}| - \frac{1}{\epsilon^2}\langle F_{0202000}| \\
&\quad - \frac{2m_t^4h(u,v,w)}{\epsilon^2}\langle F_{0220101}| - \frac{6um_t^2}{\epsilon^2}\langle F_{0220001}|\,, \\
\langle\varphi_{22}| &= -\frac{4R_2m_t^2}{\epsilon}\langle F_{0110102}|\,,
\end{aligned}
$$

$$
\begin{aligned}
\langle\varphi_{23}| &= \frac{16vwm_t^4}{\epsilon^2}\langle F_{0220110}| + \frac{2vm_t^2}{\epsilon^2}\langle F_{0202010}| + \frac{2wm_t^2}{\epsilon^2}\langle F_{0220010}| + \frac{1}{4\epsilon^2}\langle F_{0202000}|\,,\\
\langle\varphi_{24}| &= \frac{2R_2m_t^2}{\epsilon}\langle F_{0110012}| + \frac{2R_2m_t^2}{\epsilon}\langle F_{0210011}|\,,\\
\langle\varphi_{25}| &= -\frac{2m_t^2f(u,w,v)}{\epsilon}\langle F_{0110012}| - \frac{(4v+1)m_t^2}{2\epsilon^2}\langle F_{0001202}| + \frac{(8v-1)m_t^2}{\epsilon^2}\langle F_{0002201}|\\
&\quad - \frac{4m_t^2f(u,w,v)}{\epsilon}\langle F_{0210011}| - \frac{4wm_t^2}{\epsilon^2}\langle F_{0220010}| - \frac{1}{\epsilon^2}\langle F_{0202000}|\\
&\quad - \frac{2m_t^4h(u,w,v)}{\epsilon^2}\langle F_{0220011}| - \frac{6um_t^2}{\epsilon^2}\langle F_{0220001}|\,,\\
\langle\varphi_{26}| &= -\frac{4R_2m_t^2}{\epsilon}\langle F_{0110012}|\,,\\
\langle\varphi_{27}| &= -\frac{2R_1m_t^2}{\epsilon^2}\langle F_{0210002}| - \frac{R_1m_t^2}{\epsilon^2}\langle F_{0220001}|\,,\\
\langle\varphi_{28}| &= \frac{um_t^2}{\epsilon^2}\langle F_{0220001}|\,,\\
\langle\varphi_{29}| &= \frac{4wm_t^2}{\epsilon^2}\langle F_{0220010}| + \frac{1}{2\epsilon^2}\langle F_{0202000}|\,,\\
\langle\varphi_{30}| &= \frac{16v^2m_t^4}{\epsilon^2}\langle F_{0202110}| + \frac{4vm_t^2}{\epsilon^2}\langle F_{0202010}| + \frac{1}{4\epsilon^2}\langle F_{0202000}|\,,\\
\langle\varphi_{31}| &= \frac{2vm_t^2}{\epsilon^2}\langle F_{0202010}| + \frac{1}{4\epsilon^2}\langle F_{0202000}|\,,\\
\langle\varphi_{32}| &= \frac{1}{\epsilon^2}\langle F_{0202000}|\,,\\
\langle\varphi_{33}| &= \frac{4R_2m_t^2}{\epsilon}\langle F_{0011201}|\,,\\
\langle\varphi_{34}| &= \frac{4R_1(2\epsilon-1)m_t^2}{\epsilon^2}\langle F_{0012101}| + \frac{4R_1(2\epsilon-1)m_t^2}{\epsilon^2}\langle F_{0021101}| + \frac{4R_1m_t^2}{\epsilon}\langle F_{0011201}|\,,\\
\langle\varphi_{35}| &= -\frac{(4w+1)m_t^2}{4\epsilon^2}\langle F_{0010202}| - \frac{(4w+1)m_t^2}{2\epsilon^2}\langle F_{0020201}|\,,\\
\langle\varphi_{36}| &= \frac{(4w+1)m_t^2}{4\epsilon^2}\langle F_{0010202}| + \frac{m_t^2}{2\epsilon^2}\langle F_{0020201}|\,,\\
\langle\varphi_{37}| &= \frac{(4v-1)m_t^2}{2\epsilon^2}\langle F_{0002201}| - \frac{(4v+1)m_t^2}{4\epsilon^2}\langle F_{0001202}|\,,\\
\langle\varphi_{38}| &= -\frac{(4v+1)m_t^2}{4\epsilon^2}\langle F_{0001202}| - \frac{(4v+1)m_t^2}{2\epsilon^2}\langle F_{0002201}|\,.
\end{aligned}
\tag{D.27}
$$

where

$$
f(u,v,w) = 1+2u+2v-2w\,, \qquad h(u,v,w) = 1+4v+4u(1+4v)-4w\,. \tag{D.28}
$$

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Alphabet of one-loop Feynman integrals *

Alphabet of one-loop Feynman integrals*

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Abstract: In this paper, we present the universal structure of the alphabet of one-loop Feynman integrals. The letters in the alphabet are calculated using the Baikov representation with cuts. We consider both convergent and divergent cut integrals and observe that letters in the divergent cases can be easily obtained from convergent cases by applying certain limits. The letters are written as simple expressions in terms of various Gram determinants. The knowledge of the alphabet enables us to easily construct the canonical differential equations of the $d\log$ form and aids in bootstrapping the symbols of the solutions.

Keywords: Baikov representation, Alphabet, UT basis

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I. INTRODUCTION

The systematic study of one-loop Feynman integrals in perturbative quantum field theories dates back to the end of the 1970s when 't Hooft and Veltman [1] calculated generic one-, two-, three-, and four-point scalar integrals in dimensional regularization (DREG) up to order ϵ^0, where $\epsilon=(4-d)/2$ with spacetime dimension d. Passarino and Veltman [2] then demonstrated that tensor integrals up to four points can be systematically reduced to scalar ones, and later studies [3, 4] demonstrated that integrals with more than four external legs in $4-2\epsilon$ dimensions can be expressed as lower-point ones up to order ϵ^0. These developments in principle solved the problem of next-to-leading order (NLO) calculations for tree-induced scattering processes.

The improvements of experimental precision and the progress of theoretical studies require the understanding of scattering amplitudes and cross sections at higher orders in perturbation theory. Hence, we must compute the one-loop integrals to higher orders in ϵ. These enable us to predict the infrared divergences appearing in two-loop amplitudes [5–13], and they are necessary for computing one-loop squared amplitudes, which are essential ingredients of next-to-next-to-leading order (NNLO) cross sections.

Unlike the terms up to order ϵ^0, generic results for the higher order terms are not available yet. Part of the reason is that integrals with more than four external legs are generally not reducible to lower-point ones when considering higher orders in ϵ. These require further calculations, which are often complicated owing to the increasing number of physical scales involved.

It is known [14–16] that one-loop integrals in a given family admit a uniform transcendentality (UT) basis satisfying canonical differential equations of the form [17]

$$d\vec{f}(\vec{x},\epsilon)=\epsilon\, d\boldsymbol{A}(\vec{x})\,\vec{f}(\vec{x},\epsilon)\,, \tag{1}$$

where $\vec{x}$ is the set of independent kinematic variables, and the matrix $d\boldsymbol{A}$ has the $d\log$-form:

$$d\boldsymbol{A}(\vec{x})=\sum_i \boldsymbol{C}_i\, d\log(W_i(\vec{x}))\,. \tag{2}$$

In the above expression, $\boldsymbol{C}_i$ are matrices consisting of rational numbers, and $W_i(\vec{x})$ are algebraic functions of the variables. The functions W_i are called the "letters" for this integral family, and the set of all independent letters is called the "alphabet."

At one loop, a canonical basis can be generically constructed by searching for $d\log$-form integrands [14–23].

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However, obtaining the $d\log$ matrix $dA(\vec{x})$ is not always a trivial task when the number of variables is large. We note that the $d\log$ matrix can be easily reconstructed if we have the knowledge of the alphabet $\{W_i(\vec{x})\}$ in advance, since the coefficient matrices $\boldsymbol{C}_i$ can then be obtained by bootstrapping.

Having the alphabet (and hence the matrix $dA(\vec{x})$) in a good form also aids in solving the differential equations (1) order-by-order in the dimensional regulator ϵ. The (suitably normalized) solution can be expressed as a Taylor series:

$$\vec{f}(\vec{x},\epsilon) = \sum_{n=0}^{\infty} \epsilon^n \vec{f}^{(n)}(\vec{x}), \tag{3}$$

where the nth-order coefficient function can be expressed as a Chen iterated integral [24]:

$$\vec{f}^{(n)}(\vec{x}) = \int_{\vec{x}_0}^{\vec{x}} dA(\vec{x}_n)\cdots\int_{\vec{x}_0}^{\vec{x}_2} dA(\vec{x}_1) + \vec{f}^{(n)}(\vec{x}_0). \tag{4}$$

Such iterated integrals can be analyzed using the language of "symbols" [25–27] that encodes the algebraic properties of the resulting functions. In certain scenarios, these iterated integrals can be solved analytically (either by direct integration or by bootstrapping). The results can often be expressed in terms of generalized polylogarithms (GPLs) [28], which enable efficient numeric evaluation [29–31]. When an analytic solution is not available, they can straightforwardly evaluated numerically through either numerical integration or series expansion [32, 33].

In this paper, we describe a generic method to construct the letters systematically from cut integrals in the Baikov representation [34, 35]. The letters can be generically expressed in terms of various Gram determinants. The letters and symbols of one-loop integrals were considered in [36–39], and our method is similar to that in [37–39]. Nevertheless, we evaluate the cut integrals differently and obtain equivalent but simpler expressions in certain cases utilizing the properties of Gram determinants. Furthermore, we consider the cases of divergent cut integrals, which were ignored in earlier studies. Using our results, all letters for a given integral family can be easily expressed even before constructing the differential equations. These letters will also appear in the corresponding two-loop integrals.

II. CANONICAL BASIS OF ONE-LOOP INTEGRALS

We use the method of [16, 23] to construct the canonical basis in the Baikov representation. In this section, we briefly review the construction procedure since it will also be relevant for obtaining the alphabet in the matrices $dA(\vec{x})$.

Consider a generic one-loop integral topology with $N=E+1$ external legs, where E is the number of independent external momenta. Integrals in this topology can be expressed as

$$I_{a_1,\cdots,a_N} = \int \frac{\mathrm{d}^d l}{\mathrm{i}\pi^{d/2}} \frac{1}{z_1^{a_1} z_2^{a_2}\cdots z_N^{a_N}}, \tag{5}$$

where z_i are the propagator denominators given by

$$z_1 = l^2 - m_1^2, \quad z_2 = (l+p_1)^2 - m_2^2, \quad \cdots,$$
$$z_N = (l+p_1+\cdots+p_E)^2 - m_N^2. \tag{6}$$

Here, $p_1,\ldots,p_E$ are external momenta, which we assume to span a space-like subspace of the d-dimensional Minkowski spacetime. This corresponds to the so-called (unphysical) Euclidean kinematics. Results in the physical phase-space region can be defined using analytic continuation.

The concept of the Baikov representation involves changing the integration variables from loop momenta l^μ to the Baikov variables z_i, and the result is given by

$$I_{a_1,\ldots,a_N} = \frac{1}{(4\pi)^{E/2}\Gamma((d-E)/2)} \times \int_C \frac{|G_N(z)|^{(d-E-2)/2}}{|\mathcal{K}_N|^{(d-E-1)/2}} \prod_{i=1}^{N} \frac{\mathrm{d}z_i}{z_i^{a_i}}, \tag{7}$$

where $z=\{z_1,\ldots,z_N\}$ is the collection of the Baikov variables. The function $G_N(z)$ is a polynomial of the N variables, while $\mathcal{K}_N$ is independent of z. They are given by

$$G_N(z) \equiv G(l,p_1,\ldots,p_E), \quad \mathcal{K}_N = G(p_1,\cdots,p_E), \tag{8}$$

where the Gram determinant is defined as

$$G(q_1,\ldots,q_n) \equiv \det\begin{pmatrix} q_1\cdot q_1 & q_1\cdot q_2 & \cdots & q_1\cdot q_n \\ q_2\cdot q_1 & q_2\cdot q_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ q_n\cdot q_1 & \cdots & \cdots & q_n\cdot q_n \end{pmatrix}. \tag{9}$$

Note that in Eq. (8), the scalar products involving the loop momentum l should be re-expressed in terms of z:

$$l^2 = z_1 + m_0^2,$$
$$l\cdot p_i = \frac{z_{i+1} + m_{i+1}^2 - p_i^2 - z_i - m_i^2}{2} - \sum_{j=1}^{i-1} p_i\cdot p_j. \tag{10}$$

The integration domain C in Eq. (7) is determined by the condition $G_N(z)/\mathcal{K}_N \le 0$ with Euclidean kinematics.

We are now ready to express the UT integrals g_N for any N according to [16]. We must distinguish between the cases of odd N and even N:

$$\begin{aligned} g_N\Big|_{N\text{-odd}} =&\frac{\epsilon^{(N+1)/2}}{(4\pi)^{(N-1)/2}\Gamma(1-\epsilon)} \\ &\times\int\left(-\frac{\mathcal{K}_N}{G_N(z)}\right)^{\epsilon}\prod_{i=1}^{N}\frac{\mathrm{d}z_i}{z_i}, \\ g_N\Big|_{N\text{-even}} =&\frac{\epsilon^{N/2}}{(4\pi)^{(N-1)/2}\Gamma(1/2-\epsilon)} \\ &\times\int\frac{\sqrt{G_N(\mathbf{0})}}{\sqrt{G_N(z)}}\left(-\frac{\mathcal{K}_N}{G_N(z)}\right)^{\epsilon}\prod_{i=1}^{N}\frac{\mathrm{d}z_i}{z_i}, \end{aligned} \tag{11}$$

where we set $\mathcal{K}_1=1$, and $\mathbf{0}$ means that all z_i's are zero. Note that g_{2n-1} and g_{2n} can be naturally identified as Feynman integrals in $2n-2\epsilon$ dimensions:

$$\begin{aligned} g_N\Big|_{N=2n-1} =&\epsilon^n\sqrt{\mathcal{K}_N}\,I_{1\times N}^{(2n-2\epsilon)}, \\ g_N\Big|_{N=2n} =&\epsilon^n\sqrt{G_N(\mathbf{0})}\,I_{1\times N}^{(2n-2\epsilon)}, \end{aligned} \tag{12}$$

where $I_{1\times N}^{(d)}$ denotes the d-dimensional N-point Feynman integral with all powers $a_i=1$:

$$I_{1\times N}^{(d)} \equiv \int\frac{\mathrm{d}^d l}{\mathrm{i}\pi^{d/2}}\frac{1}{z_1z_2\cdots z_N}. \tag{13}$$

They can be related to Feynman integrals in $4-2\epsilon$ dimensions using dimensional recurrence relations [40, 41]. Applying the above to all sectors of a family, we can build a complete canonical basis satisfying ϵ-form differential equations.

III. LETTERS IN DIFFERENTIAL EQUATIONS: CONVERGENT CASES

Given a basis of Feynman integrals, calculating the derivatives with respect to a kinematic variable x_i is straightforward. For a UT basis $\vec{f}(\vec{x},\epsilon)$, we write

$$\frac{\partial}{\partial x_i}\vec{f}(\vec{x},\epsilon)=\epsilon\,\boldsymbol{A}_i(\vec{x})\,\vec{f}(\vec{x},\epsilon), \tag{14}$$

where the elements in the matrix $\boldsymbol{A}_i(\vec{x})$ have the property that they contain only simple poles. In principle, we may already attempt to solve these differential equations using direct integration. However, this is often difficult when $\boldsymbol{A}_i(\vec{x})$ contains many irrational functions (square roots). Therefore, a very useful method is to combine the partial derivatives into a total derivative and rewrite the differential equations in the form of Eq. (1). Hence, we must know the alphabet (i.e., the set of independent letters $W_i(\vec{x})$) in the matrix $d\boldsymbol{A}(\vec{x})$. With the knowledge of the alphabet, we can easily reconstruct the entire matrix $d\boldsymbol{A}(\vec{x})$ by comparing the coefficients in the partial derivatives.

In principle, we may obtain the letters by directly integrating the matrices $\boldsymbol{A}_i(\vec{x})$ over the variables x_i and manipulating the resulting expressions. However, in the presence of many square roots (containing high-degree polynomials) in multi-scale problems, these integrations are not easy to perform, and the results are often extremely complicated. Examples are available for various one-loop and multi-loop calculations, e.g., Refs. [42–44]. With such types of expressions, it is highly non-trivial to decide whether a set of letters are independent. There is a package `SymBuild` [45] which can carry out such a task, but the computational burden is rather heavy when there are many square roots. Furthermore, from experience, we know that letters involving square roots can often be expressed in the form

$$\frac{P(\vec{x})-\sqrt{Q(\vec{x})}}{P(\vec{x})+\sqrt{Q(\vec{x})}}, \tag{15}$$

where P and Q are polynomials. Such letters have useful properties under analytic continuation: they are real when $Q(\vec{x})>0$ and become pure phases when $Q(\vec{x})<0$. However, recovering this structure from direct integration is difficult.

Given the above considerations, we now describe a novel method of obtaining the letters, particularly those with square roots and multiple scales. Our method is based on the $d\log$-form integrals in the Baikov representation under various cuts. We will utilize the generic propagator denominators in Eq. (II) and the Baikov representation (7). Without loss of generality, we define the Baikov cut on the first r variable $z_1,\ldots,z_r$ as [35]

$$\begin{aligned} I_{a_1,\ldots,a_N}\Big|_{r\text{-cut}} =&\frac{1}{(4\pi)^{E/2}\Gamma((d-E)/2)} \\ &\times\int\prod_{j=r+1}^{N}\frac{\mathrm{d}z_j}{z_j^{a_j}}\prod_{i=1}^{r}\oint_{z_i=0}\frac{\mathrm{d}z_i}{z_i^{a_i}}\frac{|G_N(z)|^{(d-E-2)/2}}{|\mathcal{K}_N|^{(d-E-1)/2}}. \end{aligned} \tag{16}$$

An important property of the Baikov cut is that if one of the powers a_i $(1\le i\le r)$ is non-positive, the cut integral vanishes according to the residue theorem. The coefficient matrices in the differential equations are invariant under the cuts, and we utilize this property to obtain the letters by imposing various cuts.

First, we express the differential equation satisfied by an N-point one-loop UT integral g_N (see Eqs. (11) and (12)) as

$$
\begin{aligned}
dg_N(\vec{x},\epsilon) =& \epsilon\, dM_N(\vec{x})\, g_N(\vec{x},\epsilon) \\
&+ \epsilon \sum_{m<N} \sum_i dM_{N,m}^{(i)}(\vec{x})\, g_m^{(i)}(\vec{x},\epsilon)\,,
\end{aligned}
\tag{17}
$$

where $g_N(\vec{x},\epsilon)$ and $g_m^{(i)}(\vec{x},\epsilon)$ are components of the canonical basis $\vec{f}(\vec{x},\epsilon)$, while $dM_N(\vec{x})$ and $dM_{N,m}^{(i)}(\vec{x})$ are entries in the matrix $d\boldsymbol{A}(\vec{x})$. The above equation clearly indicates that the derivative of g_N cannot depend on higher-point integrals as well as on other N-point integrals. It may depend on several m-point integrals for each $m<N$, and we use a superscript as in $g_m^{(i)}$ and $dM_{N,m}^{(i)}$ to distinguish them. These m-point integrals can be obtained by "squeezing" some of the propagators in the N-point diagram.

From Eq. (17), we observe that it is possible to focus on a particular entry of the $d\boldsymbol{A}$ matrix by imposing some cuts. We elaborate on this in the following. In this section, we assume that the master integrals (after imposing cuts) have no divergences such that the integrands can be expanded as Taylor series in ϵ before integration. We can show that in this scenario, only g_N, $g_{N-1}^{(i)}$, and $g_{N-2}^{(i)}$ appear on the right side of Eq. (17). We observe that the most complicated letters are given by these cases. Occasionally, we encounter divergences in the cut integrals, and we must expand the integrands as Laurent series in terms of distributions. We discuss these cases in the next section.

A. Self-dependence dM_N

The self-dependent term in Eq. (17) is easy to extract by imposing the "maximal-cut", i.e., cut on all variables z. All the lower-point integrals vanish under this cut, and the differential equation becomes

$$
d\tilde{g}_N(\vec{x},\epsilon) = \epsilon\, dM_N(\vec{x})\, \tilde{g}_N(\vec{x},\epsilon)\,,
\tag{18}
$$

where $\tilde{g}_N$ denotes the cut integral. Using the generic form of UT integrals in Eq. (11), we observe that

$$
dM_N(\vec{x}) = d\log\left(-\frac{\mathcal{K}_N(\vec{x})}{\widetilde{G}_N(\vec{x})}\right),
\tag{19}
$$

where

$$
\widetilde{G}_N(\vec{x}) \equiv G_N(\mathbf{0})\,.
\tag{20}
$$

Hence, the corresponding letter can be selected as

$$
W_N(\vec{x}) = \frac{\widetilde{G}_N(\vec{x})}{\mathcal{K}_N(\vec{x})}\,.
\tag{21}
$$

We note that two letters are equivalent if they only differ by a constant factor or constant power, i.e.,

$$
W(\vec{x}) \sim c\, W(\vec{x}) \sim [W(\vec{x})]^n\,.
\tag{22}
$$

Therefore, in practice, we may select a form that is convenient for the particular case at hand.

It is possible that $G_N(\mathbf{0})=0$ such that $W_N(\vec{x})=0$ and cannot be a letter. In this case, the integral $\tilde{g}_N$ itself vanishes under the maximal cut. This means that the integral is reducible to integrals in sub-sectors, and we do not require to consider it as a master integral.

B. Dependence on sub-sectors with one fewer propagator

We now consider the dependence of the derivative of g_N on sub-sectors with $N-1$ propagators. We may have N such sub-sectors, corresponding to "squeezing" one of the N propagators. Focusing on one sub-sector integral $g_{N-1}^{(i)}$, we can always reorganize the propagators (by shifting the loop momentum and relabel the external momenta) such that the squeezed one is z_N. We can then impose a cut on the first $N-1$ variables and express the differential equation as

$$
\begin{aligned}
d\tilde{g}_N(\vec{x},\epsilon) =& \epsilon\, dM_N(\vec{x})\, \tilde{g}_N(\vec{x},\epsilon) \\
&+ \epsilon\, dM_{N,N-1}(\vec{x})\, \tilde{g}_{N-1}(\vec{x},\epsilon)\,,
\end{aligned}
\tag{23}
$$

where we have suppressed the superscript since only one sub-sector survives the cut. The letter in $dM_N(\vec{x})$ has been obtained in the previous step, and we now must calculate the letter in $dM_{N,N-1}(\vec{x})$.

1. *Odd number of propagators*

We first consider the case in which N is an odd number. Using the generic form of one-loop UT integrals Eq. (11), we can write

$$
\begin{aligned}
& d\int_{r_-}^{r_+} \left(-\frac{\mathcal{K}_N}{G_N(\mathbf{0}',z_N)}\right)^{\epsilon} \frac{\mathrm{d}z_N}{z_N} \\
=& \epsilon\, dM_N \int_{r_-}^{r_+} \left(-\frac{\mathcal{K}_N}{G_N(\mathbf{0}',z_N)}\right)^{\epsilon} \frac{\mathrm{d}z_N}{z_N} \\
&+ dM_{N,N-1}\, \frac{2^{1-2\epsilon}\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left(-\frac{\mathcal{K}_{N-1}}{\widetilde{G}_{N-1}}\right)^{\epsilon},
\end{aligned}
\tag{24}
$$

where the integration boundary is determined by the two roots $r_\pm$ of the polynomial $G_N(\mathbf{0}',z_N)$, and $\mathbf{0}'$ means that the vector $z' \equiv \{z_1,\ldots,z_{N-1}\}$ is zero.

If both r_+ and r_- are non-zero, the integration over z_N is convergent for $\epsilon \to 0$. We can then set $\epsilon=0$ in the equation and obtain

$$dM_{N,N-1} = \frac{1}{2} d \int_{r_-}^{r_+} \frac{\mathrm{d}z_N}{z_N} = \frac{1}{2} d \log \frac{r_+}{r_-}. \tag{25}$$

We may already set the letter to r_+/r_- and stop at this point. However, expressing $r_\pm$ in terms of certain Gram determinants would be useful. This simplifies the procedure to compute the letter and informs us about the physics in the divergent scenarios $r_+ = 0$ or $r_- = 0$.

Given the propagator denominators (II) and the definition of the Gram determinant (9), we observe that z_N only appears in the top-right and bottom-left corners of the Gram matrix. Using the expansion of the determinant in terms of cofactors, we can write

$$G_N(\mathbf{0}', z_N) = -\frac{1}{4}\mathcal{K}_{N-1} z_N^2 - \widetilde{B}_N z_N + \widetilde{G}_N, \tag{26}$$

where $\widetilde{B}_N \equiv B_N(\mathbf{0})$ with (recall that $E = N-1$)

$$B_N(z) \equiv G(l, p_1, \ldots, p_{E-1}; p_E, p_1, \ldots, p_{E-1}), \tag{27}$$

Here, we have defined an extended Gram determinant

$$\begin{aligned} &G(q_1, \ldots, q_n; k_1, \ldots, k_n) \\ &= \det \begin{pmatrix} q_1 \cdot k_1 & q_1 \cdot k_2 & \cdots & q_1 \cdot k_n \\ q_2 \cdot k_1 & q_2 \cdot k_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ q_n \cdot k_1 & \cdots & \cdots & q_n \cdot k_n \end{pmatrix}. \end{aligned} \tag{28}$$

We may further use the geometric picture of Gram determinants to simplify the two roots. The Gram determinants can be expressed as

$$\begin{aligned} G(q_1, \ldots, q_n) &= \det\left(q_i^\mu q_j^\nu g_{\mu\nu}\right) \\ &= \det(g_{\mu\nu}) \left[V(q_1, \ldots, q_n)\right]^2, \end{aligned} \tag{29}$$

where q_i^μ is the μth component of q_i in the subspace spanned by $\{q_1, \ldots, q_n\}$ (with an arbitrary coordinate system), and $g_{\mu\nu}$ is the metric tensor of this subspace. $V(q_1, \ldots, q_n)$ is the volume of the parallelotope formed by the vectors $q_1, \ldots, q_n$ (in the Euclidean sense).

Let $l^\star$ denote a solution to the equation $z = 0$ (recall that z_i contains scalar products involving the loop momentum l); we can write

$$\begin{aligned} \widetilde{G}_{N-1} &= G(l^\star, p_1, \ldots, p_{E-1}), \quad \widetilde{G}_N = G(l^\star, p_1, \ldots, p_E), \\ \widetilde{B}_N &= G(l^\star, p_1, \ldots, p_{E-1}; p_E, p_1, \ldots, p_{E-1}). \end{aligned} \tag{30}$$

We let $l_\perp^\star$ and $p_{E\perp}$ denote the components of $l^\star$ and p_E perpendicular to the subspace spanned by $p_1, \ldots, p_{E-1}$, respectively. We are interested in the region in which the subspace of external momenta is space-like, and $l_\perp^\star$ must be time-like (since $l^\star$ is either time-like or light-like owing to $(l^\star)^2 - m_1^2 = 0$). We can express the components of $l_\perp^\star$ perpendicular and parallel to $p_{E\perp}$ as $|l_\perp^\star|\cosh(\eta)$ and $|l_\perp^\star|\sinh(\eta)$, respectively, where $|l_\perp^\star| \equiv \sqrt{(l_\perp^\star)^2}$. We also denote $|p_{E\perp}| \equiv \sqrt{-p_{E\perp}^2}$. These enables us to write

$$\begin{aligned} \frac{\widetilde{B}_N}{\mathcal{K}_{N-1}} &= -|l_\perp^\star||p_{E\perp}|\sinh(\eta), \quad \frac{\mathcal{K}_N}{\mathcal{K}_{N-1}} = -|p_{E\perp}|^2, \\ \frac{\widetilde{G}_N}{\mathcal{K}_{N-1}} &= -|l_\perp^\star|^2 |p_{E\perp}|^2 \cosh^2(\eta), \quad \frac{\widetilde{G}_{N-1}}{\mathcal{K}_{N-1}} = |l_\perp^\star|^2. \end{aligned} \tag{31}$$

Thus,

$$\widetilde{B}_N^2 + \mathcal{K}_{N-1}\widetilde{G}_N = -\mathcal{K}_{N-1}^2 |l_\perp^\star|^2 |p_{E\perp}|^2 = \mathcal{K}_N \widetilde{G}_{N-1}. \tag{32}$$

Note that the above relation can also be obtained from Sylvester's determinant identity applied to Gram determinants (for other applications of this relation, see, e.g., [16, 23, 46]). We encounter further instances of this relation later in this paper.

Expressing $r_\pm$ in terms of the Gram determinants, we can finally express the letter in $dM_{N,N-1}$ (for odd N) as

$$W_{N,N-1}(\vec{x}) = \frac{\widetilde{B}_N - \sqrt{\widetilde{G}_{N-1}\mathcal{K}_N}}{\widetilde{B}_N + \sqrt{\widetilde{G}_{N-1}\mathcal{K}_N}}. \tag{33}$$

We emphasize that the ingredients $\widetilde{B}_N$, $\widetilde{G}_{N-1}$, and $\mathcal{K}_N$ can be very complicated functions of the kinematic variables $\vec{x}$ when N and the length of $\vec{x}$ are large, and it is difficult to obtain the letter through direct integration in multi-scale problems.

If one of $r_\pm$ is zero, the integration over z_N is divergent when $\epsilon \to 0$, and we cannot expand the integrand as a Taylor series. Actually, we observe that $W_{N,N-1}(\vec{x})$ in Eq. (33) becomes zero in this scenario. However, this requires $\widetilde{G}_N = 0$, which means that g_N vanishes under the maximal cut and hence is not a master integral. It is also possible that $\widetilde{G}_{N-1} = 0$ and g_{N-1} is not a master. In this case, $\log W_{N,N-1} = \log(1) = 0$ drops out of the differential equations. Therefore, we do not require to consider these cases here. Similar considerations apply to the N-even case, described in the next section.

2. *Even number of propagators*

We now analyse the scenario in which N is an even number. We proceed similarly as the odd case, and arrive at the cut differential equation

$$d\int_{r_-}^{r_+}\frac{\mathrm{d}z_N}{z_N}\frac{\sqrt{\widetilde{G}_N}}{\sqrt{G_N(\mathbf{0}',z_N)}}\left[-\frac{\mathcal{K}_N}{G_N(\mathbf{0}',z_N)}\right]^{\epsilon}$$

$$=\epsilon\, dM_N\int_{r_-}^{r_+}\frac{\mathrm{d}z_N}{z_N}\frac{\sqrt{\widetilde{G}_N}}{\sqrt{G_N(\mathbf{0}',z_N)}}\left[-\frac{\mathcal{K}_N}{G_N(\mathbf{0}',z_N)}\right]^{\epsilon}$$
$$+2\pi\epsilon\frac{2^{2\epsilon}\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)}dM_{N,N-1}\left(-\frac{\mathcal{K}_{N-1}}{\widetilde{G}_{N-1}}\right)^{\epsilon}. \tag{34}$$

We again assume that the integration over z_N is convergent for $\epsilon\to 0$. We can then expand the integrands on both sides of the above equation. At order ϵ^0, the integral on the left side is

$$\int_{r_-}^{r_+}\frac{\mathrm{d}z_N}{z_N}\frac{\sqrt{\widetilde{G}_N}}{\sqrt{G_N(\mathbf{0}',z_N)}}=\mathrm{i}\pi. \tag{35}$$

Hence, its derivative is zero. Comparing the order ϵ^1 coefficients, and plugging in the form of dM_N obtained earlier in Eq. (19), we obtain

$$dM_{N,N-1}=-\frac{1}{2\pi}d\int_{r_-}^{r_+}\frac{\mathrm{d}z_N}{z_N}\frac{\sqrt{\widetilde{G}_N}}{\sqrt{G_N(\mathbf{0}',z_N)}}\log\frac{G_N(\mathbf{0}',z_N)}{\widetilde{G}_N}. \tag{36}$$

The above integrand involves multi-valued functions such as square roots and logarithms. To define the integral, we must select a convention including branch cuts for these functions and also the path from r_- to r_+. Different conventions will cause results to differ by some constants or an overall minus sign, but these do not affect the letter up to the equivalence mentioned in Eq. (22).

We denote $G_N(\mathbf{0}',z_N)$ as $(r_+-z_N)(z_N-r_-)\mathcal{K}_{N-1}/4$ with $\mathcal{K}_{N-1}>0$, and express the integral as

$$M_{N,N-1}=-\frac{1}{2\pi}\int_{r_-}^{r_+}\frac{\mathrm{d}z_N}{z_N}\sqrt{\frac{r_+r_-}{(z_N-r_+)(z_N-r_-)}}$$
$$\times\log\frac{(z_N-r_+)(z_N-r_-)}{r_+r_-}. \tag{37}$$

The branch cuts involve the points $r_\pm$ and ∞ on the complex z_N plane. To represent the cuts more clearly, we perform the change of variable:

$$z_N=\frac{1}{t},\quad t_\pm=\frac{1}{r_\mp}. \tag{38}$$

The branch points then become $t_\pm$ and 0, and we express the integral as

$$M_{N,N-1}=-\frac{1}{2\pi}\int_{t_-}^{t_+}I(t)\,\mathrm{d}t, \tag{39}$$

with the integrand

$$I(t)=\frac{1}{\sqrt{(t-t_+)(t-t_-)}}\left[\log\frac{t-t_+}{t}+\log\frac{t-t_-}{t}\right]. \tag{40}$$

With this form of the integrand, we select the branch cut for the square root to be the line segment between t_+ and t_-, and the branch cuts for the two logarithms to be the line segments between 0 and $t_\pm$, respectively. These branch cuts are depicted as the wiggly lines in Fig. 1, together with several paths $C_{i\pm}$ thatt lie infinitesimally close to the cuts. We define the square root following the convention that $\sqrt{(t-t_+)(t-t_-)}\to t$ when $t\to\infty$.

We select the integration path in Eq. (39) to along the line segment C_{1+}, and express the integral as

$$M_{N,N-1}=-\frac{1}{4\pi}\left[\int_{C_{1+}}I(t)\,\mathrm{d}t-\int_{C_{1-}}I(t)\,\mathrm{d}t\right], \tag{41}$$

where we have used the characteristic that the values of $I(t)$ on $C_{1\pm}$ differ by a sign. Since no other singularities exist in the complex t plane (including ∞), we may deform the paths as long as we do not go across the branch cuts. Hence, we know that

$$M_{N,N-1}=\frac{1}{4\pi}\left[\int_{C_{2+}}I(t)\,\mathrm{d}t-\int_{C_{2-}}I(t)\,\mathrm{d}t\right]$$
$$+\frac{1}{4\pi}\left[\int_{C_{3+}}I(t)\,\mathrm{d}t-\int_{C_{3-}}I(t)\,\mathrm{d}t\right]. \tag{42}$$

On the paths C_{2+} and C_{2-}, a $2\pi\mathrm{i}$ difference results from the first logarithm in Eq. (40). A similar difference of $-2\pi\mathrm{i}$ resulting from the second logarithm occurs

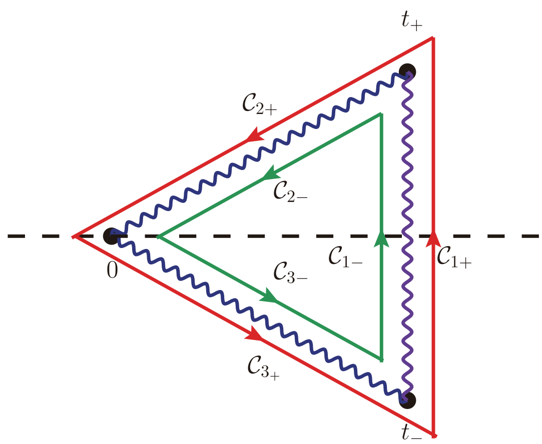

Fig. 1. (color online) Branch cuts and integration paths for $M_{N,N-1}$ with even N.

between C_{3+} and C_{3-}. Therefore, we obtain

$$\begin{aligned} dM_{N,N-1} =& -\frac{\mathrm{i}}{2}d\int_0^{t_+}\frac{\mathrm{d}t}{\sqrt{(t-t_+)(t-t_-)}} \\ & -\frac{\mathrm{i}}{2}d\int_0^{t_-}\frac{\mathrm{d}t}{\sqrt{(t-t_+)(t-t_-)}} \\ =& -\mathrm{i}\,d\log\frac{\sqrt{r_+}-\sqrt{r_-}}{\sqrt{r_+}+\sqrt{r_-}}. \end{aligned} \tag{43}$$

Note that with the above convention, we obtain

$$\begin{aligned} &\int_{r_-}^{r_+}\frac{\mathrm{d}z_N}{z_N}\frac{\sqrt{\widetilde{G}_N}}{\sqrt{G_N(\mathbf{0}',z_N)}} \\ &= \int_{t_-}^{t_+}\frac{\mathrm{d}t}{\sqrt{(t-t_+)(t-t_-)}} = \mathrm{i}\pi. \end{aligned} \tag{44}$$

We can now express the roots $r_\pm$ in terms of Gram determinants. The result can be expressed as

$$dM_{N,N-1} = \frac{\mathrm{i}}{2}d\log\frac{\widetilde{B}_N-\sqrt{-\widetilde{G}_N\mathcal{K}_{N-1}}}{\widetilde{B}_N+\sqrt{-\widetilde{G}_N\mathcal{K}_{N-1}}}, \tag{45}$$

where the definitions of $\widetilde{B}_N$, $\widetilde{G}_N$, and $\mathcal{K}_{N-1}$ are similar as before. Hence, we can express the letter in $dM_{N,N-1}$ (for even N) as

$$W_{N,N-1}(\vec{x}) = \frac{\widetilde{B}_N-\sqrt{-\widetilde{G}_N\mathcal{K}_{N-1}}}{\widetilde{B}_N+\sqrt{-\widetilde{G}_N\mathcal{K}_{N-1}}}. \tag{46}$$

As mentioned earlier, we do not require to consider the divergent case $\widetilde{G}_{N-1}=0$ or the trivial case $\widetilde{G}_N=0$ here.

C. Dependence on sub-sectors with two fewer propagators

As in the previous subsection, we consider the dependence of the derivative of g_N on sub-sectors with $N-2$ propagators. Without loss of generality, we cut on the variables $z'=\{z_1,\ldots,z_{N-2}\}$. Now, we remain with two sub-sectors with $N-1$ propagators: one with z',z_{N-1} and the other with z',z_N. We use a superscript to distinguish these two, and the differential equation then becomes

$$\begin{aligned} d\tilde{g}_N =& \epsilon\Big(dM_N\,\tilde{g}_N + dM^{(1)}_{N,N-1}\,\tilde{g}^{(1)}_{N-1} \\ & + dM^{(2)}_{N,N-1}\,\tilde{g}^{(2)}_{N-1} + dM_{N,N-2}\,\tilde{g}_{N-2}\Big), \end{aligned} \tag{47}$$

where we have suppressed the arguments of the functions for simplicity.

1. *Odd number of propagators*

If N is an odd number, assuming convergence and expanding the integrands, we obtain

$$\begin{aligned} d\int_C\frac{\mathrm{d}z_{N-1}}{z_{N-1}}\frac{\mathrm{d}z_N}{z_N} =& 4\pi\,dM_{N,N-2} \\ & + 2\,dM^{(1)}_{N,N-1}\int_{r_-^{(1)}}^{r_+^{(1)}}\frac{\mathrm{d}z_{N-1}}{z_{N-1}}\frac{\sqrt{\widetilde{G}^{(1)}_{N-1}}}{\sqrt{G^{(1)}_{N-1}(\mathbf{0}',z_{N-1})}} \\ & + 2\,dM^{(2)}_{N,N-1}\int_{r_-^{(2)}}^{r_+^{(2)}}\frac{\mathrm{d}z_N}{z_N}\frac{\sqrt{\widetilde{G}^{(2)}_{N-1}}}{\sqrt{G^{(2)}_{N-1}(\mathbf{0}',z_N)}}, \end{aligned} \tag{48}$$

where the domain C is determined by $G_N(\mathbf{0}',z_{N-1},z_N)\geq 0$, and $r^{(i)}_\pm$ are the two roots of the polynomial $G^{(i)}_{N-1}(\mathbf{0}',z)$.

The two integrals on the right-hand side can be easily performed using Eq. (35), and we obtain

$$dM_{N,N-2} = dI_{N,N-2} - \frac{\mathrm{i}}{2}\left(dM^{(1)}_{N,N-1}+dM^{(2)}_{N,N-1}\right), \tag{49}$$

where $I_{N,N-2}$ is the double integral:

$$I_{N,N-2} = \frac{1}{4\pi}\int_C\frac{\mathrm{d}z_{N-1}}{z_{N-1}}\frac{\mathrm{d}z_N}{z_N}. \tag{50}$$

The integration domain C is controlled by the positivity of the polynomial

$$\begin{aligned} G_N(\mathbf{0}',z_{N-1},z_N) =& -\frac{1}{4}\mathcal{K}_{N-1}z_N^2 - B_N(\mathbf{0}',z_{N-1},0)\,z_N \\ & + G_N(\mathbf{0}',z_{N-1},0). \end{aligned} \tag{51}$$

The integration over z_N can be easily performed to yield

$$\begin{aligned} I_{N,N-2} =& \frac{1}{4\pi}\int_{r_{N-1,-}}^{r_{N-1,+}} I(z_{N-1})\,\mathrm{d}z_{N-1} \\ \equiv& \frac{1}{4\pi}\int_{r_{N-1,-}}^{r_{N-1,+}}\frac{\mathrm{d}z_{N-1}}{z_{N-1}} \\ & \times\log\frac{B_N(\mathbf{0}',z_{N-1},0)-\sqrt{\Delta(z_{N-1})}}{B_N(\mathbf{0}',z_{N-1},0)+\sqrt{\Delta(z_{N-1})}}, \end{aligned} \tag{52}$$

where $r_{N-1,\pm}$ are the two roots of the polynomial

$$G^{(1)}_{N-1}(z',z_{N-1}) = G(l,p_1,\ldots,p_{E-1}), \tag{53}$$

and

$$
\begin{aligned}
\Delta(z_{N-1}) =& [B_N(\mathbf{0}', z_{N-1}, 0)]^2 + \mathcal{K}_{N-1} G_N(\mathbf{0}', z_{N-1}, 0) \\
=& \mathcal{K}_N G^{(1)}_{N-1}(\mathbf{0}', z_{N-1}) .
\end{aligned} \tag{54}
$$

We are now interested in the singularities of the integrand $I(z_{N-1})$ in Eq. (52). Two poles exist, at 0 and ∞, respectively. There is a branch cut between $r_{N-1,-}$ and $r_{N-1,+}$ for the square root. There is also a branch cut between $R_{N-1,-}$ and $R_{N-1,+}$ for the logarithm, where $R_{N-1,\pm}$ are the two roots of the polynomial $G_N(\mathbf{0}', z_{N-1}, 0)$. These singularities are depicted in Fig. 2. We define the integral path of Eq. (52) to be the upper half of the contour C_1. Hence, we obtain

$$
\begin{aligned}
I_{N,N-2} =& \frac{1}{8\pi} \int_{C_1} I(z_{N-1}) \mathrm{d}z_{N-1} \\
=& -\frac{1}{8\pi} \int_{C_2+C_3+C_4} I(z_{N-1}) \mathrm{d}z_{N-1} .
\end{aligned} \tag{55}
$$

The integration around C_3 is simply $(-2\pi\mathrm{i})$ multiplying the residue at $z_{N-1} = 0$, i.e.,

$$
\begin{aligned}
-\frac{1}{8\pi} d \int_{C_3} I(z_{N-1}) \mathrm{d}z_{N-1} =& \frac{\mathrm{i}}{4} d \log \frac{\widetilde{B}_N - \sqrt{\mathcal{K}_N \widetilde{G}^{(1)}_{N-1}}}{\widetilde{B}_N + \sqrt{\mathcal{K}_N \widetilde{G}^{(1)}_{N-1}}} \\
=& \frac{\mathrm{i}}{2} dM^{(1)}_{N,N-1} .
\end{aligned} \tag{56}
$$

On the two sides of C_2, the logarithm differs by $2\pi\mathrm{i}$, and

$$
\begin{aligned}
&-\frac{1}{8\pi} d \int_{C_2} I(z_{N-1}) \mathrm{d}z_{N-1} = \frac{\mathrm{i}}{4} d \log \frac{R_{N-1,+}}{R_{N-1,-}} \\
=& \frac{\mathrm{i}}{4} d \log \frac{\widetilde{B}_N - \sqrt{\mathcal{K}_N \widetilde{G}^{(2)}_{N-1}}}{\widetilde{B}_N + \sqrt{\mathcal{K}_N \widetilde{G}^{(2)}_{N-1}}} = \frac{\mathrm{i}}{2} dM^{(2)}_{N,N-1} .
\end{aligned} \tag{57}
$$

From the above, we observe that the genuine contribution to $dM_{N,N-2}$ results only from the integration along C_4. For that, we must investigate the behavior of the logarithm in Eq. (52) in the limit $z_{N-1} \to \infty$. We first note

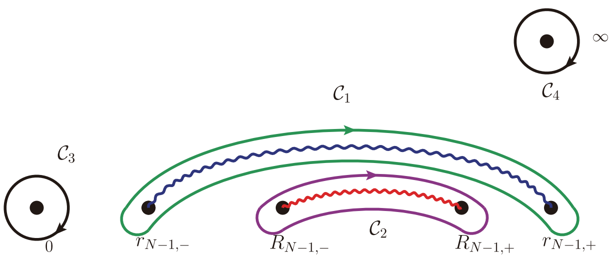

Fig. 2. (color online) Branch cuts and integration paths for $M_{N,N-2}$ with odd N.

that $G^{(1)}_{N-1}(\mathbf{0}', z_{N-1}) \sim -\mathcal{K}_{N-2} z^2_{N-1}/4$ in that limit. For $B_N(\mathbf{0}', z_{N-1}, 0)$, it is a linear function of z_{N-1}, and the coefficient can be extracted as

$$
\begin{aligned}
&\frac{\partial B_N(\mathbf{0}', z_{N-1}, 0)}{\partial z_{N-1}} \\
=& \frac{\partial G(l, p_1, \ldots, p_{E-1}; p_E, p_1, \ldots, p_{E-1})}{\partial z_{N-1}} \\
=& \frac{\partial l \cdot p_E}{\partial z_{N-1}} \frac{\partial G(l, p_1, \ldots, p_{E-1}; p_E, p_1, \ldots, p_{E-1})}{\partial l \cdot p_E} \\
&+ \frac{\partial l \cdot p_{E-1}}{\partial z_{N-1}} \frac{\partial G(l, p_1, \ldots, p_{E-1}; p_E, p_1, \ldots, p_{E-1})}{\partial l \cdot p_{E-1}} \\
=& \frac{1}{2} G(p_1, \ldots, p_{E-1}; p_1, \ldots, p_{E-1}) \\
&+ \frac{1}{2} G(p_1, \ldots, p_{E-2}, p_{E-1}; p_1, \ldots, p_{E-2}, p_E) \\
=& \frac{1}{2} G(p_1, \ldots, p_{E-2}, p_{E-1}; p_1, \ldots, p_{E-2}, p_{E-1} + p_E) .
\end{aligned} \tag{58}
$$

Hence, we obtain

$$
\begin{aligned}
dM_{N,N-2} =& -\frac{1}{8\pi} d \int_{C_4} I(z_{N-1}) \mathrm{d}z_{N-1} \\
=& \frac{\mathrm{i}}{4} d \log \frac{C_N - \sqrt{-\mathcal{K}_N \mathcal{K}_{N-2}}}{C_N + \sqrt{-\mathcal{K}_N \mathcal{K}_{N-2}}} ,
\end{aligned} \tag{59}
$$

where

$$
C_N = G(p_1, \ldots, p_{E-2}, p_{E-1}; p_1, \ldots, p_{E-2}, p_{E-1} + p_E) . \tag{60}
$$

The letter $W_{N,N-2}$ can be readily read off. Note that the Gram determinants in this letter only involve external momenta. Hence, the letter has a well-defined limit when $\widetilde{G}_{N-2} = 0$ and g_{N-2} is not a master. We explain the meaning of this later.

2. *Even number of propagators*

If N is an even number, assuming no divergence, we obtain the differential equation

$$
d \int_C \frac{\mathrm{d}z_{N-1}}{z_{N-1}} \frac{\mathrm{d}z_N}{z_N} \frac{\sqrt{\widetilde{G}_N}}{\sqrt{G_N(\mathbf{0}', z_{N-1}, z_N)}} = 4\pi \, dM_{N,N-2} , \tag{61}
$$

where the domain C is determined by $G_N(\mathbf{0}', z_{N-1}, z_N) \geq 0$. Note that the dependence on $g^{(i)}_{N-1}$ vanishes in this case. We select to integrate over z_N first and obtain

$$
dM_{N,N-2} = \frac{1}{4\pi} d \int_{r_{N-1,-}}^{r_{N-1,+}} \frac{\mathrm{d}z_{N-1}}{z_{N-1}} \frac{\sqrt{\widetilde{G}_N}}{\sqrt{G_N(\mathbf{0}', z_{N-1}, 0)}}
$$

$$\times\int_{r_{N,-}}^{r_{N,+}}\frac{\mathrm{d}z_N}{z_N}\frac{\sqrt{G_N(\mathbf{0}',z_{N-1},0)}}{\sqrt{G_N(\mathbf{0}',z_{N-1},z_N)}},\tag{62}$$

where $r_{N,\pm}$ are the two roots of the polynomial $G_N(\mathbf{0}',z_{N-1},z_N)$ with respect to z_N (treating z_{N-1} as a constant). Consequently, the integration range of z_{N-1} is determined by the discriminant Δ of $G_N(\mathbf{0}',z_{N-1},z_N)$ (with respect to the variable z_N). Expressing $\Delta=\mathcal{K}_4G^{(1)}_{N-1}\times(\mathbf{0}',z_{N-1})$, we know that the bounds $r_{N-1,\pm}$ are simply the two roots of the polynomial $G^{(1)}_{N-1}(\mathbf{0}',z_{N-1})$. Here, we define

$$\begin{aligned}G^{(1)}_{N-1}(z',z_{N-1})=&G(l,p_1,\ldots,p_{E-1}),\\ G^{(2)}_{N-1}(z',z_N)=&G(l,p_1,\ldots,p_{E-1}+p_E).\end{aligned}\tag{63}$$

The integration over z_N can be performed using Eq. (35). We then obtain

$$dM_{N,N-2}=\frac{\mathrm{i}}{4}dI_{N,N-2},\tag{64}$$

where

$$I_{N,N-2}=\int_{r_{N-1,-}}^{r_{N-1,+}}\frac{\mathrm{d}z_{N-1}}{z_{N-1}}\frac{\sqrt{\widetilde{G}_N}}{\sqrt{G_N(\mathbf{0}',z_{N-1},0)}},\tag{65}$$

where $r_{N-1,\pm}$ are the two roots of $G^{(1)}_{N-1}(\mathbf{0}',z_{N-1})$. We denote the two roots of $G_N(\mathbf{0}',z_{N-1},0)$ as $R_{N-1,\pm}$. We can then write

$$G_N(\mathbf{0}',z_{N-1},0)=-\frac{1}{4}\mathcal{K}^{(2)}_{N-1}(z_{N-1}-R_{N-1,+})(z_{N-1}-R_{N-1,-}),\tag{66}$$

where

$$\mathcal{K}^{(2)}_{N-1}=G(p_1,\ldots,p_{E-2},p_{E-1}+p_E).\tag{67}$$

We define

$$t=\frac{1}{z_{N-1}},\quad t_\pm=\frac{1}{r_{N-1,\mp}},\quad T_\pm=\frac{1}{R_{N-1,\mp}}.\tag{68}$$

The integral can then be expressed as

$$\begin{aligned}I_{N,N-2}=&\int_{t_-}^{t_+}\frac{\mathrm{d}t}{\sqrt{(t-T_+)(t-T_-)}}\\ =&2\log\frac{\sqrt{t_+-T_+}+\sqrt{t_+-T_-}}{\sqrt{tT_+}+\sqrt{tT_-}}.\end{aligned}\tag{69}$$

We now aim to rewrite the above expression in terms of Gram determinants. Hence, we first write

$$\begin{aligned}G_N(\mathbf{0}',z_{N-1},0)=&-\frac{1}{4}\mathcal{K}^{(2)}_{N-1}z^2_{N-1}-\widetilde{B}_Nz_{N-1}+\widetilde{G}_N,\\ G^{(1)}_{N-1}(\mathbf{0}',z_{N-1})=&-\frac{1}{4}\mathcal{K}_{N-2}z^2_{N-1}-\widetilde{B}^{(1)}_{N-1}z_{N-1}+\widetilde{G}^{(1)}_{N-1},\end{aligned}\tag{70}$$

where

$$\begin{aligned}\mathcal{K}_{N-2}=&G(p_1,\ldots,p_{E-2}),\\ B^{(1)}_{N-1}(z)=&G(l,p_1,\ldots,p_{E-2};p_{E-1},p_1,\ldots,p_{E-2}).\end{aligned}\tag{71}$$

The roots are given by

$$\begin{aligned}t_\pm=&\frac{\widetilde{B}^{(1)}_{N-1}\pm\sqrt{\mathcal{K}^{(1)}_{N-1}\widetilde{G}_{N-2}}}{2\widetilde{G}^{(1)}_{N-1}},\\ T_\pm=&\frac{\widetilde{B}_N\pm\sqrt{\mathcal{K}_N\widetilde{G}^{(2)}_{N-1}}}{2\widetilde{G}_N},\end{aligned}\tag{72}$$

where

$$G^{(2)}_{N-1}=G(l,p_1,\ldots,p_{E-2},p_{E-1}+p_E),\tag{73}$$

and we have used the relations

$$\begin{aligned}B_N^2+\mathcal{K}^{(2)}_{N-1}G_N=&\mathcal{K}_NG^{(2)}_{N-1},\\ \left(B^{(1)}_{N-1}\right)^2+\mathcal{K}_NG^{(1)}_{N-1}=&\mathcal{K}^{(1)}_{N-1}G_{N-2}.\end{aligned}\tag{74}$$

We can now employ the geometric representations of the Gram determinants in Eq. (31) to simplify the expressions. Let $l^\star$ be the solution to $z=0$; we are interested in the components of $l^\star$, p_{E-1} and $p_{E-1}+p_E$ orthogonal to the subspace spanned by $\{p_1,\ldots,p_{E-2}\}$. For convenience, we denote these components as k^μ (for $l^\star$), p^μ (for p_{E-1}) and q^μ (for $p_{E-1}+p_E$). We note that k^μ is time-like, while p^μ and q^μ are space-like. Hence, we can define the norms $|k|=\sqrt{k^2}$, $|p|=\sqrt{-p^2}$, and $|q|=\sqrt{-q^2}$. We further denote the components of k^μ and p^μ perpendicular to q as $k^\mu_\perp$ and $p^\mu_\perp$, respectively, and define the corresponding norms as $|k_\perp|$ and $|p_\perp|$. We can finally write

$$t_\pm=\frac{\sinh(\eta_1)\pm\mathrm{i}}{2|k||p|\cosh^2(\eta_1)},\quad T_\pm=\frac{\sinh(\eta_2)\pm\mathrm{i}}{2|k_\perp||p_\perp|\cosh^2(\eta_2)},\tag{75}$$

where η_1 is the hyperbolic angle between k and p, and η_2 is the hyperbolic angle between $k_\perp$ and $p_\perp$. It will be convenient to define the imaginary angle $\theta_{kp}\equiv\pi/2-\mathrm{i}\eta_1$ such that $\cosh(\eta_1)=\sin\theta_{kp}$ and $\mathrm{i}\sinh(\eta_1)=\cos\theta_{kp}$; similarly,

$\theta_{kp,\perp q} \equiv \pi/2 - \mathrm{i}\eta_2$.

We use θ_{pq} to denote the angle between p and q and define ξ as the hyperbolic angle between k and q (with the corresponding imaginary angle $\theta_{kq} \equiv \pi/2 - \mathrm{i}\xi$). We then obtain the relations

$$\begin{aligned} |p_\perp| &= |p|\sin\theta_{pq}, \quad |k_\perp| = |k|\sin\theta_{kq}, \\ \cos\theta_{kp} &= \cos\theta_{kq}\cos\theta_{pq} + \cos\theta_{kp,\perp q}\sin\theta_{kq}\sin\theta_{pq}. \end{aligned} \tag{76}$$

Thus,

$$t_\pm - T_\pm \equiv \frac{P_{\pm\pm}}{2|k_\perp||p_\perp|\sin^2\theta_{kp}\sin^2\theta_{kp,\perp q}}, \tag{77}$$

where

$$\begin{aligned} P_{\pm\pm} =& (-\mathrm{i}\cos\theta_{kp} \pm \mathrm{i})\sin^2\theta_{kp,\perp q}\sin\theta_{pq}\sin\theta_{kq} \\ &- (-\mathrm{i}\cos\theta_{kp,\perp q} \pm \mathrm{i})\sin^2\theta_{kp}. \end{aligned} \tag{78}$$

Substituting the relation (76), we may express the functions $P_{\pm\pm}$ as

$$\begin{aligned} P_{++} &= -8\mathrm{i}\sin^2\left(\frac{\theta_{kp}}{2}\right)\cos^2\left(\frac{\theta_{kq}+\theta_{pq}}{2}\right)\sin^2\left(\frac{\theta_{kp,\perp q}}{2}\right), \\ P_{+-} &= 8\mathrm{i}\sin^2\left(\frac{\theta_{kp}}{2}\right)\cos^2\left(\frac{\theta_{kq}-\theta_{pq}}{2}\right)\cos^2\left(\frac{\theta_{kp,\perp q}}{2}\right), \\ P_{-+} &= -8\mathrm{i}\cos^2\left(\frac{\theta_{kp}}{2}\right)\sin^2\left(\frac{\theta_{kq}+\theta_{pq}}{2}\right)\sin^2\left(\frac{\theta_{kp,\perp q}}{2}\right), \\ P_{--} &= 8\mathrm{i}\cos^2\left(\frac{\theta_{kp}}{2}\right)\sin^2\left(\frac{\theta_{kq}-\theta_{pq}}{2}\right)\cos^2\left(\frac{\theta_{kp,\perp q}}{2}\right). \end{aligned} \tag{79}$$

Using trigonometry identities together with the relations

$$\begin{aligned} \cos\theta_{pq} &= \cos\theta_{kp}\cos\theta_{kq} + \cos\theta_{pq,\perp k}\sin\theta_{kp}\sin\theta_{kq}, \\ \sin\theta_{pq} &= \sin\theta_{pq,\perp k}\frac{\sin\theta_{kp}}{\sin\theta_{kp,\perp q}}, \end{aligned} \tag{80}$$

we obtain a surprisingly simple result:

$$I_{N,N-2} = 2\log \mathrm{e}^{-\mathrm{i}\theta_{pq,\perp k}} = \log\frac{\cos\theta_{pq,\perp k} - \mathrm{i}\sin\theta_{pq,\perp k}}{\cos\theta_{pq,\perp k} + \mathrm{i}\sin\theta_{pq,\perp k}}, \tag{81}$$

where $\theta_{pq,\perp k}$ is the angle between $p_{\perp k}$ and $q_{\perp k}$. It is straightforward to rewrite the above expression in terms of Gram determinants, and we finally obtain

$$dM_{N,N-2} = \frac{\mathrm{i}}{4} d\log\frac{\widetilde{D}_N - \sqrt{-\widetilde{G}_N\widetilde{G}_{N-2}}}{\widetilde{D}_N + \sqrt{-\widetilde{G}_N\widetilde{G}_{N-2}}}, \tag{82}$$

where $\widetilde{D}_N = D_N(\mathbf{0})$ and

$$D_N(z) = G(l, p_1, \ldots, p_{E-1}; l, p_1, \ldots, p_{E-1} + p_E). \tag{83}$$

D. Dependence on further lower sub-sectors

In the convergent case, dg_N cannot depend on g_{N-3} or integrals with even fewer propagators. For odd N, this can be easily observed from the powers of ϵ in Eq. (11). However, for even N, dg_N and g_{N-3} are multiplied by the same power of ϵ in the differential equations. Subsequently, we must examine the three-fold integrals appearing in the differential equations under the $(N-3)$-cut. The first two folds can be performed using the calculations in Section III.C.2, and the last fold can be studied similar to those in Section III.C.1. Finally, we can arrive at the conclusion that $dM_{N,N-3} = 0$ in the convergent case. However, note that such dependence can be present in the divergence cases, as discussed in the next section.

IV. LETTERS IN DIFFERENTIAL EQUATIONS: DIVERGENT CASES

We now consider the scenario in which some cut integrals become divergent and we cannot perform a Taylor expansion for the integrands. As discussed earlier, this occurs when certain Gram determinants vanish under the maximal cut, and the corresponding integrals are reducible to lower sectors. A classical example is the massless three-point integral that can be reduced to two-point integrals. Reducible higher-point integrals can occur with specific configurations of external momenta, which appear, e.g., at boundaries of differential equations or in some effective field theories. Divergent cut integrals can have two types of consequences, which we discuss in the following.

A. $N, N-2$ dependence with a reducible $(N-1)$-point integral

We consider the dependence of dg_N on g_{N-2} when $g_{N-1}^{(1)}$ is reducible, where N is even. Following the derivation in Section III.C.2, we observe that now one of $r_{N-1,\pm}$ is zero and $G_{N-1}^{(1)}(\mathbf{0}', 0) = 0$. Hence, integration over z_{N-1} is divergent and we cannot perform Taylor expansion of the integrand in ϵ. Moreover, we observe that the entry $dM_{N,N-2}$ obtained in Section III.C.2 is divergent. To proceed, we can maintain the regulator in the differential equation:

$$\begin{aligned} & d\int_C \frac{\mathrm{d}z_{N-1}}{z_{N-1}}\frac{\mathrm{d}z_N}{z_N}\frac{\sqrt{\widetilde{G}_N}}{\sqrt{G_N(\mathbf{0}', z_{N-1}, z_N)}}\left[-\frac{\mathcal{K}_N}{G_N(\mathbf{0}', z_{N-1}, z_N)}\right]^\epsilon \\ =& \epsilon\, dM_N \int_C \frac{\mathrm{d}z_{N-1}}{z_{N-1}}\frac{\mathrm{d}z_N}{z_N} \end{aligned}$$

$$\times\frac{\sqrt{\widetilde{G}_N}}{\sqrt{G_N(\mathbf{0}',z_{N-1},z_N)}}\left[-\frac{\mathcal{K}_N}{G_N(\mathbf{0}',z_{N-1},z_N)}\right]^{\epsilon}+4\pi\, dM^{\star}_{N,N-2}\left(-\frac{\mathcal{K}_{N-2}}{\widetilde{G}_{N-2}}\right)^{\epsilon}+O(\epsilon), \tag{84}$$

where $dM^{\star}_{N,N-2}$ denotes the entry in the divergent case. Note that $g^{(1)}_{N-1}$ is not a master integral and does not contribute to the right-hand side, while the last $O(\epsilon)$ denotes a suppressed contribution from another $(N-1)$-point integral $g^{(2)}_{N-1}$. Here, we assume that $G^{(2)}_{N-1}(\mathbf{0}',0)$ is non-zero and the integration over z_N is convergent for $\epsilon\to 0$.

We now must perform Laurent expansions of the integrands in terms of distributions. We write

$$G^{(1)}_{N-1}(\mathbf{0}',z_{N-1})=\frac{1}{4}\mathcal{K}_{N-2}z_{N-1}(t-z_{N-1}),\qquad t=-\frac{4\widetilde{B}^{(1)}_{N-1}}{\mathcal{K}_{N-2}}. \tag{85}$$

We can then use

$$\int_0^t\frac{\mathrm{d}z}{z^{1+\epsilon}}f(z)=-\frac{t^{-\epsilon}}{\epsilon}f(0)+\int_0^t\frac{\mathrm{d}z}{z^{1+\epsilon}}[f(z)-f(0)], \tag{86}$$

to perform the series expansion. In particular, we obtain

$$\begin{aligned}&\int_C\frac{\mathrm{d}z_{N-1}}{z_{N-1}}\frac{\mathrm{d}z_N}{z_N}\frac{\sqrt{\widetilde{G}_N}}{\sqrt{G_N(\mathbf{0}',z_{N-1},z_N)}}\left[-\frac{\mathcal{K}_N}{G_N(\mathbf{0}',z_{N-1},z_N)}\right]^{\epsilon}\\&=\mathrm{i}\pi\int_0^t\frac{\mathrm{d}z_{N-1}}{z_{N-1}^{1+\epsilon}}\frac{\sqrt{\widetilde{G}_N}}{\sqrt{G_N(\mathbf{0}',z_{N-1},0)}}\times\left[1+\epsilon h(z_{N-1})+O(\epsilon^2)\right]\\&=\mathrm{i}\pi\left[-\frac{1}{\epsilon}+\log(t)-h(0)+\int_0^t\frac{\mathrm{d}z_{N-1}}{z_{N-1}}\left(\frac{\sqrt{\widetilde{G}_N}}{\sqrt{G_N(\mathbf{0}',z_{N-1},0)}}-1\right)\right]+O(\epsilon),\end{aligned} \tag{87}$$

where the function $h(z_{N-1})$ results from the expansion in ϵ after integrating over z_N. When $z_{N-1}\to 0$, it reduces to

$$h(0)=\log\left(\frac{4\mathcal{K}^{(1)}_{N-1}}{\widetilde{B}^{(1)}_{N-1}}\right)+4\log(2). \tag{88}$$

The last integral in Eq. (87) can be obtained by obtaining the limit $\widetilde{G}^{(1)}_{N-1}\to 0$ in the difference between Eq. (82) and a simple integral of $1/z_{N-1}$:

$$\begin{aligned}&\int_0^t\frac{\mathrm{d}z_{N-1}}{z_{N-1}}\left(\frac{\sqrt{\widetilde{G}_N}}{\sqrt{G_N(\mathbf{0}',z_{N-1},0)}}-1\right)\\&=\lim_{\widetilde{G}^{(1)}_{N-1}\to 0}\left(\log\frac{\widetilde{D}_N-\sqrt{-\widetilde{G}_N\widetilde{G}_{N-2}}}{\widetilde{D}_N+\sqrt{-\widetilde{G}_N\widetilde{G}_{N-2}}}-\log\frac{\widetilde{B}^{(1)}_{N-1}+\sqrt{\widetilde{G}_{N-2}\mathcal{K}^{(1)}_{N-1}}}{\widetilde{B}^{(1)}_{N-1}-\sqrt{\widetilde{G}_{N-2}\mathcal{K}^{(1)}_{N-1}}}\right).\end{aligned} \tag{89}$$

Using the relations

$$\begin{aligned}-G_NG_{N-2}&=D_N^2-G^{(1)}_{N-1}G^{(2)}_{N-1},\\ G_{N-2}\mathcal{K}^{(1)}_{N-1}&=\left(B^{(1)}_{N-1}\right)^2+G^{(1)}_{N-1}\mathcal{K}_{N-2},\end{aligned} \tag{90}$$

we can simplify the expression and obtain

$$\int_0^t\frac{\mathrm{d}z_{N-1}}{z_{N-1}}\left(\frac{\sqrt{\widetilde{G}_N}}{\sqrt{G_N(\mathbf{0}',z_{N-1},0)}}-1\right)=\log\frac{\widetilde{G}_N\mathcal{K}_{N-2}}{\mathcal{K}^{(1)}_{N-1}\widetilde{G}^{(2)}_{N-1}}. \tag{91}$$

Now, we can combine everything, and we observe that in the divergent case (for even N),

$$W^{\star}_{N,N-2}=\frac{\widetilde{G}_{N-2}\mathcal{K}_N}{\mathcal{K}^{(1)}_{N-1}\widetilde{G}^{(2)}_{N-1}}. \tag{92}$$

Comparing to Eq. (92), we note that the letter in the divergent case is simpler (without square roots) than that in the convergent case. Interestingly, this simple letter can be obtained without using the tedious calculation above. We observe that in the divergent case $\widetilde{G}^{(1)}_{N-1}\to 0$, we have the relation

$$\tilde{g}^{(1)}_{N-1}=-\frac{1}{2}\tilde{g}_{N-2}. \tag{93}$$

This hints that we should combine $dM^{(1)}_{N,N-1}$ and $dM_{N,N-2}$ to obtain $dM^{\star}_{N,N-2}$:

$$\begin{aligned}dM^{\star}_{N,N-2}&=\lim_{\widetilde{G}^{(1)}_{N-1}\to 0}\left(-\frac{1}{2}dM^{(1)}_{N,N-1}+dM_{N,N-2}\right)\\&=\frac{\mathrm{i}}{4}\lim_{\widetilde{G}^{(1)}_{N-1}\to 0}\left(\log\frac{\widetilde{D}_N-\sqrt{-\widetilde{G}_N\widetilde{G}_{N-2}}}{\widetilde{D}_N+\sqrt{-\widetilde{G}_N\widetilde{G}_{N-2}}}-\log\frac{\widetilde{B}^{(1)}_N-\sqrt{-\widetilde{G}_N\mathcal{K}^{(1)}_{N-1}}}{\widetilde{B}^{(1)}_N+\sqrt{-\widetilde{G}_N\mathcal{K}^{(N)}_3}}\right).\end{aligned} \tag{94}$$

Using the relations in Eq. (90) as well as

$$-G_N\mathcal{K}_{N-1}^{(1)}=\left(B_N^{(1)}\right)^2+G_{N-1}^{(1)}\mathcal{K}_N\,, \tag{95}$$

we can easily arrive at Eq. (92).

Further divergences may occur if $\widetilde{G}_{N-1}^{(2)}=0$ in Eq. (92). In this case, both $g_{N-1}^{(1)}$ and $g_{N-1}^{(2)}$ are reducible to lower-point integrals. The corresponding letter can be obtained by including $dM_{N,N-1}^{(2)}$, but we do not elaborate on the calculation here. We finally note that the above considerations can also be applied to the N-odd cases, although here $g_{N-1}^{(i)}$ can only be reducible for specific configurations of external momenta. We discuss similar scenarios in the next subsection.

B. $N,N-3$ dependence with a reducible $(N-2)$-point integral

In the convergent case, we have observed that dg_N can only depend on g_N, $g_{N-1}^{(i)}$, and $g_{N-2}^{(i)}$. This picture changes in the divergent case when one of $g_{N-2}^{(i)}$ is reducible, and dg_N may develop dependence on some $(N-3)$-point integrals. As a practical example, we consider the dependence of 5-point integrals on 2-point ones. According to Eq. (17), we obtain

$$\begin{aligned}d\tilde{g}_5=&\epsilon dM_5\,\tilde{g}_5+\epsilon\sum_i dM_{5,4}^{(i)}\,\tilde{g}_4^{(i)}+\epsilon\sum_i dM_{5,3}^{(i)}\,\tilde{g}_3^{(i)}\\&+\epsilon\, dM_{5,2}\,\tilde{g}_2\,,\end{aligned} \tag{96}$$

where the cut on z_1 and z_2 is imposed. Using Eq. (11), we arrive at

$$\begin{aligned}dM_{5,2}+O(\epsilon)=&\frac{\epsilon}{8\pi}dI_{5,2}(\epsilon)-\frac{\epsilon}{4\pi}\sum_{i=3}^{5}dM_{5,4}^{(i)}I_{4,2}^{(i)}(\epsilon)\\&-\frac{\epsilon}{2}\sum_{i=3}^{5}dM_{5,3}^{(i)}I_{3,2}^{(i)}(\epsilon)\,,\end{aligned} \tag{97}$$

where

$$\begin{aligned}I_{5,2}(\epsilon)&=\int\frac{dz_3}{z_3}\frac{dz_4}{z_4}\frac{dz_5}{z_5}\left(-\frac{\mathcal{K}_5}{G_5(0,0,z_3,z_4,z_5)}\right)^{\epsilon},\\I_{4,2}^{(i)}(\epsilon)&=\int\frac{dz_j}{z_j}\frac{dz_k}{z_k}\frac{\sqrt{G_4^{(i)}(0,0,0,0)}}{\sqrt{G_4^{(i)}(0,0,z_j,z_k)}}\\&\quad\times\left(-\frac{\mathcal{K}_4^{(i)}}{G_4^{(i)}(0,0,z_j,z_k)}\right)^{\epsilon},\\I_{3,2}^{(i)}(\epsilon)&=\int\frac{dz_i}{z_i}\left(-\frac{\mathcal{K}_3^{(i)}}{G_3^{(i)}(0,0,z_i)}\right)^{\epsilon},\end{aligned} \tag{98}$$

where $j<k$ and $j,k\neq i$. We note that each term on the right-hand side of Eq. (97) has a factor of ϵ. Therefore, the term can only contribute if the integral is divergent in the limit $\epsilon\to 0$. For that to occur, at least one of $G_3^{(i)}(0,0,0)$ must vanish. For simplicity, we assume $G_3^{(3)}(0,0,0)=0$, while the other two $G_3^{(i)}(0,0,0)$'s are non-zero. Generally, it is clear that the $I_{3,2}^{(i)}(\epsilon)$ terms do not contribute since they are either zero or non-divergent. The integrals $I_{4,2}^{(4)}(\epsilon)$ and $I_{4,2}^{(5)}(\epsilon)$ are similar to Eq. (87) with the result $-\mathrm{i}\pi/\epsilon+O(\epsilon^0)$. Therefore, we only require to address the divergent part of $I_{5,2}(\epsilon)$:

$$\begin{aligned}I_{5,2}(\epsilon)=&\int\frac{dz_3}{z_3}\frac{dz_4}{z_4}\,[\Delta(z_3,z_4)]^{-\epsilon}\\&\times\log\frac{B_5^{(5)}(0,0,z_3,z_4,0)-\sqrt{\Delta(z_3,z_4)}}{B_5^{(5)}(0,0,z_3,z_4,0)+\sqrt{\Delta(z_3,z_4)}}+O(\epsilon^0)\,,\end{aligned} \tag{99}$$

where

$$\Delta(z_3,z_4)=\mathcal{K}_5G_4^{(5)}(0,0,z_3,z_4)\,. \tag{100}$$

The integration over z_4 is similar to Eq. (52), except for the additional factor $\Delta^{-\epsilon}$, which regularizes the divergence as $z_3\to 0$. Since we are only interested in the leading term in ϵ, it is equivalent to replacing this factor by $z_3^{-\epsilon}$. We can then expand $z_3^{-1-\epsilon}$ in terms of distributions. Maintaining only the $1/\epsilon$ terms, we obtain

$$\begin{aligned}&dI_{5,2}(\epsilon)+O(\epsilon^0)\\&=-\frac{1}{\epsilon}d\int\frac{dz_4}{z_4}\log\frac{B_5^{(5)}(0,0,0,z_4,0)-\sqrt{\Delta(0,z_4)}}{B_5^{(5)}(0,0,0,z_4,0)+\sqrt{\Delta(0,z_4)}}\\&=-\frac{1}{\epsilon}\left(2\pi\mathrm{i}\,dM_{5,4}^{(4)}+2\pi\mathrm{i}\,dM_{5,4}^{(5)}+4\pi\,dM_{5,3}^{(3)}\right),\end{aligned} \tag{101}$$

where the second line follows from the calculation of Eq. (52). We finally arrive at

$$dM_{5,2}=-\frac{1}{2}dM_{5,3}^{(3)}=-\frac{\mathrm{i}}{8}d\log\frac{C_5-\sqrt{-\mathcal{K}_5\mathcal{K}_3}}{C_5+\sqrt{-\mathcal{K}_5\mathcal{K}_3}}, \tag{102}$$

where

$$C_5=G(p_1,p_2,p_3,p_4;\,p_1,p_2,p_3,p_4+p_5)\,. \tag{103}$$

The result in Eq. (102) is unsurprising owing to the relation $g_3^{(3)}=-g_2/2$. Similar behaviors are observed when more than one $\widetilde{G}_3$ vanish. The corresponding $dM_{5,2}$ is then a linear combination of several $dM_{5,3}$'s. Hence, we conclude that letters in these cases can also be obtained straightforwardly without tedious calculations.

The above discussion relates the appearance of $dM_{N,N-3}$ to the reducibility of one or more $g^{(i)}_{N-2}$'s. We may consider that, if in addition, one or more $g^{(i)}_{N-3}$'s becomes reducible, $dM_{N,N-4}$ can appear in the differential equations. This is impossible for integrals with generic external momenta (i.e., the E external momenta are indeed independent). However, such cases may occur at certain boundaries of kinematic configurations. When this occurs, the corresponding letters can be easily obtained using the reduction rules among the integrals, as is conducted in the previous paragraph.

V. SUMMARY AND OUTLOOK

In summary, we have studied the alphabet for one-loop Feynman integrals. The alphabet governs the form of the canonical differential equations and provides important information on the analytic solution of these equations. We observe that the letters in the alphabet can be generically constructed using UT integrals in the Baikov representation under various cuts. We have first considered cases in which all the cut integrals are convergent in the limit $\epsilon \to 0$. The corresponding letters coincide with the results in [37–39], while our expressions are simpler in certain cases. We have also thoroughly studied the cases of divergent cut integrals. We observe that letters in the divergent cases can be easily obtained from the convergent cases by applying certain limits. The letters admit universal expressions in terms of various Gram determinants. We have checked our general results for several known examples and observed agreements. We have also applied our results to the complicated case of a $2 \to 3$ amplitude with seven physical scales. The details of that is presented in Ref. [44].

We expect that our results will be useful in many calculations of $2 \to 3$ and $2 \to 4$ amplitudes, which are theoretically and/or phenomenologically interesting. It is also interesting to observe whether similar universal structures can be obtained at higher loop orders using the UT integrals in the Baikov representation of [16, 23].

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